STEP Support Programme

STEP 1 Specification Mechanics and Probability Notes

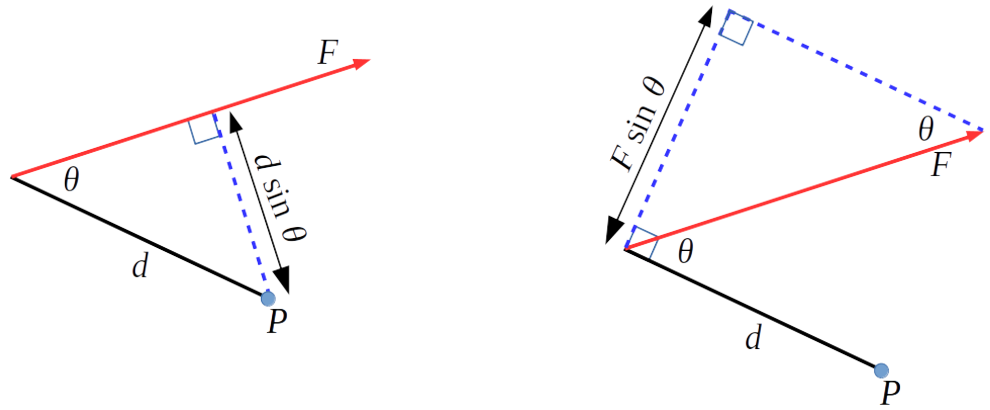
These notes are designed to help students in preparing for STEP 1. They cover the “bold and italic” sections of the STEP 1 specification which are not covered in the A-level single Mathematics specifications. Many of these topics will be covered in A and AS level Further Mathematics.

Moments

The *moment* of a force F about a point P is the product of the magnitude of the force and perpendicular distance of the line of action of the force from the point P. Point P is sometimes called the *pivot*.

Alternatively you can find the component of the force which acts perpendicularly to the pivot and then multiply this by the distance from the pivot.

The diagram below shows these two methods. They both result in the moment $Fd\sin\theta$ where θ is the angle between the line of action of the force and the line joining where the force is applied and the pivot P. I tend to find the second one easier to use, but this is a matter of personal taste!



If an object is in equilibrium then the sum of the clockwise moments will equal the sum of the anticlockwise moments.

Centres of mass

The centre of mass of an object (or a collection of objects) is the point where the mass of the object can be thought to be concentrated. When considering the force due to gravity (weight) it can be assumed that this acts vertically downwards from the centre of mass. If you hang an object from a point then the centre of mass will be vertically below this point (otherwise the weight will form a resultant moment and swing the object).

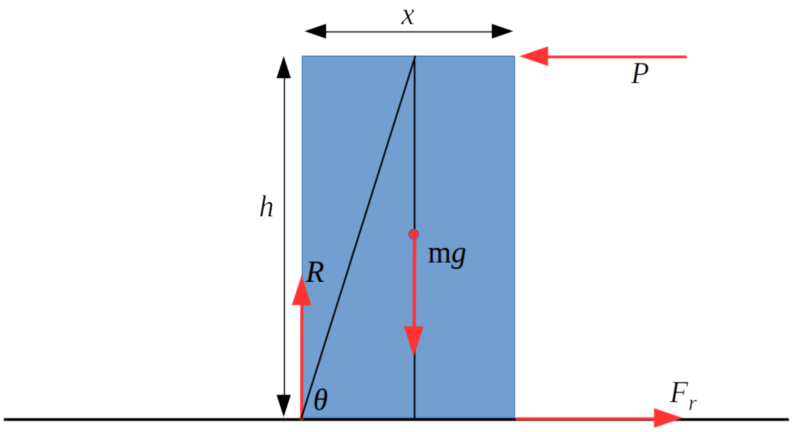
For STEP 1 the centres of mass will either be given, or can be found by symmetry. If an object has a plane of reflective symmetry then the centre of mass will lie on this line (and if the object has three planes of symmetry that meet at a point then the centre of mass will be at this point). Common shapes whose centres of mass are deducible by symmetry are cuboids, cylinders, rectangular and circular laminae.

Toppling or Sliding (Tip or Slip)

If we give a tall, thin cuboid a push near its top then it would probably tip over. If we tried the same on a short, wide cuboid then it would probably slide instead.

Example 1

A block pushed horizontally by a force P at the top of the block. It will slide (or slip) if $P > F_{r\max} = \mu R = \mu mg$. Note that this does not depend on the shape and size of the block, or where the horizontal pushing force is applied.



For the box to topple (or tip), the anticlockwise moments must be greater than the clockwise moment. When the block is on the point of tipping then the normal reaction force will be acting at the front corner of the block.

Taking moments about the bottom left corner then the block will tip if $Ph > mg \times \frac{x}{2} \implies P > \frac{mgx}{2h}$. Note that as h increases and x decreases (i.e. the block gets taller and thinner) the force needed to tip the block decreases. This is what we would expect — tall thin shapes are easier to topple!

The block will slip if the force needed to overcome friction is less than the force needed to topple the block, so we have:

- If $\mu mg < \dfrac{mgx}{2h}$ the block will slide
- If $\mu mg > \dfrac{mgx}{2h}$ the block will tip

Note that we can cancel the "mg" to give:

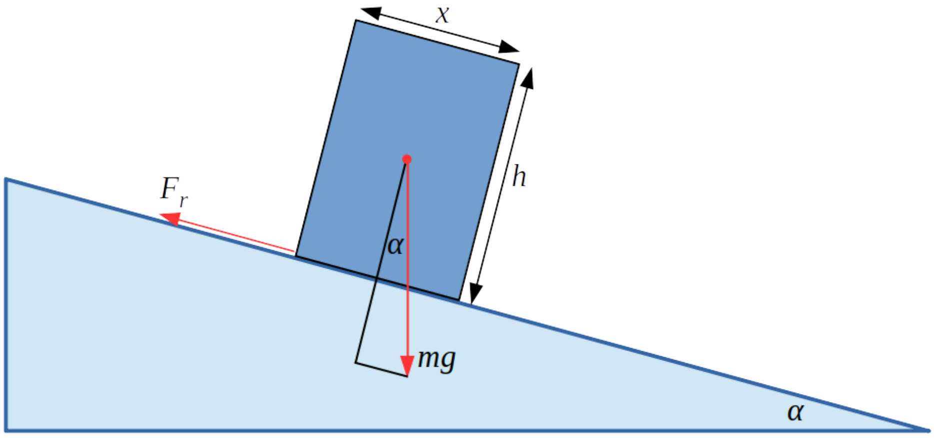
- If $\mu < \dfrac{x}{2h}$ the block will slide
- If $\mu > \dfrac{x}{2h}$ the block will tip

Another way of thinking about this is that if $\mu < \frac{1}{\tan\theta}$ (where θ is the angle shown on the diagram) then the box will slide and if $\mu > \frac{1}{\tan\theta}$ then the box will tip.

It is a very good idea to make sure that your conclusions agree with reality. In my first draft of this document I said that the boundary condition for topping was $\mu > \tan\theta$, which means that as the box gets taller and thinner it is harder to tip (a larger frictional force is needed to stop it sliding first). This doesn't make sense — I had messed up my right angled trig ratios.

Example 2

In this case a block is placed on a ramp and the angle is increased until the block either slips or tips.



The box will start to slide when $mg\sin\alpha > \mu mg\cos\alpha$, so when $\tan\alpha > \mu$.

The box will start to tip when there is a clockwise moment about the bottom right corner. This will happen when the force due to the weight of the block acts beyond the bottom right corner. This will happen when $\tan\alpha > \frac{x}{h}$. (Note that the frictional force and normal reaction force act through this bottom corner so do not produce a moment).

So if $\mu < \frac{x}{h}$ the box will slide before it tips, otherwise the box will tip first.

Triangle of Forces

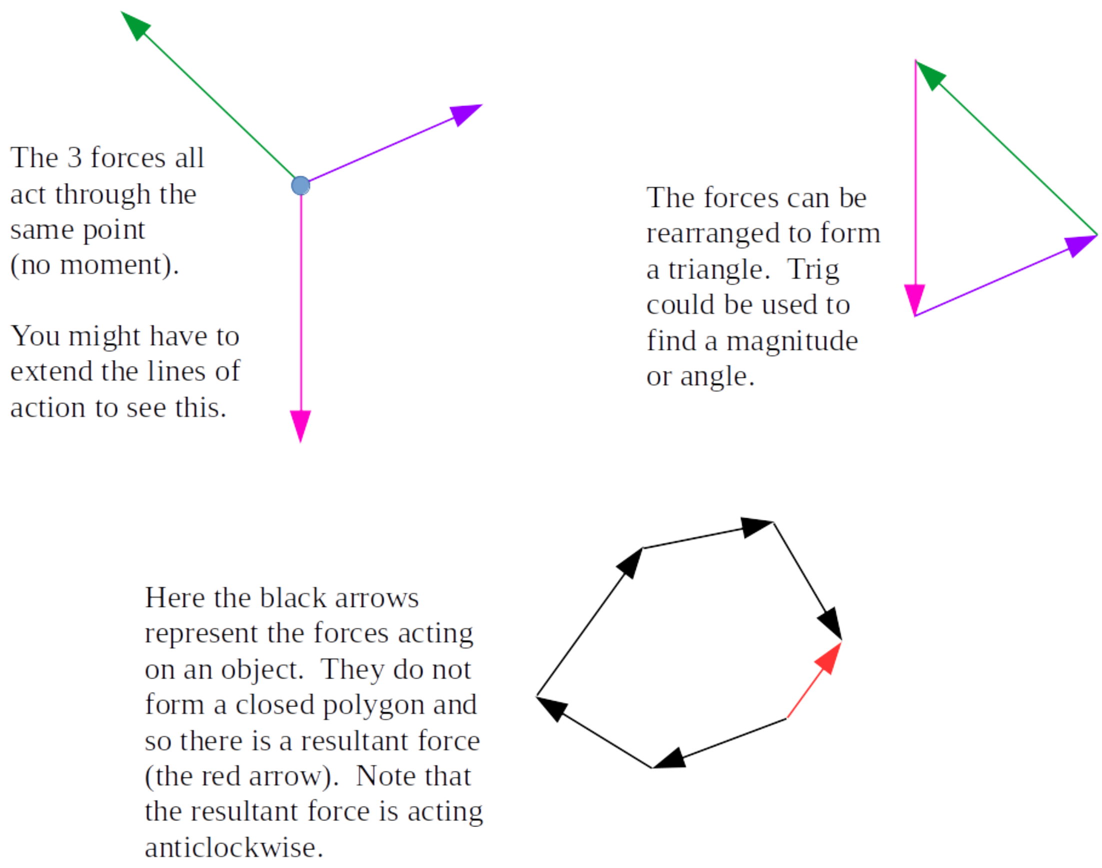
This is sometimes useful, and might avoid the need to resolve forces into components.

If an object is resting in equilibrium by the action of three co-planar, non-parallel forces then:

- The lines of actions of the three forces pass through a common point (otherwise there would be a resultant moment)
- You can draw a triangle of vectors to represent the three forces. The direction of the forces must describe a “path” around the triangle (they go “nose to tail”). This could be used to find the magnitude of one of the forces, or a direction, using trigonometry.

This can be extended to a *polygon of forces*. If the object is not in equilibrium then the missing side of the polygon will give the resultant force (but the direction of this missing side will be opposite to the rest).

The top part of the picture below shows how three forces in equilibrium can be rearranged into a triangle of forces. The lower part shows how a polygon of forces can be used to find the resultant force.



Probability

Random sample

In the simplest form of a random sample, every member of a population (which might be people, but might be light bulbs etc.) has the same probability of being picked — essentially names in a hat. There are other methods of random sampling but detailed knowledge of stratified or cluster samples etc is unlikely to be needed!

Complementary events

The complement of any event "A" (written as A^c or A') is the event "not A", i.e. that event A does not happen.

Complementary events are necessarily mutually exclusive (you cannot have both "A" and "not A" happening simultaneously!). We also have:

$$\mathrm{P}(A^c) = 1 - \mathrm{P}(A)\,.$$

Complementary events can make probability calculations easier. For example, assume a dice is thrown 10 times. What is the probability that I get at least one "6"?

$$\begin{aligned}\mathrm{P}(\text{at least one } 6) &= 1 - \mathrm{P}(\text{no } 6\text{s}) \\ &= 1 - \left(\frac{5}{6}\right)^{10}\end{aligned}$$

Addition rule for probability

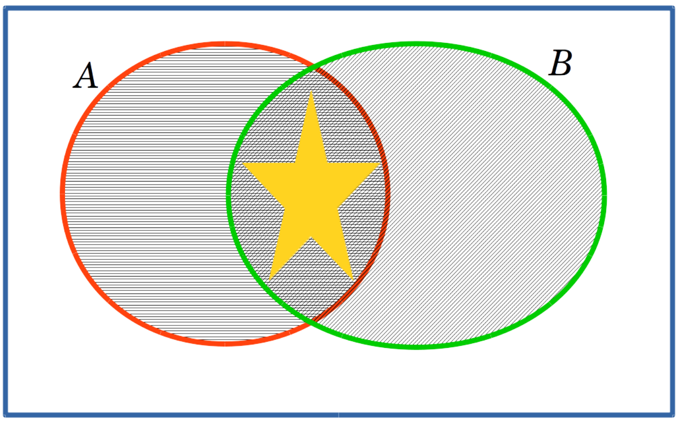
If events A and B are *mutually exclusive* (that is, they cannot both occur at the same time) then:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$$

Otherwise we have:

$$\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B) - \mathrm{P}(A \cap B)$$

This can be shown using a Venn diagram, as below:



$\mathrm{P}(A)$ is represented by the area in the red circle, and $\mathrm{P}(B)$ is the area in the green circle. $\mathrm{P}(A \cap B)$ is the bit common to both, and is the part with the star in it.

If we do $\mathrm{P}(A) + \mathrm{P}(B)$ then we have "double-counted" the bit in the middle (i.e. $\mathrm{P}(A \cap B)$), so we need to take one of these away.

Note that if A and B *are* mutually exclusive then we have $\mathrm{P}(A \cap B) = 0$.

Combinatorial arguments

Remember that ${}^nC_r$ is the number of ways of choosing r objects from a total of n objects (where order does not matter). Also, the number of ways of arranging n objects is $n!$.

An example: a team of 4 players is needed for a tennis competition, and is selected at random from a year group of 15 pupils. There are a total of 9 girls and 6 boys in the year, find the probability that the tennis team will have 2 boys and 2 girls.

The total number of ways of picking the team of 4 from the 15 pupils is ${}^{15}C_4$. The number of ways of picking 2 out of the 9 girls **and** 2 out of the 6 boys is ${}^9C_2 \times {}^6C_2$. Therefore the probability that there are 2 boys and 2 girls is:

$$\begin{aligned}
\frac{{}^9C_2 \times {}^6C_2}{{}^{15}C_4} &= \frac{\frac{9!}{2!7!} \times \frac{6!}{2!4!}}{\frac{15!}{4!11!}} \\
&= \frac{9! \times 6! \times \cancel{4!}11!}{2!7! \times 2!\cancel{4!} \times 15!} \\
&= \frac{9 \times 8 \times 6!}{2! \times 2! \times 15 \times 14 \times 13 \times 12} \\
&= \frac{9 \times 8 \times 6 \times 5 \times \cancel{4} \times 3 \times 2}{\cancel{2 \times 2} \times 15 \times 14 \times 13 \times 12} \\
&= \frac{9 \times 8 \times \cancel{6} \times \cancel{5} \times \cancel{3} \times \cancel{2}}{\cancel{15} \times 14 \times 13 \times \cancel{12}} \\
&= \frac{36}{91}
\end{aligned}$$

You can find some STEP questions using these ideas by searching for "Combinatorics" on the STEP database. There are some more notes on arrangements and combinatorics in the Pure STEP 1 Specification notes.

Statistical distributions

Random variables

The most memorable definition of a random variable I have heard is "It's not random, and it's not a variable". More usefully a random variable is a way of attaching a numerical value to the outcome of an event. For example you could let your random variable X be equal to the score shown when a fair dice is rolled, but you could instead define X to be equal to 1 if a six is rolled, and 0 if a number other than six is rolled.

More formally, a random variable is a function which maps the sample space (Ω) into the real numbers ($\mathbb{R}$). If you consider two coins being tossed then $\Omega = HH, HT, TH, TT$ and we could define X to be the number of heads so that $X(HH) = 2$ etc. You could define other random variables, e.g. Y could be equal to 1 if the coins show the same side and 0 if they are different.

Expectation and variance of discrete random variables

The *Expectation* of a discrete random variable X is given by $\mathrm{E}(X) = \displaystyle\sum_{\text{all } i} i \times \mathrm{P}(X = i)$.

Here I have used "all i", as the range of i varies from situation to situation, and it might even be the case that i is not restricted to integers.

Example: suppose that a biased coin has probability p, where $0 < p < 1$, of landing on heads. Let X by the number of coin tosses up to and including the first head. Let $q = 1 - p$ be the probability that the coin turns up tails (and note that we have $0 < q < 1$).

Then $\mathrm{E}(X)$ is given by:

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{i=1}^{\infty} i \times \mathrm{P}(X = i) \\
&= 1 \times \mathrm{P}(X = 1) + 2 \times \mathrm{P}(X = 2) + 3 \times \mathrm{P}(X = 3) + \cdots \\
&= p + 2qp + 3q^2p + 4q^3p + \cdots \\
&= p(1 + 2q + 3q^2 + 4q^3 + \cdots)
\end{aligned}$$

The only problem now is how to find the sum of the sequence in q. We know that:

$$1 + q + q^2 + q^3 + q^4 + \cdots = \frac{1}{1-q} = (1-q)^{-1}$$

by using the sum of a geometric sequence (remember that $0 < q < 1$).

Differentiating both sides of this with respect to q gives:

$$0 + 1 + 2q + 3q^2 + 4q^3 + \cdots = (1-q)^{-2} = p^{-2}$$

Hence the expectation is equal to $p \times p^{-2} = \dfrac{1}{p}$.

This then means that if the probability of getting heads was $p = \frac{1}{3}$, then the expected number of tosses up to and including the first head is $\dfrac{1}{p} = 3$.

The *Variance* of a discrete random variable is given by:

$$\text{Var}(X) = \sum i^2 \text{P}(X = i) - [\text{E}(x)]^2$$

Example: Find the Variance of the score when you roll a normal dice.

Let X be the score on the dice. Then:

$$\begin{aligned} \text{E}(X) &= 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} \\ &= \frac{21}{6} \\ &= 3\tfrac{1}{2} \end{aligned}$$

We also have:

$$\begin{aligned} \text{E}(X^2) &= 1^2 \times \frac{1}{6} + 2^2 \times \frac{1}{6} + 3^2 \times \frac{1}{6} + 4^2 \times \frac{1}{6} + 5^2 \times \frac{1}{6} + 6^2 \times \frac{1}{6} \\ &= \frac{91}{6} \end{aligned}$$

So the Variance is

$$\begin{aligned} \text{Var}(X) &= \frac{91}{6} - \left[\frac{7}{2}\right]^2 \\ &= \frac{91}{6} - \frac{49}{4} \\ &= \frac{182 - 147}{12} \\ &= \frac{35}{12} \end{aligned}$$

Note that $\text{Var}(X) \geqslant 0$, so if you get something negative then you have gone wrong somewhere!

Discrete uniform distribution

The Discrete uniform distribution is one where there are a finite set of elements, and they each have the same probability of occurring. If there are N elements then the probability that each individual element occurs is $\frac{1}{N}$. The score on a fair normal dice is an example of a discrete uniform distribution.

Non-standard Normal distributions

A general Normal distribution has mean μ and variance σ^2, and we can write $X \sim N(\mu, \sigma^2)$. The general normal is a translation and stretch of the standard normal, and we can use this to convert back to a standard normal (and then use statistical tables or calculators to find probabilities).

If $X \sim N(\mu, \sigma^2)$ then $Z = \dfrac{X - \mu}{\sigma} \sim N(0, 1)$. So if we have $X \sim N(100, 25^2)$ and we want to find the probability that $X < 120$ we can do:

$$
\begin{aligned}
\mathrm{P}(X < 120) &= \mathrm{P}\left(Z < \frac{120 - 100}{25}\right) \\
&= \mathrm{P}(Z < 0.8) \\
&= \Phi(0.8) \\
&= 0.7881
\end{aligned}
$$