STEP Support Programme

STEP 3 Hyperbolic Functions Topic Notes

Definitions

$$\cosh x = \tfrac{1}{2}\left(e^x + e^{-x}\right)$$
$$\sinh x = \tfrac{1}{2}\left(e^x - e^{-x}\right)$$
$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$
$$\operatorname{sech} x = \frac{1}{\cosh x}$$
$$\operatorname{cosech} x = \frac{1}{\sinh x}$$
$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Graphs and ranges

The graphs can be sketched by using your understanding of e^x, odd/even functions etc. For example:

$\cosh x$ is an even function (as $\cosh(-x) = \cosh x$) and so it is symmetrical in the y-axis. It passes through $(0, 1)$ and as $x \to \pm\infty$, $y \to \infty$. You can also use Desmos to sketch these.

We have:
$$1 \leqslant \cosh x < \infty$$
$$-\infty < \sinh x < \infty$$
$$-1 < \tanh x < 1$$

Relationship with sin/cos

$$\cos x = \tfrac{1}{2}\left(e^{ix} + e^{-ix}\right) \quad = \cosh(ix)$$
$$\cos(iy) = \tfrac{1}{2}\left(e^{i^2 y} + e^{-i^2 y}\right) = \cosh(y)$$
$$\sin x = \tfrac{1}{2i}\left(e^{ix} - e^{-ix}\right) \quad = \tfrac{1}{i}\sinh(ix) = -i\sinh(ix)$$
$$\sin(iy) = \tfrac{1}{2i}\left(e^{i^2 y} - e^{-i^2 y}\right) = -\tfrac{1}{i}\sinh(y) = i\sinh(y)$$
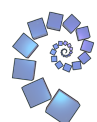
Inverse hyperbolics[1]

$$\operatorname{arcosh} x = \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \quad \text{for } x \geqslant 1$$

$$\operatorname{arsinh} x = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$$

$$\operatorname{artanh} x = \tanh^{-1} x = \tfrac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right) \quad \text{for } |x| < 1$$

Identities

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2\sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}$$

$$\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\coth^2 x - 1 = \operatorname{cosech}^2 x$$

$$\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B$$

$$\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A$$

$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

Note that the identities for $\sinh 2x$, $\cosh 2x$, $\tanh 2x$ can be derived from the identities for $\sinh(A+B)$ etc.

You can use Osborn's Rule to covert trigonometric identities written in terms of sine and cosine into a corresponding hyperbolic identity. Essentially you replace any $\sin^2 \theta$ with $-\sinh^2 \theta$.
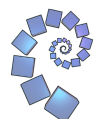
Maclaurin's Series

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots + \frac{x^{2r+1}}{(2r+1)!} + \ldots \quad \text{for all } x$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2r}}{(2r)!} + \ldots \quad \text{for all } x$$

$$\tanh^{-1} x = x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots + \frac{x^{2r+1}}{(2r+1)} + \ldots \quad \text{for } -1 < x < 1$$

---

[1]You can find the inverse of $\cosh x$ by setting $\tfrac{1}{2}\left(e^y + e^{-y}\right) = x$ and then solving to get y in terms of x. The other inverse functions can be found in a similar way. Note that for 2019 onwards the $\cosh^{-1} x$ notation will be used, but older papers may use $\operatorname{arcosh} x$.

Calculus

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh x = \operatorname{sech}^2 x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1} x = \frac{1}{\sqrt{1+x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\cosh^{-1} x = \frac{1}{\sqrt{x^2-1}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1} x = \frac{1}{1-x^2}$$

Note that you can find $\frac{\mathrm{d}}{\mathrm{d}x}\coth x$ by using the quotient rule. The chain rule can be used to find the derivatives of $\operatorname{sech} x$ and $\operatorname{cosech} x$.

$$\int \sinh x \,\mathrm{d}x = \cosh x + k$$

$$\int \cosh x \,\mathrm{d}x = \sinh x + k$$

$$\int \tanh x \,\mathrm{d}x = \ln\left(\cosh x\right) + k$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,\mathrm{d}x = \cosh^{-1}\left(\frac{x}{a}\right) + k \quad\text{or}\quad \ln\left(x+\sqrt{x^2-a^2}\right) + k \quad\text{for } x > a$$

$$\int \frac{1}{\sqrt{a^2+x^2}}\,\mathrm{d}x = \sinh^{-1}\left(\frac{x}{a}\right) + k \quad\text{or}\quad \ln\left(x+\sqrt{x^2+a^2}\right) + k$$

$$\int \frac{1}{a^2-x^2}\,\mathrm{d}x = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + k \quad\text{or}\quad \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + k \quad\text{for } |x| < a$$