STEP Support Programme

STEP 3 Mechanics Notes

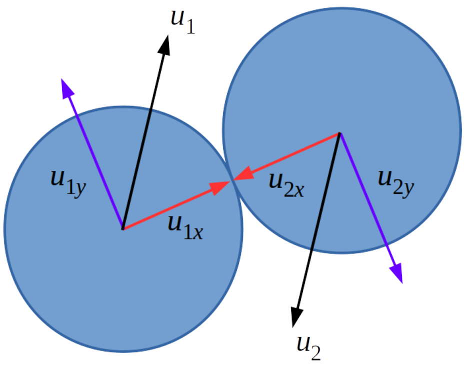
These notes are designed to help students in preparing for STEP 3. They cover the "bold and italic" sections of the STEP 3 specification which are not covered in the A-level single Mathematics specifications, or A-level Further Maths Common Core. Many of these topics will be covered in the optional A Level Further Mathematics modules.

There are more notes on the various sections of the specification in the STEP 3 modules.

Further Collisions

Oblique collisions occur when you are considering objects other than point masses (usually smooth spheres — think of snooker balls) and the spheres are not heading directly towards each other — they hit a "glancing blow". Another example is when a sphere hits a wall but was not travelling perpendicularly to the wall.

The picture below shows two smooth spheres colliding. The black arrows show the velocities of the spheres and the red and purple arrows show the components of velocity along the line connecting the centres and the components of velocity perpendicular to this.



Since the spheres are smooth, the components of velocity perpendicular to the line connecting the centres (LoC) are unchanged by the collision. The components along the LoC obey conservation of momentum and Newtons Law of restitution as in 1D collisions.

This means we have:

$$\begin{aligned} u_{1y} &= v_{1y} \\ u_{2y} &= v_{2y} \\ m_1u_{1x} - m_2u_{2x} &= m_1v_{1x} + m_2v_{2x} \\ v_{2x} - v_{1x} &= e(u_{1x} + u_{2x}) \end{aligned}$$

I have taken into account the directions of my arrows when considering u_{2x} – for the other velocities parallel to LoC I have taken them to be going in the same direction as u_{1x}.

Note that if the spheres were not smooth then there would be friction to take into account which would affect the perpendicular velocities.

Generally speaking it is best to resolve the velocities into the directions parallel and perpendicular to the LoC, but if you are considering what happens when an impulse is applied to an object a different set of perpendicular directions might be better.

An *impulse* applied to an object will will cause a change in the objects momentum. We have $\mathbf{I} = m\mathbf{v} - m\mathbf{u}$ where $\mathbf{I}$ is the impulse, $\mathbf{u}$ is the velocity beforehand and $\mathbf{v}$ is the velocity afterwards. If a smooth sphere collides with a smooth wall then the impulse will be perpendicular to the wall.

If a rough sphere collides with rough wall then there will be friction providing an impulse parallel to the wall which will cause a change in momentum in this direction as well (which will have the affect of slowing the ball down in this direction!).

Relative Motion

If the position vector of particle A is $\mathbf{a}$ and the position vector of particle B is $\mathbf{b}$ then the position of A relative to B is given by $\mathbf{x}_R = \mathbf{a} - \mathbf{b}$. The two particles will collide when $\mathbf{a} - \mathbf{b} = 0$.

The relative velocity of particle A with respect to particle B is $\mathbf{v}_R = \dot{\mathbf{a}} - \dot{\mathbf{b}}$. The particles will collide if $\mathbf{v}_R = k\overrightarrow{A_0B_0}$ (where $k > 0$ and A_0, B_0 are the initial positions of A and B).

If the particles do not collide, you can find when they are closest by when finding when $|\mathbf{x}_R|$ is minimised. This can be done by differentiating with respect to t (might be easiest to consider $|\mathbf{x}_R|^2$ in this case) or by finding when $\mathbf{x}_R \cdot \mathbf{v}_R = 0$.

Centre of Mass

The effect of gravity on an object can be thought of as a single force acting at the object's centre of mass.

If you have a *uniform* rigid body then the centre of mass will lie on any lines of symmetry (for a lamina) or planes of symmetry (for a 3-D shape). "Uniform" in this case means that the mass is evenly spread out throughout the rigid body.

If you have a system of particles, or a rigid body made up of several rigid bodies whose centres of mass are known then the coordinates of the centre of mass of the whole system are the weighted mean of the coordinates of the separate parts.

For example the x coordinate will be given by:

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

Or if you describe the position of the individual masses with vectors, $\mathbf{r}_i$:

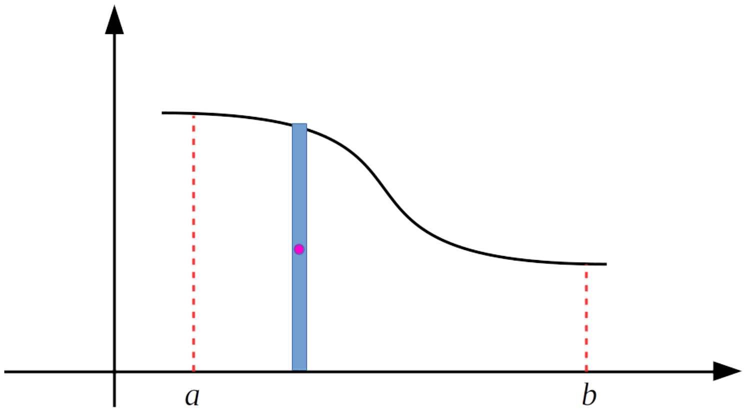
$$\bar{\mathbf{r}} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}$$

Using integration to find centres of mass

Uniform lamina

Consider a uniform lamina bounded by the x-axis, $x = a$, $x = b$ and $y = \mathrm{f}(x)$ (you may want to sketch this!). Let the mass per unit area of the lamina be ρ.

You can think of this lamina as being made up of small rectangles of width δx. The area of one of these is $y\delta x$, and the centre of mass will by at $\left(x + \frac{1}{2}\delta x, \frac{1}{2}y\right)$. Again, draw a sketch to help convince you of this.



Depending on where you draw the rectangles the y coordinate of the centre of mass might be at $\frac{1}{2}y$, $\frac{1}{2}y + \delta y$ or $\frac{1}{2}(y + \delta y)$. For the diagram above I have used rectangles with coordinates $(x, 0); (x, y); (x + \delta x, y); (x + \delta x, 0)$.

We therefore have a set of small rectangles of mass $\rho y\delta x$ and centre of mass $\left(x + \frac{1}{2}\delta x, \frac{1}{2}y\right)$. Using the weighted mean approach for a composite lamina we have:

$$\bar{x} = \frac{\displaystyle\sum_{x=a}^{b} (\rho y\delta x)\left(x + \tfrac{1}{2}\delta x\right)}{\displaystyle\sum_{x=a}^{b} \rho y\delta x} \qquad \text{and} \qquad \bar{y} = \frac{\displaystyle\sum_{x=a}^{b} (\rho y\delta x)\tfrac{1}{2}y}{\displaystyle\sum_{x=a}^{b} \rho y\delta x}$$

Cancelling the ρ's and ignoring the δx^2 term as being very, very small we have:

$$\bar{x} = \frac{\displaystyle\sum_{x=a}^{b} xy\delta x}{\displaystyle\sum_{x=a}^{b} y\delta x} \qquad \text{and} \qquad \bar{y} = \frac{\displaystyle\sum_{x=a}^{b} \tfrac{1}{2}y^2\delta x}{\displaystyle\sum_{x=a}^{b} y\delta x}$$

Taking the limit as $\delta x \to 0$ gives:

$$\bar{x} = \frac{\displaystyle\int_{x=a}^{b} xy\,\mathrm{d}x}{\displaystyle\int_{x=a}^{b} y\,\mathrm{d}x} \qquad \text{and} \qquad \bar{y} = \frac{\displaystyle\int_{x=a}^{b} \tfrac{1}{2}y^2\,\mathrm{d}x}{\displaystyle\int_{x=a}^{b} y\,\mathrm{d}x}$$

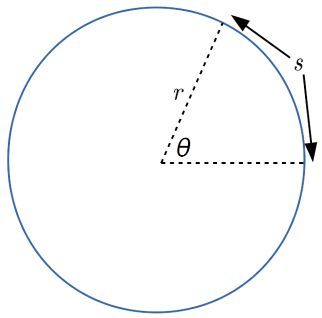
Circular Motion

Angular velocity is a measure of how fast something is moving around in a circle. If θ is the angle swept through by the object as it moves round then the angular velocity is the rate of change of this, i.e.:

$$\text{angular velocity} = \dot{\theta} = \frac{\mathrm{d}\theta}{\mathrm{d}t}$$

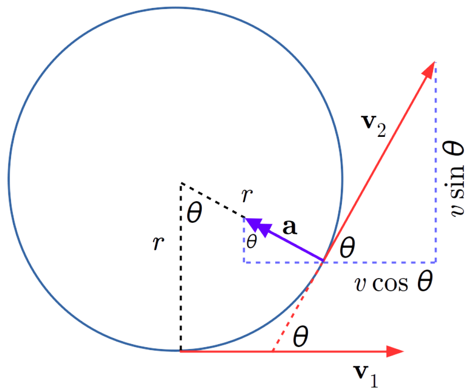
The symbol ω is sometimes used to represent $\dot{\theta}$.

As a particle moves around the circle and "sweeps out" an angle θ it will have travelled a distance $s = r\theta$. Since r is constant the linear speed v and angular velocity are related by $v = r\dot{\theta}$.



Constant speed in a horizontal circle

Even if the particle moves around at a constant speed, the velocity will be changing as the direction of the particle is changing. Therefore there is an acceleration on the particle. To find this acceleration consider what happens as the particle moves around the circle.



When the particle has moved through an angle θ the velocity will be given by:

$$\mathbf{v} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$$

Differentiating this with respect to time gives the acceleration as:

$$\mathbf{a} = \begin{pmatrix} -v\dot{\theta}\sin\theta \\ v\dot{\theta}\cos\theta \end{pmatrix}$$

This means that the acceleration is directed towards the centre of the circle as shown on the diagram above and has magnitude $a = v\dot{\theta}$ which can (more usefully) be written as $a = r\dot{\theta}^2$ or $a = \dfrac{v^2}{r}$.

Motion in a vertical circle

When a particle moves in a vertical circle the speed v (as well as the velocity $\mathbf{v}$) changes. In this case when we differentiate $\mathbf{v} = \begin{pmatrix} v\cos\theta \\ v\sin\theta \end{pmatrix}$ we get:

$$\mathbf{a} = \begin{pmatrix} \dot{v}\cos\theta - v\dot{\theta}\sin\theta \\ \dot{v}\sin\theta + v\dot{\theta}\cos\theta \end{pmatrix} = \dot{v}\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} + v\dot{\theta}\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

This means that as well as an acceleration of $v\dot{\theta}$ towards the centre of the circle there is also an acceleration of $\dot{v} = r\ddot{\theta}$ tangentially to the circle.

Note that:

$$\ddot{\theta} = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t} \times \frac{\mathrm{d}\omega}{\mathrm{d}\theta} = \omega\frac{\mathrm{d}\omega}{\mathrm{d}\theta}$$

which can sometimes be useful when integrating!

Another useful thing to note is that:

$$\frac{\mathrm{d}\left(\dot{\theta}\right)^2}{\mathrm{d}t} = 2\ddot{\theta}\dot{\theta} \implies \int \ddot{\theta}\dot{\theta}\,\mathrm{d}\theta = \tfrac{1}{2}\left(\dot{\theta}\right)^2$$

Differential Equations

If the force is given as a function of distance, then it might be more helpful to use:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x}$$

Simple Harmonic Motion

This is the situation where a particle moves so that the acceleration is always directed towards a fixed point, O, and the acceleration is proportional to the distance from this fixed point. Examples include particles hanging from springs.

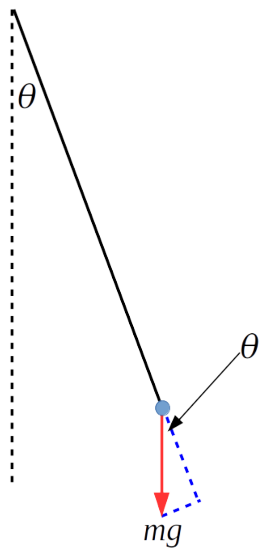
- Equation of motion $\ddot{x} = -\omega^2 x$.
- The maximum displacement from O is the amplitude, A.
- The period of the motion is $\dfrac{2\pi}{\omega}$.
- The speed is given by $v^2 = \omega^2\left(A^2 - x^2\right)$.
- The displacement is given by $x = A\sin\left(\omega t + \alpha\right)$, where α is determined by the value of x when $t = 0$.

If an additional force, $F = ma$, is applied to the system then the equation of motion becomes $\ddot{x}+\omega^2 x = a$. You may need to solve some second order differential equations (see STEP 3 Differential Equations).

The simple pendulum

Consider a simple pendulum which consists of a particle P with mass m attached to a light inextensible string of length l. It swings through a **small** angle each side of the equilibrium point.

We assume that the only force acting is gravity (so no air resistance etc). When the pendulum makes an angle of θ **radians** with the vertical the force can be resolved into components parallel and perpendicular to the string. The perpendicular force is the one pulling the pendulum back to the centre and is equal to $-mg\sin\theta$ (the negative sign is because it is always acting back towards the vertical).



We have $a = r\ddot{\theta}$, and (in this case) $r = l$, so we have:

$$\begin{aligned} ma &= -mg\sin\theta \\ l\ddot{\theta} &= -g\sin\theta \end{aligned}$$

Then, since we have said "a small angle" (and the angle is in radians) so we have $\sin\theta \approx \theta$ and we end up with the equation $\ddot{\theta} = -\frac{g}{l}\theta$. Therefore the motion of the pendulum is (approximately) simple harmonic motion with a period of $2\pi\sqrt{\frac{l}{g}}$. Note that the period does not depend on the mass of the pendulum!

Damped Harmonic Motion

Usually there is a force resisting the motion of the particle. This could be friction, air resistance or other factors. Assuming that the damping force is proportional to the velocity of the particle, and acts in the opposite direction of the movement of the particle we have:

$$m\ddot{x} = -kx - c\dot{x} \implies m\ddot{x} + c\dot{x} + kx = 0$$

See the STEP 3 Differential Equations module for methods to solve second order differential equations.