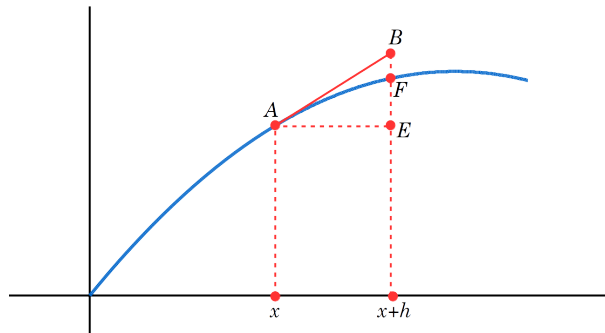# STEP Support Programme

Hints and partial solutions for Assignment 22

Warm-up

1 **(i)** It doesn't much matter what $f(x)$ looks like, though a straight line would not be suitable. Note that although h is "small" **your** h should not be so small as to make your diagram cramped. Your sketch might look something like this:

A has coordinates $(x, f(x))$, and F has coordinates $(x+h, f(x+h))$. AB is tangent to the curve and hence has gradient $f'(x)$. This means that the length EB will be $h \times f'(x)$ (by using the right-angled triangle AEB) and so B has coordinates $(x+h, f(x)+h \times f'(x))$.

Since h is small, B is close to F and therefore $f(x+h) \approx f(x) + hf'(x)$.

**(ii)**

$$
\begin{aligned}
g'(x) &= \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \to 0} \frac{f_1(x+h)f_2(x+h) - f_1(x)f_2(x)}{h} \\
&= \lim_{h \to 0} \frac{\left[f_1(x) + hf_1'(x)\right]\left[f_2(x) + hf_2'(x)\right] - f_1(x)f_2(x)}{h} \\
&= \lim_{h \to 0} \frac{\cancel{f_1(x)f_2(x)} + hf_1'(x)f_2(x) + hf_1(x)f_2'(x) + h^2 f_1'(x)f_2'(x) - \cancel{f_1(x)f_2(x)}}{h} \\
&= f_1'(x)f_2(x) + f_1(x)f_2'(x)
\end{aligned}
$$

Note that since we are taking the limit as h tends to 0 we can replace $f_1(x+h)$ with $f_1(x) + hf_1'(x)$.

This is not a rigorous proof. It would a little more rigorous if we wrote $f_1(x+h) = f_1(x) + hf_1'(x) + O(h^2)$. $O(h^2)$ means terms with h^2 or higher powers of h.

At the risk of repeating ourselves, don't assume what you are trying to prove. You may well already know the product rule, but using it in a proof of the product rule is not allowed!
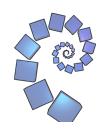
2   **(i)** & **(ii)** You may already know the answer to these, and you can use your knowledge to check your answers **but** you must use the definition $(*)$ to obtain the results, as this is what the question tells you to do!

Part **(i)** is a case of differentiating term by term and noting that (after cancelling some common factors) you get back the series that you started with. For part **(ii)** differentiate term by term and then factorise out k to obtain $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{kx}\right) = k\mathrm{e}^{kx}$.

(iii) Using the product rule we have $\mathrm{f}'(x) = 1 \times \mathrm{e}^x + x \times \mathrm{e}^x$. Using the definition $(*)$ we have:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \dots\right) = 1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \dots$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + \left(x + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \dots\right)$$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + x\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$$

$$= \mathrm{e}^x + x\mathrm{e}^x$$

This sort of "mucking around" with series is a reasonably common STEP technique.

(iv) The question tells you to use the definition $(*)$, so you should **not** assume that $\mathrm{e}^{ax}\mathrm{e}^{bx} = \mathrm{e}^{(a+b)x}$, unless you can prove it using $(*)$. To do this, you would have to show that

$$\left(1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \cdots\right) \times \left(1 + bx + \frac{(bx)^2}{2!} + \frac{(bx)^3}{3!} + \cdots\right)$$

$$= \left(1 + (a+b)x + \frac{((a+b)x)^2}{2!} + \frac{((a+b)x)^3}{3!} + \cdots\right)$$

which is possible but not recommended.

Using the product rule gives the derivative as $a\mathrm{e}^{ax} \times \mathrm{e}^{bx} + \mathrm{e}^{ax} \times b\mathrm{e}^{bx} = (a+b)\mathrm{e}^{ax}\mathrm{e}^{bx}$.

For the next part you can use the result just shown (with $a = 1$ and $b = -1$) to get:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^x\mathrm{e}^{-x}\right) = (1-1)\mathrm{e}^x\mathrm{e}^{-x} = 0\,.$$

This means that $\mathrm{e}^x\mathrm{e}^{-x} = c$ for some constant c. Substituting $x = 0$ into $(*)$ shows that $c = 1$. This shows that $\mathrm{e}^{-x} = \dfrac{1}{\mathrm{e}^x}$, which is again not at all obvious from $(*)$.

To show that $x\mathrm{e}^x \to 0$ as $x \to -\infty$ you can't just substitute $-\infty$ into $(*)$!

Instead, you can use the fact that $\mathrm{e}^{-x} = \dfrac{1}{\mathrm{e}^x}$. Let $x = -t$ (where $t > 0$). Then:

$$x\mathrm{e}^x = -t\mathrm{e}^{-t} = \frac{-t}{\mathrm{e}^t}$$

Then, using $(*)$

$$\frac{-t}{\mathrm{e}^t} = \frac{-t}{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots} = \frac{-1}{\frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots}$$

which tends to 0 as $t \to = +\infty$.

Preparation

3 You should **not** be using a calculator for any question we set (unless we specifically tell you to). If you do use a calculator to check your work after completing the questions make sure that you leave your answers in exact form, e.g. write e^2 rather than 7.39.

(i) One way of solving quadratic inequalities is to sketch a graph. In this case you would want to sketch $y = (3x - 2)(x + 1)$ and then see where this lies below the x axis. Answer: $-1 < x < \frac{2}{3}$.

(ii) You should explain why the graph passes through $(0, 1)$, ideally using $(*)$.

(iii) In each of these parts you will need to state $\mathrm{e}^x \neq 0$ or $\mathrm{e}^x > 0$ at some point.

(a) Differentiation gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = (x - 2)\mathrm{e}^x$ and since $\mathrm{e}^x > 0$ we can divide by e^x so there is one stationary point at $(2, -\mathrm{e}^2)$. The second derivative is $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2} = (x - 1)\mathrm{e}^x$ so when $x = 2$ this is positive and the curve has a minimum point here.

(b) The curve intersects the axes at $(0, -3)$ and $(3, 0)$. Since $\mathrm{e}^x > 0$, $(x - 3)\mathrm{e}^x$ is negative when $x < 3$ (and it is positive for $x > 3$).

(c) You now have most of the information you need. The curve will tend to the x axis (from below) as $x \to -\infty$ and as $x \to \infty$ we have $y \to \infty$. Try using Desmos to sketch the graph as a check.

(d) Draw $y = k$ on to your graph for some k and then consider how many intersections there are with your original graph for different values of k. The number of roots will depend on whether the line $y = k$ lies above or below the minimum, and whether it lies above or below the horizontal asymptote. Be particularly careful when it passes through the minimum point of the curve.
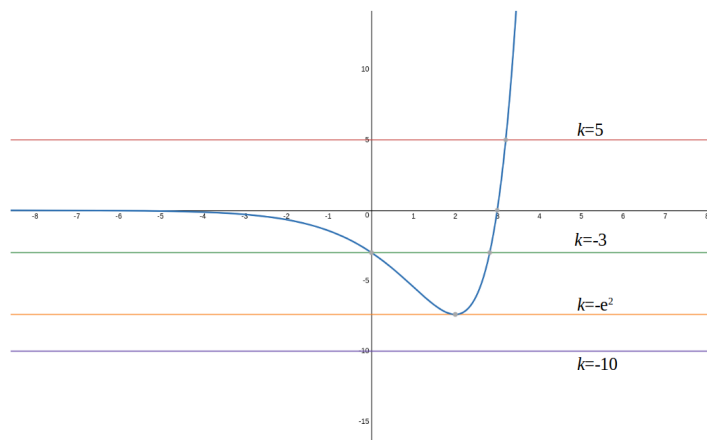
There are two roots when $-\mathrm{e}^2 < k < 0$ and one root when $k \geqslant 0$ or $k = -\mathrm{e}^2$.

(iv) **(b)** The roots occur at $x^2 = 0, \pi, 2\pi, \cdots$ i.e. for $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi} \cdots$. Note that 0 is "non-negative", so the answer is $x = 0, \sqrt{\pi}, \sqrt{2\pi}, \sqrt{3\pi}$.

(c) You should find that $f(-a) = f(a)$ and hence $f(x) = \sin(x^2)$ is an *even* function, i.e. it is symmetrical about the y axis.

**(d)** Use Desmos (or something similar) to check your answer.
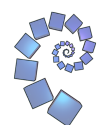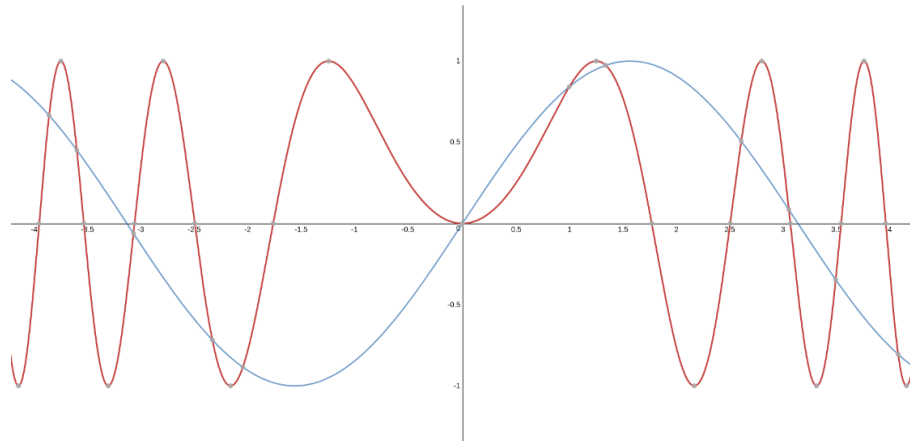
The STEP question (2015 STEP I Q1)

4 **(i)** You should do the usual things: find where the graph crosses the axes; find the coordinates of the stationary points; look at the behaviour as $x \to \pm\infty$; etc. Make sure that you label all these points on your diagram, showing the relevant values.

When you have your sketch of the graph, you can consider where the line $y = k$ intersects it for various values of k.

The curve crosses the axes at $(0, 2)$, $\left(\frac{1}{2}, 0\right)$ and $(2, 0)$. The derivative is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x \left(2x^2 - x - 3\right)$ which gives the stationary points as $\left(-1, 9\mathrm{e}^{-1}\right)$ and $\left(\frac{3}{2}, -\mathrm{e}^{3/2}\right)$ (remember to state that $\mathrm{e}^x > 0$).

The curve will lie below the x axis when $\frac{1}{2} < x < 2$ and will tend to the x axis as $x \to -\infty$. Finally as $x \to \infty$ we have $y \to \infty$.
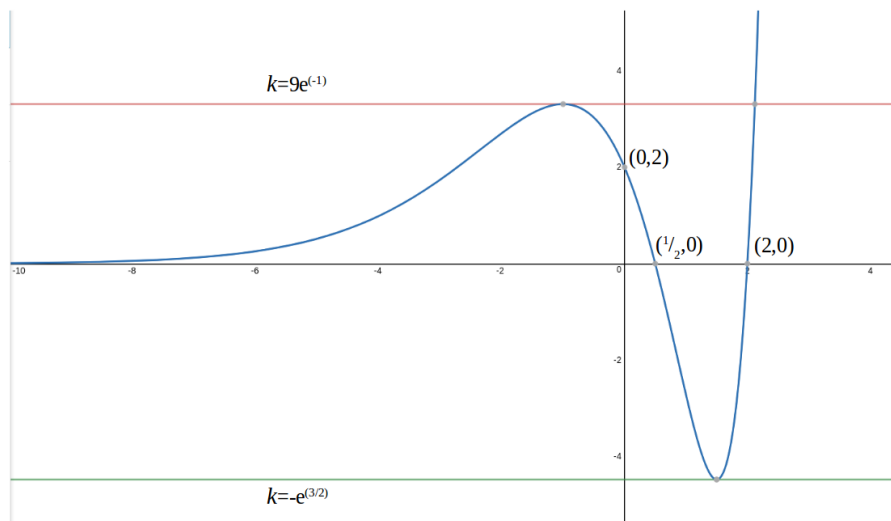
The number of roots are as follows:

$$
\begin{aligned}
k &< -\mathrm{e}^{3/2} && \text{no solutions} \\
k &= -\mathrm{e}^{3/2} && \text{one solution} \\
-\mathrm{e}^{3/2} &< k \leqslant 0 && \text{two solutions} \\
0 &< k < 9\mathrm{e}^{-1} && \text{three solutions} \\
k &= 9\mathrm{e}^{-1} && \text{two solutions} \\
k &> 9\mathrm{e}^{-1} && \text{one solution}
\end{aligned}
$$

(ii) If the original graph is $y = f(x)$ then this one is $y = f(x^2)$. It will be symmetrical about the y axis, so start by considering positive x only and then reflect to get the rest of the curve.

The y intercept will remain the same, but an x intercept at $x = a$ (where $a \geqslant 0$) will move to the point $x = \sqrt{a}$ (and there will be another at $x = -\sqrt{a}$). Similarly, if the graph has a stationary point at (p, q), then this will move to $(\sqrt{p}, q)$ — with a mirror image occurring at $(-\sqrt{p}, q)$. Make sure that you label all these points on your diagram.

You should end up with a maximum turning point at $(0, 2)$ and minimum turning points at $\left(\pm\sqrt{\frac{3}{2}}, -e^{3/2}\right)$. The curve will intersect the x axis at $x = \pm\sqrt{\frac{1}{2}}$ and $x = \pm\sqrt{2}$.

You can (if you really want to) work through all the stages of sketching a curve from scratch, but this part is much simpler if you exploit the connection between this curve and the one in part **(i)**.
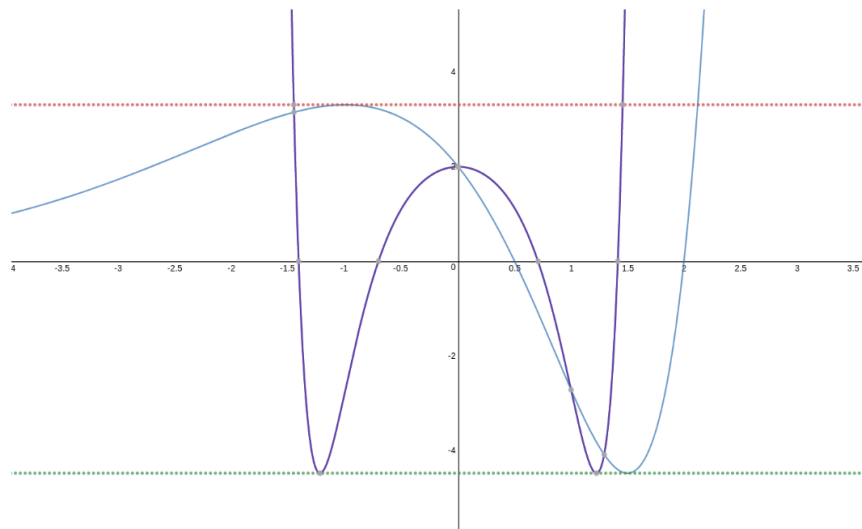
Again, use Desmos to check what the graph should look like (but note that Desmos will not give you exact values).

Warm down

5 The first three shapes all obey Euler's Theorem for *convex* polyhedra which is $F - E + V = 2$. A convex polyhedron is one for which the line segment joining any two points on the surface of the polyhedron lies inside the polyhedron.

You can find the number of vertices and edges of the icosahedron by considering how many faces meet at a vertex etc. For vertices we have 5 faces meeting at each vertex, and if we consider the 20 triangular faces these have a "total" of 60 corners. The number of vertices on the icosahedron is therefore $60 \div 5 = 12$. Similarly the number of edges is $20 \times 3 \div 2 = 30$. Or you can sketch a net!

The last shape is not a convex polyhedron which means $F - E + V$ might not be 2 (in fact it is 3). If you take the indent all the way through (so that a hole runs all the way through the big cube) you get $F - E + V = 2$ even though this is not a convex polyhedron.