STEP Support Programme

Hints and Partial Solutions for Assignment 10

Warm-up

1 **(i)** $AP = \tan\alpha$, $PC = \tan\beta$, and $AB = \frac{1}{\cos\alpha}$.
The result $\sin(90° - \beta) = \cos\beta$ will be helpful.

Using the sine rule we have:

$$\begin{aligned}\frac{\sin B}{AC} &= \frac{\sin C}{AB}\\ \frac{\sin(\alpha+\beta)}{AP+PC} &= \frac{\sin(90°-\beta)}{AB}\\ \frac{\sin(\alpha+\beta)}{\tan\alpha+\tan\beta} &= \frac{\cos\beta}{\frac{1}{\cos\alpha}}\\ \sin(\alpha+\beta) &= \cos\alpha\cos\beta\,(\tan\alpha+\tan\beta)\\ \sin(\alpha+\beta) &= \cos\alpha\cos\beta\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}\right)\\ \sin(\alpha+\beta) &= \sin\alpha\cos\beta+\cos\alpha\sin\beta\end{aligned}$$

Do be careful with brackets, write

$$\cos\alpha\cos\beta \times (\tan\alpha + \tan\beta)$$

rather than

$$\cos\alpha\cos\beta \times \tan\alpha + \tan\beta.$$

Using $\sin(-\beta) = -\sin\beta$ and $\cos(-\beta) = \cos\beta$:

$$\begin{aligned}\sin(\alpha-\beta) &= \sin\alpha\cos(-\beta)+\sin(-\beta)\cos\alpha\\ &= \sin\alpha\cos\beta-\sin\beta\cos\alpha\end{aligned}$$

$$\begin{aligned}\cos(\alpha+\beta) &= \sin(90°-(\alpha+\beta))\\ &= \sin((90°-\alpha)-\beta)\\ &= \sin(90°-\alpha)\cos\beta-\sin\beta\cos(90°-\alpha)\\ &= \cos\alpha\cos\beta-\sin\beta\sin\alpha\end{aligned}$$

$$\begin{aligned}\cos(\alpha-\beta) &= \cos(\alpha)\cos(-\beta)-\sin(\alpha)\sin(-\beta)\\ &= \cos\alpha\cos\beta+\sin\beta\sin\alpha\end{aligned}$$

Note that when you have the expression for $\sin(\alpha+\beta)$, you can derive $\sin(\alpha-\beta)$ and $\cos(\alpha\pm\beta)$ with hardly any extra work. With these you can also derive $\tan(\alpha\pm\beta)$ fairly painlessly.

(ii) At the risk of repeating ourselves: when a question asks for **values** then decimal approximations are not what are wanted.

$$\begin{aligned}\sin 75^\circ &= \sin(45^\circ + 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{1+\sqrt{3}}{2\sqrt{2}}\end{aligned}$$

$$\begin{aligned}\sin 15^\circ &= \sin(45^\circ - 30^\circ)\\ &= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}\\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}\end{aligned}$$

(iii) $\cos 3A = \cos(2A + A) = \cos 2A \cos A - \sin 2A \sin A$

Since $\cos 2A = \cos(A + A) = \cos^2 A - \sin^2 A$, and $\sin 2A = \sin(A + A) = 2 \sin A \cos A$, this becomes:

$$\begin{aligned}\cos 3A &= (\cos^2 A - \sin^2 A) \cos A - (2 \sin A \cos A) \sin A\\ &= \cos^3 A - \sin^2 A \cos A - 2 \sin^2 A \cos A\\ &= \cos^3 A - 3 \sin^2 A \cos A\\ &= \cos^3 A - 3(1 - \cos^2 A) \cos A\\ &= 4 \cos^3 A - 3 \cos A\end{aligned}$$

It is a good idea to check your answer, for example by substituting $A = 30^\circ$.

Preparation

2 **(i)** $120 = 2^3 \times 3 \times 5$, so the prime factors of 120 are $2, 3$ and 5.

$$\begin{aligned}120\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) &= 120 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5}\\ &= 120 \times \frac{4}{15}\\ &= 32\end{aligned}$$

(ii) Note that if a is a *positive* integer then $a-1$ is an integer satisfying $a-1 \geqslant 0$. If you expand you get $x^a - x^{a-1}$, and since x and a are integers and $a-1 \geqslant 0$ this is an integer subtracted from an integer, hence is an integer!

(iii) $39600 = 2^4 \times 3^2 \times 5^2 \times 11$ and $52920 = 2^3 \times 3^3 \times 5 \times 7^2$ so the HCF is $2^3 \times 3^2 \times 5 = 360$. To find the HCF pick the lowest power of each prime number that occurs in the two prime decompositions. Remember that 0 is smaller than 1! If you wanted the LCM you would need to pick the highest power of each prime number.

(iv) These sorts of statements can be a bit confusing. I usually re-order them in my head, so

"$a^2 = b^2$ **if** $a = b$" becomes "**if** $a = b$ **then** $a^2 = b^2$" which I can now see is true

and

"$a^2 = b^2$ **only if** $a = b$" becomes "**only if** $a = b$ **is it the case that** $a^2 = b^2$" which I can now see is false (e.g. $a = 2$ and $b = -2$).

(a) True: ab even $\Leftarrow$ a and b both even.
(b) False as ab would be even if just one of a and b were even, so ab even $\not\Rightarrow$ a and b both even.
(c) False: either $a = b$ or $a = -b$ if $a^2 = b^2$, so $a = b \not\Leftarrow a^2 = b^2$.
(d) True: $a = b \Rightarrow a^2 = b^2$.
(e) True: equilateral $\Leftarrow$ three equal sides.
(f) True: equilateral $\Rightarrow$ three equal sides.

(v) **(a)** True.
(b) False. An odd number is prime **if** it is three, but this is not an **only if**.
(c) False. $x = 3$ only if $x^2 - 9 = 0$ (i.e. this is not an **if**).
(d) False. It is certainly the case that the triangle is right-angled **if** $a^2 + b^2 = c^2$, but this is not a necessary condition (it is not **only if**): the triangle is also right-angled if $a^2 = b^2 + c^2$ (i.e. a is the hypotenuse). This is possibly a bit mean, but it highlights the importance of not assuming things you are not told. If you knew that c was the longest side then this would be an **iff**.
(e) True.

The STEP question

3 **(i)** **(a)** $f(12) = 12 \times \frac{1}{2} \times \frac{2}{3} = 4$
$f(180) = 180 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 48.$

(b) Let $N = p_1^{a_1} \times p_2^{a_2} \times \cdots \times p_k^{a_k}$.
Then

$$\begin{aligned} f(N) &= N \times \frac{p_1 - 1}{p_1} \times \frac{p_2 - 1}{p_2} \cdots \times \frac{p_k - 1}{p_k} \\ &= {p_1}^{a_1} \times {p_2}^{a_2} \times \cdots \times {p_k}^{a_k} \times \frac{p_1 - 1}{p_1} \times \frac{p_2 - 1}{p_2} \cdots \times \frac{p_k - 1}{p_k} \\ &= p_1^{a_1 - 1}(p_1 - 1)p_2^{a_2 - 1}(p_2 - 1) \cdots p_k^{a_k - 1}(p_k - 1) \end{aligned}$$

Since each of the a_i is an integer greater than or equal to 1, $a_i - 1$ is an integer greater than or equal to 0. Each of the p_i is an integer, so $p_i^{a^i - 1}(p_i - 1)$ is always an integer, and hence $f(N)$ is an integer for all N.

(ii) **(a)** A simple counterexample (remember the simpler the better!) would be $f(12) \neq f(2) \times f(6)$.
We have $f(12) = 4$ from above. $f(2) = 2 \times \frac{1}{2} = 1$, and $f(6) = 6 \times \frac{1}{2} \times \frac{2}{3} = 2$
Thus $f(2) \times f(6) = 2 \neq f(12)$

(b) If p is prime, then $f(p) = p(1 - \frac{1}{p}) = p - 1$.
If p and q are distinct primes, then $f(pq) = pq(1 - \frac{1}{p})(1 - \frac{1}{q}) = (p-1)(q-1) = f(p)f(q)$.
When p and q are not distinct ($p = q$), the result is not true.

(c) Let $p = 4$ and $q = 15$.
$f(p) = 4 \times \frac{1}{2} = 2$, and $f(q) = 15 \times \frac{2}{3} \times \frac{4}{5} = 8$.
$f(pq) = f(60) = 60 \times \frac{1}{2} \times \frac{2}{3} \times \frac{4}{5} = 16$
So $f(p)f(q) = f(pq)$ even though p and q are not prime, hence the statement is not true.
In actual fact, the statement $f(p)f(q) = f(pq)$ holds as long as p and q are *co-prime*, that is that they have no prime factors in common. When p and q have a prime factor in common then $f(pq) \neq f(p)f(q)$.

(iii) $f(p^m) = p^m \left(1 - \dfrac{1}{p}\right) = p^{m-1}(p - 1)$.

$146410 = 2 \times 5 \times 11^4 = 11^4 \times 10$. Thus, $p = 11$ and $m = 5$.

Warm down

4 **(i)** Leave answers in fractions (unless specifically asked for a decimal approximation). At normal working rate, the workmen work 66 hours. If they increase this by $\frac{1}{12}$, they will only require $\frac{12}{13}$ of the time to do the same amount of work. If they only work $5\frac{1}{2}$ days, at a rate of N hours per day, $\frac{11N}{2} = 66 \times \frac{12}{13}$ so $N = \frac{66\times12\times2}{11\times13} = \frac{144}{13}$.

(ii) There is nothing wrong with the negative solution (nothing in the question says that the numbers have to be positive) so put down both solutions. We have $a - b = 3$ and $a^3 - b^3 = 279$. This gives:

$$\begin{aligned} a^3 - b^3 &= 279 \\ (b+3)^3 - b^3 &= 279 \\ b^3 + 9b^2 + 27b + 27 - b^3 &= 279 \\ b^2 + 3b + 3 - 31 &= 0 \\ (b+7)(b-4) &= 0 \end{aligned}$$

Answer: $(7, 4)$ or $(-4, -7)$.

(iii) Two of the terms simplify nicely, but the other two do not.
Answer: $ax + by - a^{\frac{1}{4}}b^{\frac{2}{5}}x^{\frac{1}{3}}y^{\frac{1}{6}} - a^{\frac{3}{4}}b^{\frac{3}{5}}x^{\frac{2}{3}}y^{\frac{5}{6}}$.

(iv)

$$\begin{aligned} \frac{\sqrt{12+6\sqrt{3}}}{\sqrt{3}+1} &= \frac{(\sqrt{3}-1)\sqrt{12+6\sqrt{3}}}{2} \\ &= \frac{1}{2}\left(\sqrt{(\sqrt{3}-1)^2(12+6\sqrt{3})}\right) \\ &= \frac{1}{2}\left(\sqrt{(4-2\sqrt{3})(12+6\sqrt{3})}\right) \\ &= \frac{1}{2}\left(\sqrt{48-24\sqrt{3}+24\sqrt{3}-36}\right) \\ &= \frac{\sqrt{12}}{2} \\ &= \sqrt{3} \end{aligned}$$