STEP Support Programme

STEP 2 Matrices Solutions

Note that several of these solutions contain discussion as well as just the bare solutions, so they are often longer than would be expected in an exam solution.

Question 1

The complex number $x + \mathrm{i}y$ is mapped into the complex number $X + \mathrm{i}Y$ where X and Y are given by the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} X \\ Y \end{pmatrix}.$$

Which numbers are invariant under the mapping?

A number $x + \mathrm{i}y$ is invariant if $X + \mathrm{i}Y = x + \mathrm{i}y$, which is the case if and only if $X = x$ and $Y = y$. From the matrix equation, we have

$$\begin{aligned} X &= 2x + y \\ Y &= x + 2y. \end{aligned}$$

From this, we see that *if* $x + \mathrm{i}y$ is invariant, then

$$\begin{aligned} x &= 2x + y \\ y &= x + 2y, \end{aligned}$$

so $x + y = 0$.

Conversely, *if* $x + y = 0$, then

$$\begin{aligned} X &= 2x + y = x + (x + y) = x \\ Y &= x + 2y = y + (x + y) = y, \end{aligned}$$

so $x + \mathrm{i}y$ is invariant.

Therefore the points which are invariant are those for which $x + y = 0$, or equivalently $y = -x$, so the points which are invariant under the mapping are those of the form $t - \mathrm{i}t$ for any real t.

Question 2

The simultaneous equations

$$x + 2y = 4,$$
$$2x - y = 0,$$
$$3x + y = 5$$

may be written in matrix form as

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}, \quad \text{or} \quad \mathbf{AX} = \mathbf{B}.$$

Carry out numerically the procedure of the three following steps:

(1) $\mathbf{A}^{\mathrm{T}}\mathbf{AX} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$;

(2) $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{AX} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}$;

(3) $\mathbf{IX} = \begin{pmatrix} x \\ y \end{pmatrix} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}$.

(1) We have

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix}$$

so

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 3 \\ 3 & 6 \end{pmatrix}$$

and

$$\mathbf{A}^{\mathrm{T}}\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 19 \\ 13 \end{pmatrix},$$

giving the equation

$$\mathbf{A}^{\mathrm{T}}\mathbf{AX} = \mathbf{A}^{\mathrm{T}}\mathbf{B} \quad \text{or} \quad \begin{pmatrix} 14 & 3 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 19 \\ 13 \end{pmatrix}.$$

(2) We have $\det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = 14 \times 6 - 3 \times 3 = 75$, so

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1} = \frac{1}{75} \begin{pmatrix} 6 & -3 \\ -3 & 14 \end{pmatrix}$$

which gives

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{A} = \frac{1}{75} \begin{pmatrix} 6 & -3 \\ -3 & 14 \end{pmatrix} \begin{pmatrix} 14 & 3 \\ 3 & 6 \end{pmatrix} = \frac{1}{75} \begin{pmatrix} 75 & 0 \\ 0 & 75 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}$$

and

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B} = \frac{1}{75}\begin{pmatrix} 6 & -3 \\ -3 & 14 \end{pmatrix}\begin{pmatrix} 19 \\ 13 \end{pmatrix} = \frac{1}{75}\begin{pmatrix} 75 \\ 125 \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{3} \end{pmatrix},$$

giving the equation

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{X} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{3} \end{pmatrix}.$$

(3) There is nothing to do for this last part, as we have now done all of the calculations: we observe that

$$(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{X} = (\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}(\mathbf{A}^{\mathrm{T}}\mathbf{A})\mathbf{X} = \mathbf{I}\mathbf{X} = \mathbf{X}$$

as claimed, giving

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{5}{3} \end{pmatrix}.$$

> Verify that the values of x, y so found do not satisfy all the original three equations. Suggest a reason for this.

Substituting $x = 1$, $y = \frac{5}{3}$ into the equations gives

$$\begin{aligned} 1 + 2 \times \tfrac{5}{3} &= \tfrac{13}{3} \\ 2 \times 1 - \tfrac{5}{3} &= \tfrac{1}{3} \\ 3 \times 1 + \tfrac{5}{3} &= \tfrac{14}{3} \end{aligned}$$

so none of the original three equations is actually satisfied.

The simplest reason we could suggest is that the calculation of step (1) has turned the three-dimensional vector $\mathbf{B}$ into a two-dimensional vector $\mathbf{A}^{\mathrm{T}}\mathbf{B}$, so we have lost information: many vectors other than the original $\mathbf{B}$ would give the same numerical result at the end of step (1), so there cannot be a guarantee that all three equations are satisfied.

Another way of thinking about this is that the three given equations are actually inconsistent: it is not possible to satisfy all three of them simultaneously. We can see this by solving the first two to get $x = \frac{4}{5}$, $y = \frac{8}{5}$, but these do not satisfy the third equation. So our purported solution may be "doing its best" to satisfy inconsistent equations, at least in some sense.

> Under what circumstances will the procedure given above, when applied to a set of three simultaneous equations in two variables, result in values which satisfy the equations?

A sensible guess would be that the procedure will work when the equations are consistent (that is, where the three equations do have a common solution). Modifying the right-hand side of the third equation to $3x + y = 4$ would make the three equations consistent, and calculating $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}\mathbf{A}^{\mathrm{T}}\mathbf{B}$ in this case gives $x = \frac{4}{5}$, $y = \frac{8}{5}$. This suggests that our guess may be correct. Can we prove it? (We do not try to work out the meaning of the value of the right-hand side of (3) in the case that the equations do not have a consistent solution: this does not seem to be at all straightforward.)

So suppose that the original simultaneous equations *do* have a (consistent) solution, say x_0 and y_0. This means that the equation $\mathbf{AX} = \mathbf{B}$ holds when $\mathbf{X} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$.

Since this equation holds for this $\mathbf{X}$, we can multiply both sides on the left by $\mathbf{A}^{\mathrm{T}}$, which shows that the equation in step (1) also holds.

If we now multiply the equation in (1) on the left by $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$, it follows that the equation in step (2) holds in this case.

Since $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = \mathbf{I}$, it then follows that the equation in step (3) also holds for this choice of $\mathbf{X}$.

Putting this together shows that if the original simultaneous equations have a consistent solution $\mathbf{X}$, then step (3) gives us this $\mathbf{X}$. But the right-hand side of (3) is entirely independent of $\mathbf{X}$ itself! So if the original equations have a consistent solution, it will be given by this procedure.

On the other hand, if there is no consistent solution, then the procedure will give some vector $\mathbf{X}$, but it cannot be a solution to the equations, as there isn't one.

There is one other matter to consider when there is a solution to the original equations, which is whether $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$ actually exists. The examiners who wrote this question probably did not expect students to think about this. The arguments below are beyond what would be expected in a STEP examination without guidance in the question itself.

Here are two different ways to approach this matter. The first is to work with the matrix $\mathbf{A}$ explicitly, so we write

$$\mathbf{A} = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

which gives

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix} = \begin{pmatrix} a^2+b^2+c^2 & ad+be+cf \\ ad+be+cf & d^2+e^2+f^2 \end{pmatrix}.$$

This matrix is invertible if and only if its determinant is non-zero, so we calculate its determinant. We obtain:

$$\begin{aligned} \det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) &= (a^2+b^2+c^2)(d^2+e^2+f^2) - (ad+be+cf)^2 \\ &= a^2d^2 + a^2e^2 + a^2f^2 + b^2d^2 + b^2e^2 + b^2f^2 + c^2d^2 + c^2e^2 + c^2f^2 \\ &\qquad - (a^2d^2 + b^2e^2 + c^2f^2 + 2adbe + 2adcf + 2becf) \\ &= a^2e^2 + a^2f^2 + b^2d^2 + b^2f^2 + c^2d^2 + c^2e^2 - 2adbe - 2adcf - 2becf. \end{aligned}$$

If we look at this carefully, we see that we can write $a^2e^2 + b^2d^2 - 2adbe$ as $(ae-bd)^2$, and similarly for the remaining terms. So we obtain

$$\det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) = (ae-bd)^2 + (af-cd)^2 + (bf-ce)^2.$$

This is a sum of squares, so it is never negative. But it will be zero when $ae = db$ and $af = cd$ and $bf = ce$. If we assume for a moment that a, b and c are all non-zero, this is equivalent to

$$\frac{d}{a} = \frac{e}{b} = \frac{f}{c}$$

which just says that (d, e, f) is a multiple of (a, b, c), say $d = ka$, $e = kb$, $f = kc$. In that case, the original three simultaneous equations become

$$\begin{aligned} ax + kay &= u \\ bx + kby &= v \\ cx + kcy &= w. \end{aligned}$$

Dividing these by a, b and c respectively (which we have assumed to be non-zero) gives

$$\begin{aligned} x + ky &= \tfrac{u}{a} \\ x + ky &= \tfrac{v}{b} \\ x + ky &= \tfrac{w}{c}. \end{aligned}$$

Recall that we are assuming that the equations do have a solution. We must therefore have $\frac{u}{a} = \frac{v}{b} = \frac{w}{c}$, and they are all the same line, giving infinitely many solutions. (If $\frac{u}{a} \neq \frac{v}{b}$, for example, then the first two lines would be distinct parallel lines, and there would be no solutions.) The case where one or more of a, b and c is zero can be dealt with in a similar fashion.

The converse is true, too: if there are infinitely many solutions, then all three equations must represent the same line, and so the determinant of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ will be zero (working backwards through the above argument).

Therefore $(\mathbf{A}^{\mathrm{T}}\mathbf{A})^{-1}$ exists if there is a *unique* solution to the original equations but not if there are infinitely many solutions.

Thus the complete answer to the question is that this method works if there exists a *unique* solution to the system of three simultaneous equations.

There is a second approach to thinking about whether the determinant of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is zero, which is significantly more sophisticated. We can consider $\mathbf{A}$ to be a pair of column vectors, and write

$$\mathbf{A} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$$

where each of $\mathbf{v}_1$ and $\mathbf{v}_2$ is a column vector with 3 entries. In the case in this question, we would have

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}.$$

Then we can write $\mathbf{A}^{\mathrm{T}}$ in terms of $\mathbf{v}_1$ and $\mathbf{v}_2$ as

$$\mathbf{A}^{\mathrm{T}} = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \end{pmatrix},$$

where $\mathbf{v}_1^{\mathrm{T}}$ and $\mathbf{v}_2^{\mathrm{T}}$ are row vectors. Then we can calculate

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} \mathbf{v}_1^{\mathrm{T}}\mathbf{v}_1 & \mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2 \\ \mathbf{v}_2^{\mathrm{T}}\mathbf{v}_1 & \mathbf{v}_2^{\mathrm{T}}\mathbf{v}_2 \end{pmatrix}.$$

In this expression, we have calculated "row times column" to give elements of the form $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j$; these are technically 1×1 matrices, but we are regarding them as just numbers. (To understand this a little better, look at what you would need to do to calculate $\mathbf{v}_1^{\mathrm{T}}\mathbf{v}_2$, for example, and what you would need to do to calculate the $(1,2)$-element of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.)

Now, since $\mathbf{v}_i^{\mathrm{T}}\mathbf{v}_j$ multiplies the corresponding elements of $\mathbf{v}_i$ and $\mathbf{v}_j$ together, and then adds the results, it is the same as the scalar product $\mathbf{v}_i.\mathbf{v}_j$, so we can write

$$\mathbf{A}^{\mathrm{T}}\mathbf{A} = \begin{pmatrix} \mathbf{v}_1.\mathbf{v}_1 & \mathbf{v}_1.\mathbf{v}_2 \\ \mathbf{v}_2.\mathbf{v}_1 & \mathbf{v}_2.\mathbf{v}_2 \end{pmatrix}.$$

Therefore the determinant of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$ is given by

$$\begin{aligned} \det(\mathbf{A}^{\mathrm{T}}\mathbf{A}) &= (\mathbf{v}_1.\mathbf{v}_1)(\mathbf{v}_2.\mathbf{v}_2) - (\mathbf{v}_1.\mathbf{v}_2)^2 \\ &= |\mathbf{v}_1|^2|\mathbf{v}_2|^2 - |\mathbf{v}_1|^2|\mathbf{v}_2|^2\cos^2\theta \\ &= |\mathbf{v}_1|^2|\mathbf{v}_2|^2(1-\cos^2\theta) \\ &= |\mathbf{v}_1|^2|\mathbf{v}_2|^2\sin^2\theta, \end{aligned}$$

where θ is the angle between the three-dimensional vectors $\mathbf{v}_1$ and $\mathbf{v}_2$. Thus the determinant is never negative, and is 0 if and only if either one of $\mathbf{v}_1$ and $\mathbf{v}_2$ is zero or $\theta = 0$ or $\theta = \pi$, which occurs if and only if one of $\mathbf{v}_1$ and $\mathbf{v}_2$ is a multiple of the other (where the multiple can be zero).

The rest of the argument then continues very much as above.

Question 3

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be real 2×2 matrices and write

$$[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}, \text{ etc.}$$

Prove that:

(i) $[\mathbf{A}, \mathbf{A}] = \mathbf{O}$, where $\mathbf{O}$ is the zero matrix,

At each step you should state clearly any properties of matrices which you use.

We calculate directly from the definition:

$$[\mathbf{A}, \mathbf{A}] = \mathbf{AA} - \mathbf{AA} = \mathbf{O}.$$

(ii) $[[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] = \mathbf{O}$,

Again, we just calculate (though this is a little longer):

$$\begin{aligned}
&[[\mathbf{A}, \mathbf{B}], \mathbf{C}] + [[\mathbf{B}, \mathbf{C}], \mathbf{A}] + [[\mathbf{C}, \mathbf{A}], \mathbf{B}] \\
&\quad = [\mathbf{AB} - \mathbf{BA}, \mathbf{C}] + [\mathbf{BC} - \mathbf{CB}, \mathbf{A}] + [\mathbf{CA} - \mathbf{AC}, \mathbf{B}] \\
&\quad = (\mathbf{AB} - \mathbf{BA})\mathbf{C} - \mathbf{C}(\mathbf{AB} - \mathbf{BA}) + (\mathbf{BC} - \mathbf{CB})\mathbf{A} - \mathbf{A}(\mathbf{BC} - \mathbf{CB}) \\
&\qquad + (\mathbf{CA} - \mathbf{AC})\mathbf{B} - \mathbf{B}(\mathbf{CA} - \mathbf{AC}) \\
&\quad = \mathbf{ABC} - \mathbf{BAC} - \mathbf{CAB} + \mathbf{CBA} + \mathbf{BCA} - \mathbf{CBA} - \mathbf{ABC} + \mathbf{ACB} \\
&\qquad + \mathbf{CAB} - \mathbf{ACB} - \mathbf{BCA} + \mathbf{BAC} \\
&\quad = \mathbf{O}.
\end{aligned}$$

During this calculation, we have used the property that matrix multiplication is associative, which means that $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$ whenever either side is defined. Therefore we can write $\mathbf{ABC}$, and this is unambiguous.

(iii) if $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$, then $[\mathbf{A}, \mathbf{B}^m] = m\mathbf{B}^{m-1}$ for all positive integers m.

The statement is given for all positive integers m, so it is likely that induction will be a good approach.

The statement is clearly true when $m = 1$, as long as we interpret $\mathbf{B}^0$ to mean $\mathbf{I}$, just as $a^0 = 1$ when a is a real number.

So assume that the statement is true for $m = k$, that is, $[\mathbf{A}, \mathbf{B}^k] = k\mathbf{B}^{k-1}$, or in full $\mathbf{AB}^k - \mathbf{B}^k\mathbf{A} = k\mathbf{B}^{k-1}$. Then for $m = k+1$, we have

$$\begin{aligned}[\mathbf{A}, \mathbf{B}^{k+1}] &= \mathbf{AB}^{k+1} - \mathbf{B}^{k+1}\mathbf{A} \\ &= \mathbf{AB}^k\mathbf{B} - \mathbf{BB}^k\mathbf{A}.\end{aligned}$$

It is not immediately clear how we can apply the induction hypothesis, so we try a standard trick of adding and subtracting terms to match what we already know. In this case, we can write $\mathbf{AB}^k$ in the first term as $\mathbf{AB}^k - \mathbf{B}^k\mathbf{A} + \mathbf{B}^k\mathbf{A}$, and similarly for the second term. This gives us

$$\begin{aligned}[\mathbf{A}, \mathbf{B}^{k+1}] &= \mathbf{AB}^k\mathbf{B} - \mathbf{BB}^k\mathbf{A} \\ &= (\mathbf{AB}^k - \mathbf{B}^k\mathbf{A} + \mathbf{B}^k\mathbf{A})\mathbf{B} - \mathbf{B}(\mathbf{B}^k\mathbf{A} - \mathbf{AB}^k + \mathbf{AB}^k) \\ &= (k\mathbf{B}^{k-1} + \mathbf{B}^k\mathbf{A})\mathbf{B} - \mathbf{B}(-k\mathbf{B}^{k-1} + \mathbf{AB}^k) \\ &= k\mathbf{B}^k + \mathbf{B}^k\mathbf{AB} + k\mathbf{B}^k - \mathbf{BAB}^k.\end{aligned}$$

This is looking hopeful, but we still have the difficult second and fourth terms. And trying the same trick on these terms using $[\mathbf{A}, \mathbf{B}^k]$ again will bring us back to where we started. Alternatively, we could try using $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$ on these terms, but it is unclear how this will help.

So let's try a different approach. We could use $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$, or $\mathbf{AB} - \mathbf{BA} = \mathbf{I}$ to move the first $\mathbf{A}$. This gives

$$\begin{aligned}[\mathbf{A}, \mathbf{B}^{k+1}] &= \mathbf{AB}^{k+1} - \mathbf{B}^{k+1}\mathbf{A} \\ &= \mathbf{ABB}^k - \mathbf{B}^{k+1}\mathbf{A} \\ &= (\mathbf{I} + \mathbf{BA})\mathbf{B}^k - \mathbf{B}^{k+1}\mathbf{A} \\ &= \mathbf{B}^k + \mathbf{BAB}^k - \mathbf{B}^{k+1}\mathbf{A}.\end{aligned}$$

This may look unhelpful, but there are some good things about it: first, we've got a $\mathbf{B}^k$ term, which we want in our final answer, and second, the other two terms both begin with a $\mathbf{B}$, so we can factorise it out:

$$\begin{aligned}[\mathbf{A}, \mathbf{B}^{k+1}] &= \mathbf{B}^k + \mathbf{BAB}^k - \mathbf{B}^{k+1}\mathbf{A} \\ &= \mathbf{B}^k + \mathbf{B}(\mathbf{AB}^k - \mathbf{B}^k\mathbf{A}) \\ &= \mathbf{B}^k + \mathbf{B}[\mathbf{A}, \mathbf{B}^k] \\ &= \mathbf{B}^k + \mathbf{B}(k\mathbf{B}^{k-1}) \\ &= (k+1)\mathbf{B}^k,\end{aligned}$$

where we have used the induction hypothesis on the penultimate line.

This might make us look back at the previous approach and realise that we could have taken out a factor of $\mathbf{B}$ from the start and end of the difficult terms, giving:

$$\begin{aligned}[\mathbf{A}, \mathbf{B}^{k+1}] &= k\mathbf{B}^k + \mathbf{B}^k\mathbf{AB} + k\mathbf{B}^k - \mathbf{BAB}^k \\ &= 2k\mathbf{B}^k + \mathbf{B}(\mathbf{B}^{k-1}\mathbf{A})\mathbf{B} - \mathbf{B}(\mathbf{AB}^{k-1})\mathbf{B} \\ &= 2k\mathbf{B}^k - \mathbf{B}(\mathbf{AB}^{k-1} - \mathbf{B}^{k-1}\mathbf{A})\mathbf{B} \\ &= 2k\mathbf{B}^k - \mathbf{B}[\mathbf{A}, \mathbf{B}^{k-1}]\mathbf{B} \\ &= 2k\mathbf{B}^k - \mathbf{B}((k-1)\mathbf{B}^{k-2})\mathbf{B} \\ &= 2k\mathbf{B}^k - (k-1)\mathbf{B}^k \\ &= (k+1)\mathbf{B}^k\end{aligned}$$

where we have used the induction hypothesis for $k-1$ here instead of for k. So if we know the result for $k=1$, we will have it for $k=3$, $k=5$ and so on by induction, and if we know the result for $k=2$ (which can be found by putting $k=1$ on the first line above: the terms $\mathbf{B}^k\mathbf{AB}$ and $-\mathbf{BAB}^k$ then cancel), we know it for $k=4$, $k=6$ and so on by induction, so we have the result for all k (or all n) as we wanted.

Therefore using either method, we have successfully shown that the induction step holds, and the desired result holds by induction.

The *trace*, $\text{Tr}(\mathbf{A})$, of a matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

is defined by

$$\text{Tr}(\mathbf{A}) = a_{11} + a_{22}.$$

Prove that:

(iv) $\text{Tr}(\mathbf{A}+\mathbf{B}) = \text{Tr}(\mathbf{A}) + \text{Tr}(\mathbf{B})$,

If we write $\mathbf{A} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, then we have

$$\begin{aligned} \text{Tr}(\mathbf{A}+\mathbf{B}) &= \text{Tr}\begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix} \\ &= (a_{11}+b_{11}) + (a_{22}+b_{22}) \\ \text{Tr}(\mathbf{A}) + \text{Tr}(\mathbf{B}) &= (a_{11}+a_{22}) + (b_{11}+b_{22}) \end{aligned}$$

and hence $\text{Tr}(\mathbf{A}+\mathbf{B}) = \text{Tr}(\mathbf{A}) + \text{Tr}(\mathbf{B})$.

(v) $\text{Tr}(\mathbf{AB}) = \text{Tr}(\mathbf{BA})$,

We can do this by multiplying out the matrices:

$$\begin{aligned} \mathbf{AB} &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \\ &= \begin{pmatrix} a_{11}b_{11}+a_{12}b_{21} & a_{11}b_{12}+a_{12}b_{22} \\ a_{21}b_{11}+a_{22}b_{21} & a_{21}b_{12}+a_{22}b_{22} \end{pmatrix} \\ \mathbf{BA} &= \begin{pmatrix} b_{11}a_{11}+b_{12}a_{21} & b_{11}a_{12}+b_{12}a_{22} \\ b_{21}a_{11}+b_{22}a_{21} & b_{21}a_{12}+b_{22}a_{22} \end{pmatrix} \end{aligned}$$

(where $\mathbf{BA}$ is obtained from $\mathbf{AB}$ by just replacing every a with b and vice versa). Therefore

$$\begin{aligned} \text{Tr}(\mathbf{AB}) &= a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22} \\ \text{Tr}(\mathbf{BA}) &= b_{11}a_{11} + b_{12}a_{21} + b_{21}a_{12} + b_{22}a_{22} \end{aligned}$$

which are seen to be equal, so $\mathrm{Tr}(\mathbf{AB}) = \mathrm{Tr}(\mathbf{BA})$.

An alternative (and more sophisticated) way to do this part without writing out the whole multiplication is to work out an expression for the ith diagonal element of $\mathbf{AB}$, which we can write as $(\mathbf{AB})_{ii}$, and then sum these over i. We see that

$$(\mathbf{AB})_{ii} = \sum_k a_{ik}b_{ki}$$

where the sum is over k from 1 to 2 (because our matrices are 2×2), and the (i, j)-th element of $\mathbf{AB}$ is obtained by multiplying the ith row of $\mathbf{A}$ by the jth column of $\mathbf{B}$. Thus we have

$$\mathrm{Tr}(\mathbf{AB}) = \sum_i \sum_k a_{ik}b_{ki}.$$

Swapping a's and b's, we have

$$\mathrm{Tr}(\mathbf{BA}) = \sum_i \sum_k b_{ik}a_{ki}.$$

Each of these expressions is the sum of every element in the matrix $\mathbf{A}$ multiplied by the corresponding element in $\mathbf{B}^{\mathrm{T}}$, so they are equal. We can also see this by algebraic manipulation:

$$\mathrm{Tr}(\mathbf{BA}) = \sum_i \sum_k b_{ik}a_{ki} = \sum_i \sum_k a_{ki}b_{ik} = \sum_k \sum_i a_{ki}b_{ik} = \sum_i \sum_k a_{ik}b_{ki} = \mathrm{Tr}(\mathbf{AB}).$$

where at the first step, we have just swapped the order of multiplication, in the second we have swapped the order of the sums and in the third we have swapped the dummy variables i and k.

The advantage of the second approach is that it shows that the result is true for $n \times n$ matrices if we define the trace of an $n \times n$ matrix to be the sum of all of its diagonal elements.

(vi) $\mathrm{Tr}(\mathbf{I}) = 2$.

This is a straightforward calculation: $\mathbf{I} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, so $\mathrm{Tr}(\mathbf{I}) = 1 + 1 = 2$.

Note that for $n \times n$ matrices, where the trace is the sum of all of the diagonal elements, $\mathrm{Tr}(\mathbf{I}_n) = n$.

Deduce that there are no matrices satisfying $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$. Does this in any way invalidate the statement in (iii)?

We have $\mathrm{Tr}([\mathbf{A}, \mathbf{B}]) = \mathrm{Tr}(\mathbf{AB} - \mathbf{BA})$. The result of part (iv) can easily be adapted to show that $\mathrm{Tr}(\mathbf{C} - \mathbf{D}) = \mathrm{Tr}(\mathbf{C}) - \mathrm{Tr}(\mathbf{D})$, so we have

$$\begin{aligned} \mathrm{Tr}([\mathbf{A}, \mathbf{B}]) &= \mathrm{Tr}(\mathbf{AB} - \mathbf{BA}) \\ &= \mathrm{Tr}(\mathbf{AB}) - \mathrm{Tr}(\mathbf{BA}) && \text{by adapted (iv)} \\ &= \mathrm{Tr}(\mathbf{AB}) - \mathrm{Tr}(\mathbf{AB}) && \text{by (v)} \\ &= 0 \end{aligned}$$

while $\text{Tr}(\mathbf{I}) = 2$. Therefore we cannot have $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$.

This does not actually invalidate (iii), but rather makes it vacuously true: we showed that *if* $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$ then a certain result follows, and the argument is still valid. It is just that we cannot find such an $\mathbf{A}$ and $\mathbf{B}$. Another approach is to say that whenever $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$ is false, the logical statement "if $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$, then ..." is true, as discussed in the hints. Since $[\mathbf{A}, \mathbf{B}] = \mathbf{I}$ is always false, the statement of (iii) is always true.

A set of elements which allows addition, subtraction and multiplication by scalars, and also has a bracket operation satisfying the conditions of (i) and (ii), is known as a *Lie algebra*, with the bracket operation being called the *Lie bracket*. These are extremely important in many areas of mathematics and physics. Part (i) and (ii) of this question show that the set of all $n \times n$ matrices with this definition of the Lie bracket forms a Lie algebra (as nowhere did we use the fact that $n = 2$).

Question 4

> Matrices $\mathbf{P}$ and $\mathbf{Q}$ are given by
>
> $$\mathbf{P} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}$$
>
> (where $\mathrm{i}^2 = -1$). Show that $\mathbf{P}^2 = \mathbf{Q}^2$, $\mathbf{PQP} = \mathbf{Q}$ and $\mathbf{P}^4 = \mathbf{I}$, the identity matrix.

We calculate the requested products:

$$\begin{aligned}
\mathbf{P}^2 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
\mathbf{Q}^2 &= \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathbf{P}^2 \\
\mathbf{PQP} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} = \mathbf{Q} \\
\mathbf{P}^4 &= (\mathbf{P}^2)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbf{I}.
\end{aligned}$$

> Deduce that, for all positive integers n, $\mathbf{P}^n\mathbf{QP}^n = \mathbf{Q}$.

Since we have $\mathbf{PQP} = \mathbf{Q}$, we can prove this result by induction: assuming that it is true for $n = k$, then

$$\mathbf{P}^{k+1}\mathbf{QP}^{k+1} = \mathbf{P}(\mathbf{P}^k\mathbf{QP}^k)\mathbf{P} = \mathbf{PQP} = \mathbf{Q}.$$

Since it is true when $n = 1$, the result holds by induction.

An alternative argument is as follows. We note that if $n > 4$, then we can divide by 4 and write $n = 4k + r$, where $1 \leqslant r \leqslant 4$. Then

$$\mathbf{P}^n = \mathbf{P}^{4k+r} = \mathbf{P}^{4k}\mathbf{P}^r = (\mathbf{P}^4)^k\mathbf{P}^r = (\mathbf{I})^k\mathbf{P}^r = \mathbf{IP}^r = \mathbf{P}^r.$$

So if we can prove the result for $n = 1$, 2, 3 and 4, it will be true for all n. It is true for $n = 1$ by the above; then

$$\mathbf{P}^2\mathbf{QP}^2 = \mathbf{P}(\mathbf{PQP})\mathbf{P} = \mathbf{PQP} = \mathbf{Q}$$

and similarly for $n = 3$; when $n = 4$, $\mathbf{P}^4 = \mathbf{I}$, so $\mathbf{P}^4\mathbf{QP}^4 = \mathbf{IQI} = \mathbf{Q}$.

The proof by induction is simpler!

> Hence, or otherwise, show that if $\mathbf{X}$ and $\mathbf{Y}$ are each matrices of the form
>
> $$\mathbf{P}^m\mathbf{Q}^n, \quad m = 1, 2, 3, 4;\ n = 1, 2$$
>
> then $\mathbf{XY}$ has the same form.

Let us say that $\mathbf{X} = \mathbf{P}^m\mathbf{Q}^n$ and $\mathbf{Y} = \mathbf{P}^r\mathbf{Q}^s$, then

$$\mathbf{XY} = \mathbf{P}^m\mathbf{Q}^n\mathbf{P}^r\mathbf{Q}^s.$$

Now if $n = 2$, we can replace $\mathbf{Q}^2$ by $\mathbf{P}^2$, giving $\mathbf{XY} = \mathbf{P}^{m+2+r}\mathbf{Q}^s$. Since $\mathbf{P}^4 = \mathbf{I}$, we can subtract 4 or 8 from $m+2+r$ if necessary so that we end up with an expression for $\mathbf{XY}$ in the required form.

If, on the other hand, $n = 1$, then we can use our previous result, taking care than we have a high enough power of $\mathbf{P}$ at the start (which we can obtain using $\mathbf{P}^4 = \mathbf{I}$), and we find

$$\begin{aligned}
\mathbf{XY} &= \mathbf{P}^m\mathbf{Q}\mathbf{P}^r\mathbf{Q}^s \\
&= \mathbf{P}^{m+4}\mathbf{Q}\mathbf{P}^r\mathbf{Q}^s \\
&= \mathbf{P}^{m+4-r}\mathbf{P}^r\mathbf{Q}\mathbf{P}^r\mathbf{Q}^s \\
&= \mathbf{P}^{m+4-r}\mathbf{Q}\mathbf{Q}^s \\
&= \mathbf{P}^{m+4-r}\mathbf{Q}^{s+1}.
\end{aligned}$$

If $s = 1$, then this resulting expression is either in the desired form, or by subtracting 4 from $m+4-r$ it will be. If, on the other hand, $s = 2$, we then have $\mathbf{Q}^{s+1} = \mathbf{Q}^3 = \mathbf{Q}^2\mathbf{Q} = \mathbf{P}^2\mathbf{Q}$, so $\mathbf{XY} = \mathbf{P}^{m+6-r}\mathbf{Q}$, and either this is in the required form, or by subtracting 4 or 8 from $m+6-r$ it will be.

Thus we are done.

We therefore have 8 invertible matrices, any two of which multiply to give another one of them. This collection of matrices is known as the *group of quaternions* (note the spelling!), and (in a related form) were discovered by Hamilton in the 19th century. They have turned out to be critical to understanding several aspects of physics, as well as being very important in some areas of mathematics.

Question 5

(a) Show that if $\mathbf{A} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$, then

$$\mathbf{A}^2 - (p+s)\mathbf{A} + (ps - qr)\mathbf{I} = \mathbf{O},$$

where $\mathbf{I}$ is the identity matrix and $\mathbf{O}$ is the zero matrix.

This can be answered just by performing the calculations:

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} p & q \\ r & s \end{pmatrix}\begin{pmatrix} p & q \\ r & s \end{pmatrix} \\ &= \begin{pmatrix} p^2+qr & pq+qs \\ pr+rs & qr+s^2 \end{pmatrix} \end{aligned}$$

so

$$\begin{aligned} \mathbf{A}^2 - (p+s)\mathbf{A} + (ps-qr)\mathbf{I} &= \begin{pmatrix} p^2+qr & pq+qs \\ pr+rs & qr+s^2 \end{pmatrix} - (p+s)\begin{pmatrix} p & q \\ r & s \end{pmatrix} + (ps-qr)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} p^2+qr & pq+qs \\ pr+rs & qr+s^2 \end{pmatrix} - \begin{pmatrix} p^2+ps & pq+qs \\ pr+rs & ps+s^2 \end{pmatrix} + \begin{pmatrix} ps-qr & 0 \\ 0 & ps-qr \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ &= \mathbf{O} \end{aligned}$$

as required.

This is an example of the Cayley–Hamilton theorem: every matrix satisfies its characteristic equation; see the topic notes for an explanation of the characteristic equation of a matrix.

(b) Given that $\mathbf{X} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and that $\mathbf{X}^2 = \mathbf{O}$, show that $\mathbf{X}$ can be written either in terms of a and b only or in terms of c only, or of b only.

One way to approach this is to start with an expression for $\mathbf{X}^2$, and equate all of the elements to zero. This looks as if it will be quite messy; it would work with enough persistence, though.

Another approach is to use the result of part (a), replacing $\mathbf{A}$ by $\mathbf{X}$. This gives

$$\mathbf{X}^2 - (a+d)\mathbf{X} + (ad-bc)\mathbf{I} = \mathbf{O}$$

But we are told that $\mathbf{X}^2 = \mathbf{O}$, so this equation becomes (after rearrangement):

$$(a+d)\mathbf{X} = (ad-bc)\mathbf{I}.$$

Assuming for a moment that $a + d \neq 0$, this gives

$$\mathbf{X} = \frac{ad - bc}{a + d}\,\mathbf{I}.$$

But we already know what $\mathbf{X}$ is (from the statement of the question), so we can now equate the elements on the two sides of this equation.

At this point, though, it is probably simpler to go back one step and not divide by $a + d$; that way, we do not need to treat $a + d = 0$ as a special case, at least to start with. We have

$$\begin{aligned} (a+d)\mathbf{X} &= (ad - bc)\mathbf{I} \\ \text{so} \quad (a+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= (ad - bc)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and equating elements on both sides now gives

$$\begin{aligned} a(a+d) &= ad - bc \\ b(a+d) &= 0 \\ c(a+d) &= 0 \\ d(a+d) &= ad - bc. \end{aligned}$$

There are several possible cases here, and it not obvious where to start, so we just start somewhere.

Subtracting the first and last equations gives $(a - d)(a + d) = 0$, so either $a - d = 0$ or $a + d = 0$.

If $a - d = 0$, so that $d = a$, the first three equations become:

$$\begin{aligned} 2a^2 &= a^2 - bc \\ 2ab &= 0 \\ 2ac &= 0. \end{aligned}$$

The fourth equation is just a repeat of the first, and the first equation simplifies to $a^2 = -bc$.

The second equation gives $a = 0$ or $b = 0$. We consider first the case that $a \neq 0$. Then $b = 0$ and the third equation gives $c = 0$. But then the first equation becomes $a^2 = 0$, so $a = 0$ itself. Therefore we cannot have $a \neq 0$. So $a = 0$, and the first equation becomes $bc = 0$. Therefore either $b = 0$ or $c = 0$, and we end up with two possibilities for $\mathbf{X}$:

$$\mathbf{X} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \mathbf{X} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

It is easy to check that both of these square to the zero matrix.

The second case was that $a + d = 0$, so that $d = -a$. Now the first three equations become

$$\begin{aligned} 0 &= -a^2 - bc \\ 0 &= 0 \\ 0 &= 0 \end{aligned}$$

(and again the fourth equation does not add anything). Therefore $bc = -a^2$. If either b or c is zero, then $a = d = 0$ and we are back in our previous cases. On the other hand, if neither b nor c is zero, then we can express c in terms of a and b as $c = -a^2/b$. We therefore have one final possibility for $\mathbf{X}$:

$$\mathbf{X} = \begin{pmatrix} a & b \\ -a^2/b & -a \end{pmatrix}.$$

Again, a quick check shows that this squares to the zero matrix.

These three possibilities for $\mathbf{X}$ also satisfy the conditions in the question: the final one is in terms of a and b, while the first two are in terms of b only and c only.

(b) (continued)

Show that when $\mathbf{X}$ is written in terms of c only, the solution can be written in the form:

$$\mathbf{X} = c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and interpret this result in terms of transformations of the plane represented by these matrices, relating your answer to the fact that $\mathbf{X}^2 = \mathbf{O}$.

In this case, we have $\mathbf{X} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$, so we can compare this to the product given in the question:

$$c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix},$$

so in this case, $\mathbf{X}$ can indeed be written in this form.

The transformations in this product are, from right to left:

- Stretching by a factor of 0 in the y-direction. This is the same as projecting onto the x-axis: a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is transformed to $\begin{pmatrix} x \\ 0 \end{pmatrix}$.
- Rotation about the origin by $\frac{\pi}{2}$.
- Multiplying by c is the same as multiplying by $c\mathbf{I} = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$, which is an enlargement centred on the origin with scale factor c.

Since composition of transformations is represented by writing their matrices in order from right to left, the effect of $\mathbf{X}$ is to project onto the x-axis, then rotate by $\frac{\pi}{2}$ and finally to enlarge by a factor of c.

Any point on the plane will therefore first be sent to the x-axis, then rotated to the y-axis. Doing this again will project this point to the origin, where it will remain. Therefore the effect of performing the transformation represented by $\mathbf{X}$ twice is to send every point on the plane to the origin.

Question 6

A mapping $(x, y) \to (u, v)$ is given by

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -8 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Show briefly that this mapping is not one to one.

Since $u = 2x + y$ and $v = -8x - 4y = -4(2x + y) = -4u$, if we take any point (x, y) on the line $2x + y = 0$, it will map to $(0, 0)$. Thus the map is not one to one.

Find the locus, L, of all points which map to $(1, -4)$.

We require $2x + y = 1$ and $-8x - 4y = -4$. Dividing the latter by -4 gives $2x + y = 1$ again, so the locus L is the line $2x + y = 1$.

Describe the locus of (u, v) as (x, y) is allowed to vary throughout the plane.

We saw in our answer to the first part of the question that $v = -4u$, so (u, v) always lies on the line $v = -4u$.

We now show that every point on this line lies on the locus, proving that the locus is the entire line. Consider the point given by $u = t$, $v = -4t$. Since $u = 2x + y$, we could take $x = \frac{1}{2}t$ and $y = 0$, and that would give $u = t$, $v = -4t$ as required.

Hence the locus of (u, v) is the entire line $v = -4u$.

Show that any given point, P, on this locus is the image of just one point on the y-axis, and describe how the set of all points with image P is related to the locus L.

If we take any point on the y-axis, say $(0, c)$, it maps to $(c, -4c)$, so different points on the y-axis map to different points on the locus. Conversely, we can also obtain every point $(c, -4c)$ on this line by taking $x = 0$, $y = c$, so the locus is the entire line $v = -4u$.

For a given point $P = (c, -4c)$, we require $2x + y = c$. The locus L was the line $2x + y = 1$, so the set of all points with image P is a line parallel to L.

This sort of behaviour always happens for singular transformation matrices such as this one.

Question 7

> You are given that $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ are 2×2 matrices, $\mathbf{I}$ is the identity matrix and $\mathbf{P}^{-1}$ exists.

For the sake of simplicity, we will decide on a notation for the whole question. We will take

$$\mathbf{P} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}.$$

We are given that $\mathbf{P}$ is invertible, so $ad - bc \neq 0$.

> (i) Prove, by expanding both sides, that
>
> $$\det(\mathbf{PQ}) = \det \mathbf{P} \det \mathbf{Q}.$$
>
> Deduce that
>
> $$\det(\mathbf{P}^{-1}\mathbf{Q} + \mathbf{I}) = \det(\mathbf{QP}^{-1} + \mathbf{I}).$$

We have

$$\mathbf{PQ} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

so

$$\begin{aligned} \det(\mathbf{PQ}) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= acef + adeh + bcfg + bdgh - acef - adfg - bceh - bdgh \\ &= adeh + bcfg - adfg - bceh, \end{aligned}$$

while

$$\begin{aligned} \det \mathbf{P} \det \mathbf{Q} &= (ad - bc)(eh - fg) \\ &= adeh - adfg + bcfg - bceh, \end{aligned}$$

so we see that $\det(\mathbf{PQ}) = \det \mathbf{P} \det \mathbf{Q}$.

This result is true in general for $n \times n$ matrices, once we have defined the determinant of an $n \times n$ matrix. The proof does not require writing out a large algebraic expression like this, though!

Then we have, using $\mathbf{P}^{-1}\mathbf{P} = \mathbf{PP}^{-1} = \mathbf{I}$:

$$\begin{aligned} \det(\mathbf{P}^{-1}\mathbf{Q} + \mathbf{I}) &= \det(\mathbf{P}^{-1}(\mathbf{Q} + \mathbf{P})) && (1) \\ &= \det(\mathbf{P}^{-1}) \det(\mathbf{Q} + \mathbf{P}) && (2) \\ &= \det(\mathbf{Q} + \mathbf{P}) \det(\mathbf{P}^{-1}) && (3) \\ &= \det((\mathbf{Q} + \mathbf{P})\mathbf{P}^{-1}) && (4) \\ &= \det(\mathbf{QP}^{-1} + \mathbf{I}) && (5) \end{aligned}$$

On lines (2) and (4), we used the above result about the determinant of a product, and on line (3), we used the fact that $ab = ba$ whenever a and b are numbers (and a determinant is a number).

(ii) If $\mathbf{PX} = \mathbf{XP}$ for *every* 2×2 matrix $\mathbf{X}$, prove that $\mathbf{P} = \lambda\mathbf{I}$, where λ is a constant.

It would make sense to start with some very simple matrices for $\mathbf{X}$. If we take $\mathbf{X} = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$, then we get

$$\begin{aligned}\mathbf{PX} &= \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}a & 0\\ c & 0\end{pmatrix},\\ \mathbf{XP} &= \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}.\end{aligned}$$

So we immediately see, if $\mathbf{PX} = \mathbf{XP}$, then $b = c = 0$.

An alternative matrix we could use to show that $b = c = 0$ is a reflection in an axis. For example, if we take $\mathbf{X} = \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right)$, then we get

$$\begin{aligned}\mathbf{PX} &= \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-a & b\\ -c & d\end{pmatrix},\\ \mathbf{XP} &= \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}-a & -b\\ c & d\end{pmatrix}.\end{aligned}$$

Comparing these, we see that if $\mathbf{PX} = \mathbf{XP}$, then $b = -b$ and $c = -c$, so $b = c = 0$.

To show that $a = d$, we will need to somehow swap them or something like that. We could try $\mathbf{X} = \left(\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right)$; this gives

$$\begin{aligned}\mathbf{PX} &= \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & a\\ 0 & c\end{pmatrix},\\ \mathbf{XP} &= \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}c & d\\ 0 & 0\end{pmatrix}.\end{aligned}$$

For these to be equal, we need $a = d$ (and $c = 0$, which we already know).

An alternative way to swap them is to use a different reflection matrix, one which swaps x and y. If we use $\mathbf{X} = \left(\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right)$, we obtain

$$\begin{aligned}\mathbf{PX} &= \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}b & a\\ d & c\end{pmatrix},\\ \mathbf{XP} &= \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix} = \begin{pmatrix}c & d\\ a & b\end{pmatrix}.\end{aligned}$$

For these to be equal, we need $a = d$ (and $b = c$, which we already know).

So if $\mathbf{PX} = \mathbf{XP}$ for all $\mathbf{X}$, then $\mathbf{P}$ must have the form $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right) = \lambda\mathbf{I}$ for some non-zero λ (non-zero because we have assumed that $\mathbf{P}$ is invertible).

We now make a few further observations on this part of the question.

This result is also true if we only require $\mathbf{PX} = \mathbf{XP}$ for all *invertible* matrices $\mathbf{X}$, as we were able to deduce it by considering just reflection matrices.

Furthermore, a very similar result is true if we do not assume that $\mathbf{P}$ is invertible: the only difference is that now we could have $\lambda = 0$, so $\mathbf{P}$ is the zero matrix.

A generalisation of this result is that if $\mathbf{P}$ is an $n \times n$ matrix which commutes with every (invertible) $n \times n$ matrix (that is, $\mathbf{PX} = \mathbf{XP}$ for every (invertible) $\mathbf{X}$), then $\mathbf{P}$ is a multiple of the $n \times n$ identity matrix. In the language of group theory, which is usually taught as part of an undergraduate course, this result says that the centre of the general linear group $\mathrm{GL}(n, \mathbb{R})$ is the set $\{c\,\mathbf{I} \colon c \in \mathbb{R}, c \neq 0\}$.

Though the question does not ask us to show it, the result is actually an "if and only if" result: we have already shown that *if* $\mathbf{PX} = \mathbf{XP}$ for every 2×2 matrix $\mathbf{X}$, *then* $\mathbf{P}$ is a multiple of the identity matrix. Conversely, *if* $\mathbf{P}$ is a multiple of the identity matrix, *then* $\mathbf{PX} = \mathbf{XP}$ for every 2×2 matrix $\mathbf{X}$. This follows immediately, as in this case $\mathbf{PX} = \lambda\mathbf{IX} = \lambda\mathbf{X}$, and $\mathbf{XP} = \mathbf{X}(\lambda\mathbf{I}) = \lambda\mathbf{XI} = \lambda\mathbf{X}$, so they are equal for every $\mathbf{X}$.

(iii) If $\mathbf{RQ} = \mathbf{QR}$, prove that

$$\mathbf{RQ}^n = \mathbf{Q}^n\mathbf{R} \quad \text{and} \quad \mathbf{R}^n\mathbf{Q}^n = \mathbf{Q}^n\mathbf{R}^n$$

for any positive integer n.

We start by proving the first statement by induction on n, the result being given when $n = 1$. Then if the result is true for $n = k$, so $\mathbf{RQ}^k = \mathbf{Q}^k\mathbf{R}$, we have

$$\mathbf{RQ}^{k+1} = \mathbf{RQ}^k\mathbf{Q} = \mathbf{Q}^k\mathbf{RQ} = \mathbf{Q}^k\mathbf{QR} = \mathbf{Q}^{k+1}\mathbf{R}$$

so the result holds for $n = k + 1$, and hence for all positive integers n by induction.

For the next result, it actually seems easier to prove a more general result, which is that for any positive integer m, $\mathbf{R}^m\mathbf{Q}^n = \mathbf{Q}^n\mathbf{R}^m$ for any positive integer n. When $m = 1$, this is the result we have just shown, so we proceed by induction on m. Assuming the result is true for $m = k$ and all values of n, we have

$$\mathbf{R}^{k+1}\mathbf{Q}^n = \mathbf{RR}^k\mathbf{Q}^n = \mathbf{RQ}^n\mathbf{R}^k = \mathbf{Q}^n\mathbf{RR}^k = \mathbf{Q}^n\mathbf{R}^{k+1}$$

and so the result is true for $m = k + 1$ and all values of n. In particular, $\mathbf{R}^n\mathbf{Q}^n = \mathbf{Q}^n\mathbf{R}^n$.

This approach of proving something more general can sometimes be very useful. It was prompted here by realising that changing both the power of $\mathbf{R}$ and the power of $\mathbf{Q}$ at the same time could be quite difficult to manage.

Another way of approaching this question is to try changing the powers of both $\mathbf{Q}$ and $\mathbf{R}$ at the same time, and see what happens.

So assuming that the result is true when $n = k$, we have

$$\mathbf{R}^{k+1}\mathbf{Q}^{k+1} = \mathbf{RR}^k\mathbf{Q}^k\mathbf{Q} = \mathbf{RQ}^k\mathbf{R}^k\mathbf{Q}.$$

Now the first half of this final expression is $\mathbf{RQ}^k = \mathbf{Q}^k\mathbf{R}$, by the first part of this question, which gets us most of the way: we now have $\mathbf{R}^{k+1}\mathbf{Q}^{k+1} = \mathbf{Q}^k\mathbf{RR}^k\mathbf{Q} = \mathbf{Q}^k\mathbf{R}^{k+1}\mathbf{Q}$. If we could swap the last two terms here, we would be done. But this looks very much like the first part of the question; if we swap $\mathbf{R}$ and $\mathbf{Q}$, the statement becomes: If $\mathbf{QR} = \mathbf{RQ}$, then $\mathbf{QR}^n = \mathbf{R}^n\mathbf{Q}$ for any positive integer n. In our case, then, we have $\mathbf{R}^{k+1}\mathbf{Q} = \mathbf{QR}^{k+1}$, so the whole argument runs:

$$\begin{aligned}\mathbf{R}^{k+1}\mathbf{Q}^{k+1} &= \mathbf{RR}^k\mathbf{Q}^k\mathbf{Q}\\ &= \mathbf{RQ}^k\mathbf{R}^k\mathbf{Q}\\ &= \mathbf{Q}^k\mathbf{RR}^k\mathbf{Q}\\ &= \mathbf{Q}^k\mathbf{R}^{k+1}\mathbf{Q}\\ &= \mathbf{Q}^k\mathbf{QR}^{k+1}\\ &= \mathbf{Q}^{k+1}\mathbf{R}^{k+1}.\end{aligned}$$

Since we are given the result in the case $n = 1$, it follows for all n by induction.

Question 8

The real 3×3 matrix $\mathbf{A}$ is such that $\mathbf{A}^2 = \mathbf{A}$.

(i) Prove that $(\mathbf{I} - \mathbf{A})^2 = \mathbf{I} - \mathbf{A}$.

We use the given property $\mathbf{A}^2 = \mathbf{A}$ to calculate:

$$(\mathbf{I} - \mathbf{A})^2 = \mathbf{I}^2 - \mathbf{IA} - \mathbf{AI} + \mathbf{AA} = \mathbf{I} - 2\mathbf{A} + \mathbf{A}^2 = \mathbf{I} - 2\mathbf{A} + \mathbf{A} = \mathbf{I} - \mathbf{A}.$$

(ii) Express $(\mathbf{I} - \mathbf{A})^3$ in the form $\mathbf{I} + k\mathbf{A}$, where k is a number to be determined.

We again calculate:

$$(\mathbf{I} - \mathbf{A})^3 = (\mathbf{I} - \mathbf{A})^2(\mathbf{I} - \mathbf{A}) = (\mathbf{I} - \mathbf{A})(\mathbf{I} - \mathbf{A}) = \mathbf{I} - \mathbf{A},$$

using the result of part (i) twice.

It now seems clear that in this case, $(\mathbf{I} - \mathbf{A})^n = \mathbf{I} - \mathbf{A}$ for any positive integer n; this will also follow from the result of part (iii).

(iii) Prove that, for all real constants λ and all positive integers n,

$$(\mathbf{I} + \lambda\mathbf{A})^n = \mathbf{I} + \big((\lambda + 1)^n - 1\big)\mathbf{A}.$$

Use this result to verify your answer to (ii).

We use induction to prove this result. It is true when $n = 1$, since $\mathbf{I} + \big((\lambda + 1)^1 - 1\big)\mathbf{A} = \mathbf{I} + \lambda\mathbf{A}$. Assuming that the result is true when $n = k$, we have (remembering that $\mathbf{A}^2 = \mathbf{A}$ to get to the third line):

$$\begin{aligned}
(\mathbf{I} + \lambda\mathbf{A})^{k+1} &= \big(\mathbf{I} + \big((\lambda + 1)^k - 1\big)\mathbf{A}\big)(\mathbf{I} + \lambda\mathbf{A}) \\
&= \mathbf{I} + \lambda\mathbf{A} + \big((\lambda + 1)^k - 1\big)\mathbf{A} + \big((\lambda + 1)^k - 1\big)\mathbf{A}(\lambda\mathbf{A}) \\
&= \mathbf{I} + \lambda\mathbf{A} + \big((\lambda + 1)^k - 1\big)\mathbf{A} + \big((\lambda + 1)^k - 1\big)\lambda\mathbf{A} \\
&= \mathbf{I} + \big(\lambda + (\lambda + 1)^k - 1 + ((\lambda + 1)^k - 1)\lambda\big)\mathbf{A} \\
&= \mathbf{I} + \big((\lambda + 1)(\lambda + 1)^k + \lambda - 1 - \lambda\big)\mathbf{A} \\
&= \mathbf{I} + \big((\lambda + 1)^{k+1} - 1\big)\mathbf{A}
\end{aligned}$$

so the induction step holds and the result follows by induction.

Applying this now to the case in (ii), we have $\lambda = -1$, so this result says that $(\mathbf{I} - \mathbf{A})^n = \mathbf{I} + \big((-1 + 1)^n - 1\big)\mathbf{A} = \mathbf{I} + (-1)\mathbf{A} = \mathbf{I} - \mathbf{A}$, as we found in parts (i) and (ii).

Acknowledgements

The exam questions are reproduced by kind permission of Cambridge Assessment Group Archives. Unless otherwise noted, the questions are reproduced verbatim, except for some differences in convention between the papers as printed and STEP, specifically: the use of j to represent $\sqrt{-1}$ has been replaced by i; variables representing matrices are written in boldface type rather than italics; transpose is denoted $\mathbf{A}^{\mathrm{T}}$ rather that A'.

In the list of sources below, the following abbreviations are used:

O&C	Oxford and Cambridge Schools Examination Board
SMP	School Mathematics Project
MEI	Mathematics in Education and Industry
QP	Question paper
Q	Question

1 O&C, A level Mathematics (SMP), 1966, QP Mathematics II, Q A3

2 O&C, A level Mathematics (SMP), 1967, QP Mathematics II, Q B22

3 O&C, A level Mathematics (MEI), 1968, QP MEI 20, Pure Mathematics III (Special Paper), Q 3; editorial changes here: the definition of **O** is inserted, the implication symbol is written in words, and the reference to the matrix ring is removed

4 O&C, A level Mathematics (MEI), 1968, QP MEI 143*, Pure Mathematics I, Q 6

5 O&C, A level Mathematics (MEI), 1980, QP 9655/1, Pure Mathematics 1, Q 2; editorial changes here: use **O** rather than **0** for the zero matrix, and define the notation.

6 O&C, A level Mathematics (MEI), 1981, QP 9655/1, Pure Mathematics 1, Q 6(b)

7 O&C, A level Mathematics (MEI), 1986, QP 9657/0, Mathematics 0 (Special Paper), Q 2; editorial change here: **I** is called the identity matrix rather than the unit matrix

8 O&C, A level Mathematics (MEI), 1987, QP 9650/2, Mathematics 2, Q 16

9 O&C, A level Mathematics (MEI), 1987, QP 9655/1, Pure Mathematics 1, Q 2