STEP Support Programme

Assignment 1

Warm-up

1 **(i)** Simplify $\sqrt{50} + \sqrt{18}$.

(ii) Express $(3 + 2\sqrt{5})^3$ in the form $a + b\sqrt{5}$ where a and b are integers.

(iii) Expand and simplify

$$(1 - \sqrt{2} + \sqrt{6})^2 \,.$$

(iv) **(a)** Expand and simplify $(1 + \sqrt{2})^2$.

(b) Find all real values of x that satisfy

$$x^2 + \frac{4}{x^2} = 12 \,.$$

Leave your answers in the form of surds.

Preparation

2 **(i)** Solve the equation:

$$\frac{2}{x+3} + \frac{1}{x+1} = 1 \,.$$

(ii) Find the value(s) of b for which the following equation has a single (repeated) root.

$$9x^2 + bx + 4 = 0 \,.$$

(iii) Find the range of (real) values of c for which the following equation has no real roots:

$$3x^2 + 5cx + c = 0 \,.$$

Probably the safest way of dealing with inequalities is to sketch a graph.

The STEP question

3 In this question a and b are distinct, non-zero real numbers, and c is a real number.

(i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

(ii) Show that, if $c \neq 1$, the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if

$$c^2 = -\frac{4ab}{(a-b)^2}\,.$$

Show that this condition can be written

$$c^2 = 1 - \left(\frac{a+b}{a-b}\right)^2$$

and deduce that it can only hold if $0 < c^2 \leqslant 1$.

Discussion

This question requires fluent algebraic manipulation and a clear head. Your layout needs to be clear — the mantra "One equal sign per line, all equal signs aligned." is a good one to follow. Be careful when using the formula "$b^2 - 4ac$" for the discriminant when one or more of the letters a b or c are defined differently in the question (as in this question). It is safer to write "$B^2 - 4AC = 0$" or "$b'^2 - 4a'c' = 0$" so that you don't get confused.

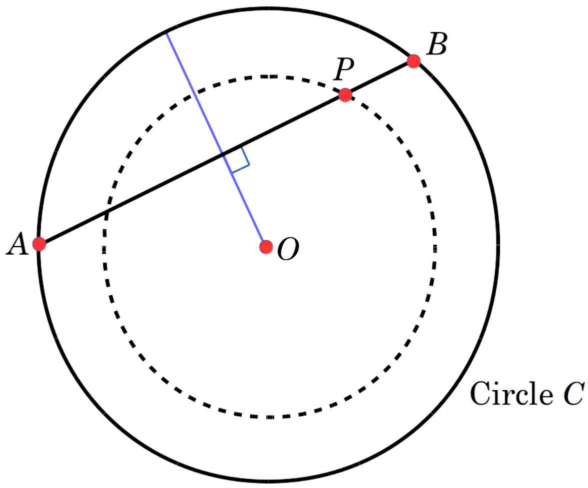
Where the answer is given ("Show that") you must be particularly careful to show every step of the argument in sufficient detail to convince yourself (and any examiners reading it) that your argument is correct and complete. Some people consider it mathematically uncouth to start with what you want to show and work backwards to the given starting point — however you can always work backwards on a scrap piece of paper and then write up your final solution reversing the steps (as long as they are truly reversible!). Or you could work forwards and backwards and meet in the middle, provided that you then construct a coherent proof.

Note that the last part uses the work "deduce" — this means that you must use the previous part and also implies that not much further working is required; however your explanations must still be clear and logical. Remember that the information given about a, b and c in the very first line of the question holds throughout the question.

Warm down

4 The diagram shows a circle C with centre O, and a rod AB the ends of which can slide round the circle C (so that AB is a chord of C). The radius of the circle is R and the length of the rod is $2a$.

As the rod slides round C the point P, which is a fixed distance b from the centre of the rod, traces out a circle with centre O of radius r.



Show that the area between the two circles is $\pi(a^2 - b^2)$.

The surprising thing about this result is that it is independent of the radius of C (assuming that it is greater than a) and depends only on the length of the stick and the position of P on the stick. It doesn't matter how big the circle is, the area between the two circles is always $\pi(a^2 - b^2)$.

If you had known that the answer was independent of R, can you think of an easy way (by choosing R) of obtaining the result?

Even more surprising is the fact that the result holds even when C is not a circle, but is any closed curve round which the rod can slide smoothly. This is Holditch's theorem, proved in about 1840, and not much seen until it was used as a STEP question in 2010.

The lengths R and r are given here for your convenience: they are required for the calculation but do not appear in the answer. In a STEP question, it would have been up to you to decide what is needed for the calculation. And you might have had to draw the diagram yourself.