# STEP Support Programme

## STEP 2 Specification Pure Notes



These notes are designed to help students in preparing for STEP 2. They cover the "bold and italic" sections of the STEP 2 specification which are not covered in the A-level single Mathematics specifications, or AS Further Maths Common Core. Many of these topics will be covered in A Level Further Mathematics, and will be covered in some AS Further Mathematics modules.

There are more notes on the various sections of the specification in the STEP 2 modules.

## Complex Numbers

Quadratic equations involving complex coefficients can be solved by using the quadratic formula. You may have to find the square root of a complex number in order to do this. To find the square root of $a+ib$ set $\sqrt{a+\mathrm{i}b} = x + \mathrm{i}y \implies a + \mathrm{i}b = (x+\mathrm{i}y)^2$ and then solve for $x$ and $y$ by equating real and imaginary parts.

Questions involving cubics and quartics will usually come with enough information for a root to be deduced, and then the corresponding factor can be divided out. I would probably use a "backwards table" rather than long division to do this.

**Note that if an equation has complex coefficients then the complex roots do not necessarily occur in complex conjugate pairs. There will be at least one complex root where the conjugate of this is not a root.**

## Matrices

### Determinants

The determinant of a $2 \times 2$ matrix is given by:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc$$

If the matrix is representing a transformation, then the determinant of the matrix gives the area scale factor of the transformation.

For example, if the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 9 \end{pmatrix}$$

is used to transform a triangle with vertices at coordinates $(0,0),\ (0,1),\ (4,0)$, then we can find the area of the resulting triangle by taking the area of the old triangle $(\frac{4\times 1}{2} = 2)$ and multiplying it by the determinant of the matrix. This gives the area of the transformed triangle as $2 \times (9-6) = 6$, which is not obvious from the vertices of the new triangle.

You might like to find the coordinates of the new triangle and use these to verify that the area of the new triangle is 6. I found it easier to sketch the vertices and use areas of rectangles and right-angled triangles rather than try and use the formula $\frac{1}{2}ab\sin C$.

If the determinant is negative then the orientation of the shape is reversed. For example, if a triangle have vertices $A$, $B$ and $C$ going clockwise then if the determinant is negative then the corresponding vertices will be $A'$, $B'$ and $C'$ in an *anticlockwise* direction.

# Further Algebra and Functions

## Roots and Coefficients

For a quadratic equation of the form $ax^2 + bx + c = 0$, we know that the sum of the roots of the equation is equal to $-\frac{b}{a}$, and that the product of the roots is equal to $\frac{c}{a}$. This can be demonstrated by writing $ax^2 + bx + c = a(x - \alpha)(x - \beta)$ and equating coefficients. For a quartic equation of the form $ax^4 + bx^3 + cx^2 + dx + e = 0$ we have:

$$\begin{aligned} \sum \alpha &= -\frac{b}{a} \\ \sum \alpha\beta &= \frac{c}{a} \\ \sum \alpha\beta\gamma &= -\frac{d}{a} \\ \alpha\beta\gamma\delta &= \frac{e}{a} \end{aligned}$$

Note that $\sum \alpha\beta$ means the sum of all the possible products of pairs of roots. In the case of a quartic equation this is the sum of 6 products.

This idea generalises to higher order polynomials. If we take the general polynomial of degree $n$

$$p(x) = \sum_{i=0}^{n} a_i x^i.$$

then the sum of the roots of $p(x) = 0$ is $-\frac{a_{n-1}}{a_n}$, and the sum of products of pairs of roots is $\frac{a_{n-2}}{a_n}$ etc...

## Partial Fractions

If the denominator contains a quadratic factor of the form $ax^2 + c$ where $c > 0$ then when you are splitting into partial fractions you need to include a fraction of the form $\dfrac{AX + B}{ax^2 + c}$.

Note that if $c < 0$ then you can factorise the quadratic e.g. $4x^2 - 1 = (2x + 1)(2x - 1)$.

**Example:**

Evaluate $I = \displaystyle\int_0^1 \frac{1}{x^3 + x^2 + 4x + 4}\,\mathrm{d}x$

Considering the cubic in the denominator we notice that $x = -1$ is a root of this and so $(x + 1)$ is a factor. This gives us:

$$x^3 + x^2 + 4x + 4 = (x + 1)(x^2 + 4)$$

We can write this in partial fractions as:

$$\frac{1}{(x + 1)(x^2 + 4)} = \frac{C}{x + 1} + \frac{Ax + B}{x^2 + 4}$$

Multiplying throughout by $(x + 1)(x^2 + 4)$ gives $1 = C(x^2 + 4) + (Ax + B)(x + 1)$, and then substituting $x = -1$ gives $C = \frac{1}{5}$. With this value of $C$ we can substitute $x = 0$ to give $B = \frac{1}{5}$ and then equating coefficients of $x^2$ we have $A = -\frac{1}{5}$.[1]

[1] Equating coefficients and substituting values of $x$ are both valid methods and can be mixed together. Go for whatever makes the arithmetic easiest!

The integral is now:

$$\begin{aligned} I &= \frac{1}{5}\int_0^1 \frac{1}{x+1} + \frac{1-x}{x^2+4}\,\mathrm{d}x \\ &= \frac{1}{5}\int_0^1 \frac{1}{x+1} + \frac{1}{x^2+4} - \frac{1}{2}\left(\frac{2x}{x^2+4}\right)\mathrm{d}x \\ &= \frac{1}{5}\left[\ln(x+1) + \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\ln(x^2+4)\right]_0^1 \\ &= \frac{1}{5}\ln(2) + \frac{1}{10}\tan^{-1}\left(\frac{1}{2}\right) - \frac{1}{10}\ln\left(\frac{5}{4}\right) \end{aligned}$$

The result $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)|$ is always worth looking for! Here it is used to find $\displaystyle\int \frac{2x}{x^2+4}\,\mathrm{d}x$. The integral $\displaystyle\int \frac{1}{x^2+4}\,\mathrm{d}x$ can be found using a tan substitution. In these sorts of cases it is ok to show the substitution and working or to just write down the general result. However if the question was a "show that" then you would need to show your working..

**Method of differences**

Partial fractions can often be used to help evaluate sums (both finite sums and in some cases infinite sums).

**Example:**

Find, in terms of $N$:

$$\sum_{n=1}^{N} \frac{1}{n^2+n}$$

and hence evaluate $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n}$.

First we can write $\displaystyle\frac{1}{n^2+n} = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$.

We then have:

$$\begin{aligned} \sum_{n=1}^{N} \frac{1}{n^2+n} &= \sum_{n=1}^{N}\left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \frac{1}{1} - \cancel{\frac{1}{2}} \\ &\quad + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} \\ &\quad\quad + \cancel{\frac{1}{3}} - \cancel{\frac{1}{4}} \\ &\quad\quad\quad \cdots \\ &\quad\quad\quad\quad + \cancel{\frac{1}{N}} - \frac{1}{N+1} \\ &= 1 - \frac{1}{N+1} \end{aligned}$$

Then letting $N \to \infty$ we have $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+n} = 1$.

If the degree of the numerator is greater than or equal to the degree of the denominator then you should divide the numerator by the denominator before using partial fractions. Long algebraic division is often not necessary:

$$\frac{x^2+4}{1-x^2} = \frac{(1-x^2)\times -1 + 1 + 4}{1-x^2} = \frac{5}{1-x^2} - 1$$

**Series expansion for** $\mathrm{e}^x$

The series expansion for $\mathrm{e}^x$ is given by:

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

It is worth noticing that "an exponential always beats a polynomial", or to put it another way $\mathrm{e}^{-x}x^n \to 0$ as $n \to \infty$. You can see this by considering:

$$\begin{aligned}
\mathrm{e}^{-x}x^n &= \frac{x^n}{\mathrm{e}^x} \\
&= \frac{x^n}{1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + \cdots} \\
&= \frac{1}{\frac{1}{x^n} + \frac{1}{x^{n-1}} + \frac{1}{2!x^{n-2}} + \cdots + \frac{1}{n!} + \frac{x}{(n+1)!} + \cdots}
\end{aligned}$$

Then as $x \to \infty$ lots of terms in the denominator tend to zero, but others tend to infinity, and so we have $\mathrm{e}^{-x}x^n \to 0$ as $n \to \infty$.

**Curves**

The curve $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ describes an ellipse which crosses the $x$-axis at $(\pm a, 0)$ and crosses the $y$-axis at $(0, \pm b)$. It can be considered to be a two way stretch of the unit circle $x^2 + y^2 = 1$.

The curve $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ describes a hyperbola which crosses the $x$-axis at $(\pm a, 0)$ and doesn't cross the $y$-axis. As $x$ and $y$ both get very large they are both much larger than 1 and so we have $\dfrac{x^2}{a^2} \approx \dfrac{y^2}{b^2}$ and the asymptotes of this curve are $y = \pm\dfrac{bx}{a}$.

You might like to investigate these curves using Desmos.

# Further Calculus

## Improper Integrals

There are two basic types of improper integrals:

1. where the range of integration extends to infinity (or negative infinity) e.g. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,\mathrm{d}x$

2. where the *integrand*[2] is undefined for some value(s) in the range of integration e.g. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x$ (in this case $\dfrac{1}{\sqrt{x}}$ is undefined when $x = 0$)

In this second case you need to be even more careful when the integrand is undefined at some point *strictly between* the limits. The usual way to deal with this is to write the integral as two separate ones so that the undefined value is equal to one of the limits in each case. For example:

$$\int_{-2}^{3} \frac{1}{\sqrt{x}}\,\mathrm{d}x = \int_{-2}^{0} \frac{1}{\sqrt{x}}\,\mathrm{d}x + \int_{0}^{3} \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

In this example we would actually get the same answer whether we naively evaluated the left hand integral or considered the right hand side more carefully.

If we consider $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x$ and evaluated this in the "usual way" we would get the value $-2$.[3]

Splitting up the integral gives $\displaystyle\int_{-1}^{1} \frac{1}{x^2}\,\mathrm{d}x = \int_{-1}^{0} \frac{1}{x^2}\,\mathrm{d}x + \int_{0}^{1} \frac{1}{x^2}\,\mathrm{d}x$.

Consider $\displaystyle\int_{s}^{1} \frac{1}{x^2}\,\mathrm{d}x$ and then see what happens as $s \to 0$. We have:

$$\begin{aligned}\int_{s}^{1} \frac{1}{x^2}\,\mathrm{d}x &= \left[-\frac{1}{x}\right]_s^1 \\ &= -1 + \frac{1}{s}\end{aligned}$$

and so as $s \to 0$ this tends to infinity and the integral is undefined.

For an integral like $\displaystyle\int_{1}^{\infty} \frac{1}{x^2}\,\mathrm{d}x$ we can consider:

$$\begin{aligned}\int_{1}^{t} \frac{1}{x^2}\,\mathrm{d}x &= \left[-\frac{1}{t}\right]_1^t \\ &= -\frac{1}{t} + 1\end{aligned}$$

and then as $t \to \infty$ then this converges to 1 and so we have:

$$\int_{1}^{\infty} \frac{1}{x^2}\,\mathrm{d}x = 1$$

[2] The bit inside the integral.

[3] This is definitely a little fishy as $\frac{1}{x^2}$ is always positive, so you would expect the integral to be positive

**Trigonometric Functions**

To differentiate $y = \sin^{-1} x$ start by re-writing this as $x = \sin y$. Then we have:

$$\begin{aligned} \sin y &= x \\ \cos y \times \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\cos y} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sqrt{1-x^2}} \end{aligned}$$

In a very similar way you can show that $\dfrac{\mathrm{d}}{\mathrm{d}x}\cos^{-1} x = -\dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1} x = \dfrac{1}{1+x^2}$.

Alternatively you can use the fact that $\cos^{-1} x = \frac{1}{2}\pi - \sin^{-1} x$ to find the derivative of $\cos^{-1} x$ from the derivative of $\sin^{-1} x$.

We therefore have:

$$\int \frac{1}{1+x^2}\,\mathrm{d}x = \tan^{-1}(x) + c$$

and

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \sin^{-1}(x) + c.$$

**Partial Fractions**

See the section on partial fractions in Further Algebra and Functions for an example of using partial fractions with integration.

**Reduction Formulae**

Sometimes we have an integral which depends on an integer $n$, which is often written as $I_n$. This might be able to be manipulated to write it in terms of $I_{n-1}$ and this process could be repeated until we reach $I_1$ (which hopefully is reasonably easy to integrate). These are also sometimes called *recurrence relations* and if you are using the STEP database the questions are usually tagged "recurrence" rather than "reduction".

For example, let:

$$I_n = \int x^n \mathrm{e}^x \,\mathrm{d}x, \;\; n \geq 0$$

Using integration by parts (with $u = x^n, \;\; v' = e^x$) gives:

$$I_n = x^n \mathrm{e}^x - \int n x^{n-1} \mathrm{e}^x \,\mathrm{d}x = x^n \mathrm{e}^x - nI_{n-1}, \;\; n \geq 1$$

and so our reduction formula is $I_n = x^n \mathrm{e}^x - nI_{n-1}$.

We can find $I_0 = \int \mathrm{e}^x \,\mathrm{d}x = \mathrm{e}^x$, and then use the reduction formula to find $I_1 \,(= x\mathrm{e}^x - \mathrm{e}^x)$, $I_2 \,\big(= x^2\mathrm{e}^x - 2(x\mathrm{e}^x - \mathrm{e}^x)\big)$ etc.

In some cases (such as 1995 STEP 3 Q2) a reduction formula can be found which enables you to calculate $I_n$ explicitly.

# Further Vectors

### Straight Lines in 3D

The vector equation for a straight line in 3D is formed from a direction vector, and a starting point. So if you know the line travels in the $\begin{pmatrix}3\\4\\2\end{pmatrix}$ direction and passes through the point $(0, -1, 1)$ then the equation of the line is

$$\mathbf{x} = \begin{pmatrix}0\\-1\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\2\end{pmatrix}, \ \lambda \in \mathbb{R}$$

where $\mathbf{x}$ is a point on the line.

Note that the vector equation of a line is not unique. We could have used any multiple of the direction vector, and any point on the line as the starting point.

To find the Cartesian form make $\lambda$ the subject of each individual coordinate equation:

$$\begin{aligned}\begin{pmatrix}x\\y\\z\end{pmatrix} &= \begin{pmatrix}0+3\lambda\\-1+4\lambda\\1+2\lambda\end{pmatrix}\\ \implies \lambda &= \frac{x}{3}\\ \lambda &= \frac{y+1}{4}\\ \lambda &= \frac{z-1}{2}\end{aligned}$$

In general, if the line passes through the point $(x_0, y_0, z_0)$, and has direction vector $\begin{pmatrix}a\\b\\c\end{pmatrix}$ then the Cartesian equation is

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c} \left( = \lambda \right)$$

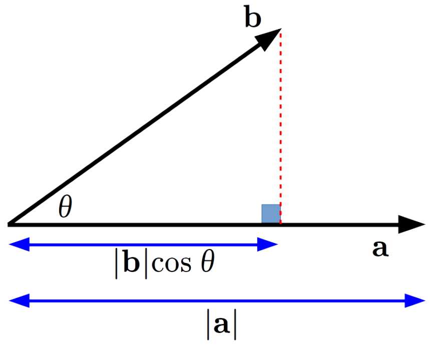
Note that if one component of the direction vector is zero, this component cannot be rearranged for $\lambda$. For example if:

$$\mathbf{x} = \begin{pmatrix}2\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}-1\\0\\2\end{pmatrix}$$

then the Cartesian equation is $\dfrac{x-2}{-1} = \dfrac{z+1}{2}$, $y = 3$.

**Scalar Product**

The scalar product of two vectors is the product of the magnitude of one of the vectors with the *scalar projection* of the second vector onto the first one. The diagram below shows the two lengths which are multiplied together in blue.




The scalar product is:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

where $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$. Note that for non-zero vectors:

$$\mathbf{a} \cdot \mathbf{b} = 0 \iff \theta = 90^\circ$$

so $\mathbf{a}$ and $\mathbf{b}$ are perpendicular if and only if the scalar product is equal to 0.

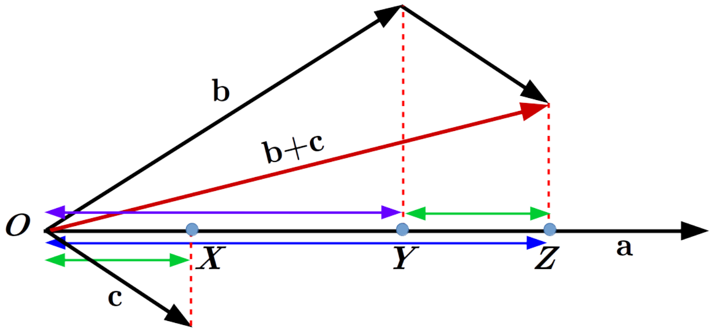
From this definition, we can show two properties of the scalar product:

- The scalar product is *commutative*[4], that is $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
- The scalar product is also *distributive*, that is: $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

The first of these can be shown by using the above diagram, but projecting $\mathbf{a}$ onto $\mathbf{b}$ to get $\mathbf{b} \cdot \mathbf{a} = |\mathbf{b}| \times |\mathbf{a}| \cos \theta$.

[4] Addition is commutative as $a + b = b + a$, but subtraction is not (In general $a - b \neq b - a$).

For the second one, consider the diagram below:




We have $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times OY$ and $\mathbf{a} \cdot \mathbf{c} = |\mathbf{a}| \times OX$.

Considering $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})$ we have:

$$\begin{aligned}
\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) &= |\mathbf{a}| \times OZ \\
&= |\mathbf{a}| \times (OY + YZ) \\
&= |\mathbf{a}| \times (OY + OX) \qquad \text{as } YZ = OX \\
&= |\mathbf{a}| \times OY + |\mathbf{a}| \times OX \\
&= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}
\end{aligned}$$

The scalar product of two vectors $\mathbf{a}, \mathbf{b}$ can also be written as:

$$\mathbf{a} \cdot \mathbf{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

where $n$ is the length of the vectors and $a_i$ is the $i^{th}$ component of $\mathbf{a}$.

This can be seen by using some of the properties of the scalar product. In three dimensions we have:

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} \qquad \text{and} \qquad \mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

and so:

$$\begin{aligned}
&(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot (b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\
=&(a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot b_1\mathbf{i} \\
&\quad + (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot b_2\mathbf{j} \\
&\qquad + (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \cdot b_3\mathbf{k} \\
=&a_1b_1\mathbf{i}\cdot\mathbf{i} + a_2b_1\mathbf{j}\cdot\mathbf{i} + a_3b_1\mathbf{k}\cdot\mathbf{i} \\
&\quad + a_1b_2\mathbf{i}\cdot\mathbf{j} + a_2b_2\mathbf{j}\cdot\mathbf{j} + a_3b_2\mathbf{k}\cdot\mathbf{j} \\
&\qquad + a_1b_3\mathbf{i}\cdot\mathbf{k} + a_2b_3\mathbf{j}\cdot\mathbf{k} + a_3b_3\mathbf{k}\cdot\mathbf{k} \\
&= a_1b_1 + a_2b_2 + a_3b_3
\end{aligned}$$

In the last line we have just the fact that if two vectors are perpendicular then their scalar product is equal to 0.