STEP Support Programme

Pure STEP 1 Solutions

2012 S1 Q4

1 Preparation

(i) Differentiating gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}x^{-\frac{1}{2}}$, so the gradient at the point $(4, 2)$ is $m = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$. Using $y - y_1 = m(x - x_1)$ gives $y - 2 = \frac{1}{4}(x - 4)$ i.e. $y = \frac{1}{4}x + 1$.

(ii) Substituting for y into the first equation gives:

$$\begin{aligned} ax + b(2ax) &= c \\ x\big(a + 2ab\big) &= c \\ x &= \frac{c}{a(1+2b)} \\ y = 2ax &= \frac{2c}{1+2b} \end{aligned}$$

(iii) The gradient of the curve at the point (p, p^2) is $2p$, so the gradient of the normal will be $\frac{-1}{2p}$. The equation of the normal is:

$$\begin{aligned} y - p^2 &= \tfrac{-1}{2p}\,(x - p) \\ 2py - 2p^3 &= -x + p \\ 2py + x &= 2p^3 + p \end{aligned}$$

(iv) Expanding in a table gives:

	a^2	ab	b^2
a	a^3	~~a^2b~~	~~ab^2~~
$-b$	~~$-a^2b$~~	~~$-ab^2$~~	$-b^3$

and so $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

2 The STEP I question

The equation for C can be written as $y = \dfrac{1}{2x}$, so $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{2x^2}$. The equation of the tangent to P is:

$$\begin{aligned} y - \tfrac{1}{2p} &= -\tfrac{1}{2p^2}\left(x - p\right) \\ y &= -\tfrac{1}{2p^2}x + \tfrac{1}{p} \end{aligned}$$

Similarly the equation of the tangent to Q is $y = -\frac{1}{2q^2}x + \frac{1}{q}$.

Where the tangents meet we have:

$$\begin{aligned} -\tfrac{1}{2p^2}x + \tfrac{1}{p} &= -\tfrac{1}{2q^2}x + \tfrac{1}{q} \\ x\left(\tfrac{1}{2q^2} - \tfrac{1}{2p^2}\right) &= \tfrac{1}{q} - \tfrac{1}{p} \\ x\left(\frac{p^2 - q^2}{2p^2q^2}\right) &= \frac{p-q}{pq} \\ x\left(\frac{(p-q)(p+q)}{2(pq)^2}\right) &= \frac{p-q}{pq} \\ x\left(\frac{p+q}{2pq}\right) &= 1 \qquad \text{since } p \neq q \\ & \qquad\qquad \text{(we are told that the points are distinct)} \\ x &= \frac{2pq}{p+q} \end{aligned}$$

The y coordinate is given by

$$\begin{aligned} y &= -\frac{1}{2p^2}x + \frac{1}{p} \\ &= -\frac{1}{\cancel{2}p^{\cancel{2}}} \times \frac{\cancel{2}\cancel{p}q}{p+q} + \frac{1}{p} \\ &= \frac{p+q}{p(p+q)} - \frac{q}{p(p+q)} \\ &= \frac{p}{p(p+q)} = \frac{1}{p+q} \end{aligned}$$

The gradient of the normal at P is $2p^2$, so the equation of the normal at P is:

$$\begin{aligned} y - \tfrac{1}{2p} &= 2p^2\left(x - p\right) \\ y &= 2p^2x + \tfrac{1}{2p} - 2p^3 \end{aligned}$$

and the normal to Q has equation $y = 2q^2x + \frac{1}{2q} - 2q^3$.

Where the normals intersect we have:

$$\begin{aligned}
2p^2x + \tfrac{1}{2p} - 2p^3 &= 2q^2x + \tfrac{1}{2q} - 2q^3 \\
(2p^2 - 2q^2)x &= 2p^3 - 2q^3 + \tfrac{1}{2q} - \tfrac{1}{2p} \\
2(p+q)(p-q)x &= 2(p-q)(p^2+pq+q^2) + \frac{p-q}{2pq} \\
2(p+q)x &= 2(p^2+pq+q^2) + \frac{1}{2pq} \qquad \text{since } p \neq q \\
x &= \frac{p^2+pq+q^2}{p+q} + \frac{1}{4pq(p+q)}
\end{aligned}$$

Since $pq = \frac{1}{2}$ we have $2pq = 1$. We also have $p^2 + q^2 + pq = (p+q)^2 - pq$. x can therefore be written as:

$$\begin{aligned}
x &= \frac{p^2+pq+q^2}{p+q} + \frac{1}{4pq(p+q)} \\
&= \frac{(p+q)^2 - pq}{p+q} + \frac{1}{2(p+q)} \\
&= \frac{2(p+q)^2 - \cancel{2pq} + \cancel{1}}{2(p+q)} \\
&= p+q
\end{aligned}$$

The y coordinate of N is given by:

$$\begin{aligned}
y &= 2p^2x + \tfrac{1}{2p} - 2p^3 \\
&= 2p^2(p+q) + \tfrac{1}{2p} - 2p^3 \\
&= 2p^2q + \tfrac{1}{2p} \\
&= 2pq \times p + q \\
&= p+q
\end{aligned}$$

Hence $N = (p+q,\ p+q)$, so it lies on the line $y = x$. Using our previous answer for the coordinates of T with the condition that $2pq = 1$ gives $T = \left(\dfrac{1}{p+q}, \dfrac{1}{p+q}\right)$, so also lies on the line $y = x$. The distance of T from the origin is $\dfrac{\sqrt{2}}{p+q}$ and the distance of N from the origin is $\sqrt{2}(p+q)$, therefore the product of the distances is constant (it is 2).

2012 S1 Q8

3 Preparation

(i) $\dfrac{\mathrm{d}}{\mathrm{d}x}(xv) = v + x\dfrac{\mathrm{d}v}{\mathrm{d}x}$. Using this in $(*)$ gives:

$$\begin{aligned} x\frac{\mathrm{d}y}{\mathrm{d}x} - y &= x^3 \\ x\left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) - xv &= x^3 \\ x^2\frac{\mathrm{d}v}{\mathrm{d}x} &= x^3 \\ \frac{\mathrm{d}v}{\mathrm{d}x} &= x \qquad \text{since } x \neq 0 \end{aligned}$$

We therefore have $v = \frac{1}{2}x^2 + c$ and so $y = \frac{1}{2}x^3 + cx$. If you then substitute this back into $(*)$ you can verify that it satisfies the original equation.

(ii) **(a)** Rearranging gives:

$$\begin{aligned} \int \frac{1}{y+1}\,\mathrm{d}y &= \int \frac{1}{x}\,\mathrm{d}x \\ \ln|y+1| &= \ln|x| + c \\ y + 1 &= Ax \\ y &= Ax - 1 \end{aligned}$$

(b) Here we have:

$$\begin{aligned} \int \frac{1}{\tan y}\,\mathrm{d}y &= \int 1\,\mathrm{d}x \\ \int \frac{\cos y}{\sin y}\,\mathrm{d}y &= x + c \\ \ln|\sin y| &= x + c \\ \sin y &= A\mathrm{e}^x \end{aligned}$$

(iii) We need A and B such that $A(x+2) + B(x+1) \equiv x$. Substituting $x = -1$ gives $A = -1$ and substituting $x = -2$ gives $B = 2$. The integral therefore becomes:

$$\begin{aligned} \int_1^2 \frac{2}{x+2} - \frac{1}{x+1}\,\mathrm{d}x &= [2\ln(x+2) - \ln(x+1)]_1^2 \\ &= (2\ln 4 - \ln 3) - (2\ln 3 - \ln 2) \\ &= \ln(4^2 \times 2) - \ln(3^3) \\ &= \ln\left(\tfrac{32}{27}\right) \end{aligned}$$

4 The STEP I question

(i) Using the substitution $y = xv$ gives:

$$\begin{aligned} x \times xv \times \left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + (xv)^2 - 2x^2 &= 0 \\ x^2v^2 + x^3v\frac{\mathrm{d}v}{\mathrm{d}x} + (xv)^2 - 2x^2 &= 0 \\ v^2 + xv\frac{\mathrm{d}v}{\mathrm{d}x} + v^2 - 2 &= 0 \qquad \text{since } x \neq 0 \\ xv\frac{\mathrm{d}v}{\mathrm{d}x} + 2v^2 - 2 &= 0\,. \end{aligned}$$

Separating variables:

$$\begin{aligned} \int \frac{v}{2-2v^2}\,\mathrm{d}v &= \int \frac{1}{x}\,\mathrm{d}x \\ \tfrac{1}{2}\int \frac{v}{1-v^2}\,\mathrm{d}v &= \ln|x| + c \\ \tfrac{1}{2} \times -\tfrac{1}{2}\ln|1-v^2| &= \ln|x| + c \\ -\ln|1-v^2| &= 4\ln|x| + k \\ \frac{1}{1-v^2} &= Ax^4 \\ 1-v^2 &= \frac{1}{Ax^4} \\ x^4 - x^4v^2 &= \frac{1}{A} \\ x^4 - x^2y^2 &= \frac{1}{A} \\ x^2(y^2-x^2) &= -\frac{1}{A} = C\,. \end{aligned}$$

There are lots of ways to tackle $\int \frac{v}{1-v^2}\,\mathrm{d}v$, one is to use partial fractions and another is to notice that the numerator is very similar to the derivative of the denominator, and then use $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$.

(ii) You are left to your own devices here a little. One possible starting point is to use $y = xv$ again (and if that doesn't work, hopefully this will give some insight into which substitution will be useful).

Using $y = xv$ we have:

$$\begin{aligned}
xv \times \left(v + x\frac{\mathrm{d}v}{\mathrm{d}x}\right) + 6x + 5xv &= 0 \\
v^2 + xv\frac{\mathrm{d}v}{\mathrm{d}x} + 6 + 5v &= 0 \qquad \text{since } x \neq 0 \\
xv\frac{\mathrm{d}v}{\mathrm{d}x} &= -\left(v^2 + 5v + 6\right) \\
\frac{v}{v^2 + 5v + 6}\frac{\mathrm{d}v}{\mathrm{d}x} &= -\frac{1}{x} \\
\int \frac{v}{(v+2)(v+3)}\,\mathrm{d}v &= -\ln|x| + c \\
\int \frac{3}{v+3} - \frac{2}{v+2}\,\mathrm{d}v &= -\ln|x| + c \\
3\ln|v+3| - 2\ln|v+2| &= -\ln|x| + c \\
\frac{(v+3)^3}{(v+2)^2} &= \frac{A}{x} \\
\frac{(xv+3x)^3}{x(xv+2x)^2} &= \frac{A}{x} \\
\frac{(y+3x)^3}{(y+2x)^2} &= A
\end{aligned}$$

The penultimate line was obtained by multiplying top and bottom of the fraction by x^3. The final answer can be written as $(y + 3x)^3 = A(y + 2x)^2$ if you like.

2010 S1 Q5

5 Preparation

(i) Using the sum of an infinite Geometric Progression we have:

$$1 + p + p^2 + p^3 + \ldots = \frac{1}{1-p}\,.$$

Substituting $p = \frac{1}{2}$ gives $1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 = \dfrac{1}{1-\frac{1}{2}} = 2$.

(ii) We have:

$$1 + p + p^2 + p^3 + \ldots = (1-p)^{-1}\,.$$

Differentiating both sides of this gives:

$$0 + 1 + 2p + 3p^2 + \ldots = (1-p)^{-2}\,.$$

Then substituting $p = \frac{1}{2}$ gives:

$$1 + \tfrac{2}{2} + \tfrac{3}{4} + \tfrac{4}{8} + \ldots = \frac{1}{\left(1-\frac{1}{2}\right)^2} = 4\,.$$

(iii) We have:

$$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5\,.$$

Integrating gives:

$$\tfrac{1}{6}(1+x)^6 = x + \tfrac{5}{2}x^2 + \tfrac{10}{3}x^3 + \tfrac{10}{4}x^4 + x^5 + \tfrac{1}{6}x^6 + c\,.$$

Substituting $x = 0$ gives $c = \frac{1}{6}$ (you can substitute any value of x you like, but the others are a lot more work!).

If you then multiply throughout by 6 you get the (hopefully not too surprising result):

$$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6\,.$$

This was a slightly contrived question, there is no real reason to do this other than to practice something that you will need for the STEP question!

6 The STEP I question

For this question you are asked to consider the expansion of $(1+x)^n$ in the "stem" of the question. This means that this will probably be useful in all of the parts of the question.

We have:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n}x^n \qquad (*)$$

(i) Substituting $x = 1$ into $(*)$ gives:

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} = (1+1)^n = 2^n\,.$$

(ii) Differentiating $(*)$ with respect to (WRT) x gives:

$$n(1+x)^{n-1} = \binom{n}{1} + \binom{n}{2} \times 2x + \binom{n}{3} \times 3x^2 + \ldots + \binom{n}{n} \times nx^{n-1}$$

and substituting $x = 1$ gives the required result.

(iii) Integrating $(*)$ WRT x gives:

$$\tfrac{1}{n+1}(1+x)^{n+1} = \binom{n}{0}x + \binom{n}{1}\tfrac{1}{2}x^2 + \binom{n}{2}\tfrac{1}{3}x^3 + \binom{n}{3}\tfrac{1}{4}x^4 + \ldots + \binom{n}{n}\tfrac{1}{n+1}x^{n+1} + c$$

Substituting $x = 0$ gives $c = \frac{1}{n+1}$. Then substitute $x = 1$ to get:

$$\begin{aligned}
\binom{n}{0} + \binom{n}{1}\tfrac{1}{2} + \binom{n}{2}\tfrac{1}{3} + \binom{n}{3}\tfrac{1}{4} + \ldots + \binom{n}{n}\tfrac{1}{n+1} + \tfrac{1}{n+1} &= \tfrac{1}{n+1}(1+1)^{n+1} \\
\binom{n}{0} + \tfrac{1}{2}\binom{n}{1} + \tfrac{1}{3}\binom{n}{2} + \tfrac{1}{4}\binom{n}{3} + \ldots + \tfrac{1}{n+1}\binom{n}{n} &= \tfrac{1}{n+1}(2)^{n+1} - \tfrac{1}{n+1} \\
&= \tfrac{1}{n+1}\left(2^{n+1} - 1\right)
\end{aligned}$$

(iv) This last part is a little trickier. The "2^{n-2}" suggests that a second derivative of $(*)$ might be useful:

$$\begin{aligned}
&2\binom{n}{2} + 3 \times 2\binom{n}{3}x + 4 \times 3\binom{n}{4}x^2 + \ldots + n(n-1)\binom{n}{n}x^{n-2} = n(n-1)(1+x)^{n-2} \\
&x = 1 \implies 2\binom{n}{2} + 3 \times 2\binom{n}{3} + 4 \times 3\binom{n}{4} + \ldots + n(n-1)\binom{n}{n} = n(n-1)(2)^{n-2}
\end{aligned}$$

The required result has a $\binom{n}{1}$ term as well, so try adding the result from part **(ii)** to this. This gives:

$$\begin{aligned}
&\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + 4\binom{n}{4} + \ldots + n\binom{n}{n} \\
&\quad + 2\binom{n}{2} + 3 \times 2\binom{n}{3} + 4 \times 3\binom{n}{4} + \ldots + n(n-1)\binom{n}{n} = n2^{n-1} + n(n-1)2^{n-2} \\
&\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + 4^2\binom{n}{4} + \ldots + n^2\binom{n}{n} = 2^{n-2}\Big[2n + n(n-1)\Big] \\
&\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + 4^2\binom{n}{4} + \ldots + n^2\binom{n}{n} = 2^{n-2}\left[n(n+1)\right]
\end{aligned}$$

Note that $n + n(n-1) = n + n^2 - n = n^2$.

2014 S1 Q1

7 Preparation

(i) We have $(2a+1)(2b+1) = 4ab+2a+2b+1 = 2\big[2ab+a+b\big]+1$ which has the form $2k+1$ and hence is odd.

(ii) $(2a+1)+(2b+1) = 2a+2b+2 = 2\big[a+b+1\big]$, which is of the form $2k$ and so is even.

(iii)

$$\begin{aligned}(2k+1)^2-(2k-1)^2 &= \big[4k^2+4k+1\big]-\big[4k^2-4k+1\big]\\ &= 8k\end{aligned}$$

Hence any number of the form $8k$ can be written in the form $(2k+1)^2-(2k-1)^2$, which is the difference of two odd (and consecutive!) squares. E.g. $8 = 3^2-1^2$, $16 = 5^2-3^2$, $88 = 23^2-21^2$.

(iv) You could start on either side to show this identity. If starting with the left hand side, using difference of two squares is a good starting point.

$$\begin{aligned}(2a+1)^2-(2b+1)^2 &= \Big[(2a+1)+(2b+1)\Big]\Big[(2a+1)-(2b+1)\Big]\\ &= (2a+2b+2)(2a-2b)\\ &= 2(a-b)\times 2(a+b+1)\\ &= 4(a-b)(a+b+1)\end{aligned}$$

The difference between two odd squares must therefore be divisible by 4 (as it has the form $4\times K$). However we are asked to show that it is divisible by 8. We need to consider different cases here.

- If a and b are both even, then $(a-b)$ is even and hence has a factor of 2.
- If a and b are both odd, then $(a-b)$ is even and hence has a factor of 2
- If one of a and b is even and the other odd, then $(a+b+1)$ is even and hence has a factor of 2.

So for all possible values of a and b, $(a-b)(a+b+1)$ has a factor of 2 and hence $4(a-b)(a+b+1)$ is divisible by 8.

(v) Looking at a few values (such as $4^2-2^2 = 12$, $10^2-4^2 = 84$), it looks as if the difference of two even squares is divisible by 4.

Consider $(2a)^2-(2b)^2$:

$$\begin{aligned}(2a)^2-(2b)^2 &= 4a^2-4b^2\\ &= 4(a+b)(a-b)\end{aligned}$$

Hence the difference of two even squares is a multiple of 4, but if one of a, b is odd and the other even then $(a+b)(a-b)$ is odd.

(vi) $58^2-42^2 = (58+42)(58-42) = 100\times 16 = 1600$.

(vii) Listing systematically gives $a = 12, b = 1$, $a = 6, b = 2$ and $a = 4, b = 3$.

8 The STEP I question

(i) $3 = 2^2 - 1^2$, $5 = 3^2 - 2^2$, $8 = 3^2 - 1^2$, $12 = 4^2 - 2^2$, $16 = 5^2 - 3^2$.

(ii) Looking at the first two numbers in part **(i)** gives a suggestion as to how to attempt this question. You could also write down a few more examples of writing odd numbers as a difference of two squares to help you.

$(k+1)^2 - (k)^2 = k^2 + 2k + 1 - k^2 = 2k + 1$ and hence any odd number $(2k+1)$ can be written as the difference of two squares $((k+1)^2 - k^2)$.

(iii) Looking at part **(i)** is appears that all the numbers of the form $4k$ are formed by considering squares of numbers of the form a and $a+2$. It is actually a bit easier to show this if we consider $k+1$ and $k-1$ instead.

$(k+1)^2 - (k-1)^2 = (k^2 + 2k + 1) - (k^2 - 2k + 1) = 4k$, hence anything of the form $4k$ can be written as the difference of two squares.

(iv) We have $a^2 - b^2 = (a+b)(a-b)$. The different cases are:

- a and b are both even. Then both $a+b$ and $a-b$ are even and so both have a factor of 2. $a^2 - b^2$ is hence divisible by 4.
- a and b are both odd. Then both $a+b$ and $a-b$ are even and so both have a factor of 2. $a^2 - b^2$ is hence divisible by 4
- One of a and b is even and the other is odd. Then both $a+b$ and $a-b$ are odd and so neither have a factor of 2. $a^2 - b^2$ is hence odd.

A number of the form $4k+2$ is even, but is not divisible by 4 so fits none of the cases above. Hence a number of the form $4k+2$ cannot be written as the difference of two squares.

(v) I started by trying an example. Let $p = 5$ and $q = 3$, then $pq = 15$ which can be $4^2 - 1^2$ or $8^2 - 7^2$.

If p and q are both primes greater than 2 then they are both odd. This means that $\frac{p+q}{2}$, $\frac{p-q}{2}$, $\frac{pq+1}{2}$ and $\frac{pq-1}{2}$ are all integers.

Consider:

$$\begin{aligned}\left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 &= \frac{p^2 + 2pq + q^2}{4} - \frac{p^2 - 2pq + q^2}{4} \\ &= pq\end{aligned}$$

And also:

$$\begin{aligned}\left(\frac{pq+1}{2}\right)^2 - \left(\frac{pq-1}{2}\right)^2 &= \frac{(pq)^2 + 2pq + 1}{4} - \frac{(pq)^2 - 2pq + 1}{4} \\ &= pq\end{aligned}$$

Hence pq can be written as a difference of two squares in two ways, as these two ways are different. You can show that they are different by considering the first square in each case. Since p and q are both greater than 2, $pq + 1 > p + q$, hence the two ways are different.[1].

I found these two ways by considering $(p + q)^2 - (p - q)^2$ and $(pq + 1)^2 - (pq - 1)^2$ and then scaling to get $1pq$.

However, we need to show that pq can be written as a difference of two squares in **exactly** two ways, i.e. that no other way is possible.

Consider $pq = a^2 - b^2 = (a + b)(a - b)$. The only factors possible are p and q, so (assuming that $p \geqslant q$) we have either:

- $a + b = p$ and $a - b = q$
- $a + b = pq$ and $a - b = 1$

These are the only options, so there are only two ways of writing pq as a difference of two squares.

Note that if $p = q$, then the ways to write $p \times p = p^2$ are $p^2 - 0^2$ and $\left(\frac{p^2+1}{2}\right)^2 - \left(\frac{p^2-1}{2}\right)^2$.

This is not the most elegant solution, a better one would be to start with $pq = a^2 - b^2$ and then find a and b in terms of p and q. However, I thought it might be useful to show an "unpolished" solution.

If $q = 2$, and p is a prime greater than 2 (hence p is odd) then $\frac{p+q}{2}$, $\frac{p-q}{2}$, $\frac{pq+1}{2}$ and $\frac{pq-1}{2}$ are not integers, so it is not possible to write pq as a difference of two squares.

(vi) 675 can be written as $1 \times 3^3 \times 5^2$, so we want to find a and b such that $(a + b)(a - b) = 1 \times 3^3 \times 5^2$. We also have $a + b \geqslant a - b$. The options are:

$a + b$	$a - b$
$3^3 \times 5^2$	1
$3^2 \times 5^2$	3
$3^3 \times 5$	5
3×5^2	3^2
$3^2 \times 5$	3×5
3^3	5^2

Hence there are 6 different ways to write 675 as the difference of two squares.

Note that we don't actually have to find the ways, we just have to find out how many there are!

[1]You could try solving $pq + 1 = p + q$ which can be rearranged to give $pq - p - q + 1 = 0$, which factorises to $(p - 1)(q - 1) = 0$, i.e. we must have either $p = 1$ or $q = 1$ (or both). This is not possible as p and q are prime numbers bigger than 2.

2005 S1 Q4

9 Preparation

(i) Let $\phi = 3\theta$, so that $0 \leqslant \phi < 6\pi$. The solutions to $\tan\phi = 1$ are $\phi = \frac{1}{4}\pi, \frac{5}{4}\pi, \frac{9}{4}\pi, \dots \frac{21}{4}\pi$ which gives $\theta = \frac{1}{12}\pi, \frac{5}{12}\pi, \frac{9}{12}\pi, \frac{13}{12}\pi, \frac{17}{12}\pi, \frac{21}{12}\pi$.

(ii) If $\sin\theta = \frac{1}{2}$ and $\cos\theta < 0$ then one solution is $\theta = \frac{5}{6}\pi$. However we can add multiples of 2π to this so the possible values are $\theta = \frac{5}{6}\pi \pm 2n\pi$.

(iii)

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta \sin\theta\\ &= 2\sin\theta\cos^2\theta + \left(\cos^2\theta - \sin^2\theta\right)\sin\theta\\ &= 3\sin\theta\cos^2\theta - \sin^3\theta\end{aligned}$$

Replacing θ with $\frac{1}{2}\pi - \theta$ gives:

$$\begin{aligned}\sin 3\left(\frac{\pi}{2} - \theta\right) &= 3\sin\left(\frac{\pi}{2} - \theta\right)\cos^2\left(\frac{\pi}{2} - \theta\right) - \sin^3\left(\frac{\pi}{2} - \theta\right)\\ \sin\left(\frac{3\pi}{2}\right)\cos 3\theta - \cos\left(\frac{3\pi}{2}\right)\sin 3\theta &= 3\cos\theta\sin^2\theta - \cos^3\theta\\ -\cos 3\theta &= 3\cos\theta\sin^2\theta - \cos^3\theta\\ \cos 3\theta &= \cos^3\theta - 3\cos\theta\sin^2\theta\end{aligned}$$

(iv) **(a)** Here one root is $x = 1$. Factorising gives $(x-1)(x^2 - x - 12) = 0$ and so $(x-1)(x-4)(x+3) = 0$ and the solutions are $x = -3, x = 1, x = 4$.

(b) Here one root is $x = -1$. Factorising gives $(x+1)(12x^2 - x - 1) = 0$ and so $(x+1)(4x+1)(3x-1) = 0$ and the solutions are $x = -1, x = -\frac{1}{4}, x = \frac{1}{3}$.

Another way to solve this equation is to note that if we substitute $x = -\frac{1}{t}$ into the equation in **(a)** we get this second equation. Hence the roots are given by $t = -\frac{1}{x}$.

(c) One root here is $x = \frac{1}{2}$, and so we have $(2x-1)(x^2 - 2x - 1) = 0$. Using the quadratic formula gives the other two root as $x = 1 \pm \sqrt{2}$.

(v) You can consider the two parts of the inequality separately. Firstly we have $8 < 5\sqrt{3}$ as $64 < 25 \times 3$ hence $8 - 5\sqrt{3} < 0$. Now consider $-1 < 8 - 5\sqrt{3}$. This is equivalent to $5\sqrt{3} < 9$, which is true as $25 \times 3 < 81$ and so we know that $-1 < 8 - 5\sqrt{3}$. Hence $-1 < 8 - 5\sqrt{3} < 0$.

10 The STEP I question

(i) Using $\cos^2\theta + \sin^2\theta = 1$ gives $\sin^2\theta = \frac{16}{25}$. For the given range of θ we know that $\sin\theta$ must be negative and so $\sin\theta = -\frac{4}{5}$.

We then have $\sin 2\theta = 2\cos\theta\sin\theta = 2\times\frac{3}{5}\times -\frac{4}{5} = -\frac{24}{25}$.

$$\begin{aligned}
\cos 3\theta &= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta\\
&= \cos\theta\left(\cos^2\theta - \sin^2\theta\right) - \sin\theta\sin 2\theta\\
&= \tfrac{3}{5}\left(\tfrac{9}{25} - \tfrac{16}{25}\right) - \left(-\tfrac{4}{5}\times -\tfrac{24}{25}\right)\\
&= -\tfrac{3\times7+4\times24}{125}\\
&= -\tfrac{117}{125}
\end{aligned}$$

(ii)

$$\begin{aligned}
\tan 3\theta &= \tan(2\theta+\theta)\\
&= \frac{\tan 2\theta + \tan\theta}{1 - \tan 2\theta\times\tan\theta}\\
&= \frac{\frac{2\tan\theta}{1-\tan^2\theta} + \tan\theta}{1 - \frac{2\tan\theta}{1-\tan^2\theta}\times\tan\theta}\\
&= \frac{2\tan\theta + \tan\theta\left(1-\tan^2\theta\right)}{1-\tan^2\theta - 2\tan^2\theta}\\
&= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}
\end{aligned}$$

A perhaps simpler way of showing this identity is to use $\tan 3\theta = \frac{\sin 3\theta}{\cos 3\theta}$ and then use the identities found question **9 (iii)**.

If $\tan 3\theta = \dfrac{11}{2}$, and if we let $\tan\theta = t$, then we have:

$$\begin{aligned}
\frac{3t-t^3}{1-3t} &= \frac{11}{2}\\
6t - 2t^3 &= 11 - 33t^2\\
2t^3 - 33t^2 - 6t + 11 &= 0\\
(2t-1)(t^2-16t-11) &= 0
\end{aligned}$$

This gives $\tan\theta = \frac{1}{2}$ or $\tan\theta = 8\pm5\sqrt{3}$. If $\dfrac{\pi}{4}\leqslant\theta\leqslant\dfrac{\pi}{2}$ then we have $\tan\theta\geqslant 1$, and hence the solution is $\tan\theta = 8+5\sqrt{3}$.

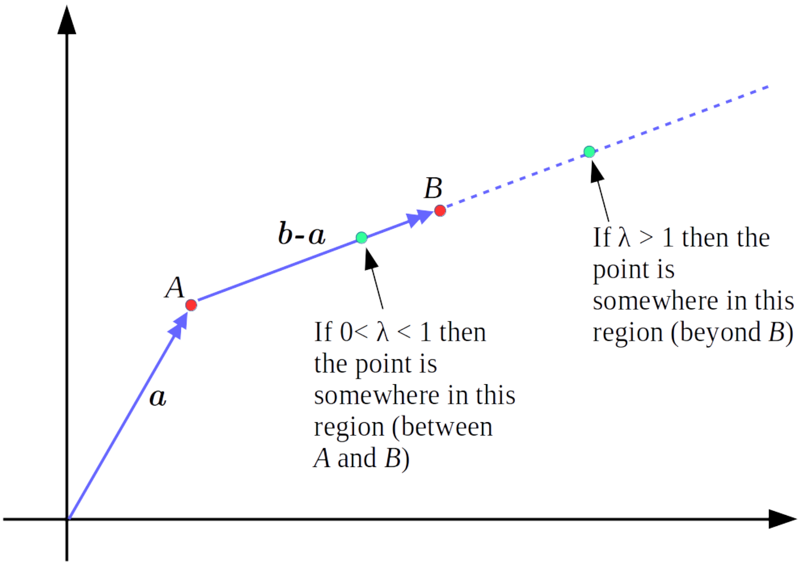
2013 S1 Q3

11 Preparation

(i) **(a)** To find $\overrightarrow{AB}$, consider how to get from point A to point B. You could travel from A to the origin and then to point B which gives us $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -\mathbf{a} + \mathbf{b}$.

(b) We have $\overrightarrow{OC} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$ and so $\mathbf{c} = \mathbf{a} + \frac{2}{3}(\mathbf{b} - \mathbf{a}) = \frac{1}{3}\mathbf{a} + \frac{2}{3}\mathbf{b}$.

(c)



(ii) **(a)** $2 \oplus 3 = 6 - 2 = 4$ and $3 \oplus 2 = 6 - 3 = 3$.

(b) $a \oplus b = b \oplus a \iff ab - a = ba - b \iff a = b$.

(c) $3 \oplus (5 \oplus 1) = -3$ and $(3 \oplus 5) \oplus 1 = 0$.

(d) We need:

$$\begin{aligned}
a \oplus (b \oplus c) &\neq (a \oplus b) \oplus c \\
a \oplus (bc - b) &\neq (ab - a) \oplus c \\
a(bc - b) - a &\neq (ab - a)c - (ab - a) \\
\cancel{abc} - \cancel{ab} - a &\neq \cancel{abc} - ac - \cancel{ab} + a \\
ac - 2a &\neq 0 \\
a(c - 2) &\neq 0
\end{aligned}$$

So the conditions are $a \neq 0$ and $c \neq 2$.

12 The STEP I question

(i) The points are distinct iff:

$$\begin{aligned} X * Y &\neq Y * X \\ \lambda\mathbf{x} + (1-\lambda)\mathbf{y} &\neq \lambda\mathbf{y} + (1-\lambda)\mathbf{x} \\ 2\lambda\mathbf{x} - 2\lambda\mathbf{y} - \mathbf{x} + \mathbf{y} &\neq 0 \\ (2\lambda - 1)(\mathbf{x} - \mathbf{y}) &\neq 0 \end{aligned}$$

We are told that X and Y are distinct, so we have $\mathbf{x} - \mathbf{y} \neq 0$. Therefore $X * Y$ and $Y * X$ are distinct unless $\lambda = \frac{1}{2}$.

(ii)

$$\begin{aligned} (X * Y) * Z &\neq X * (Y * Z) \\ \Big[\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\Big] * Z &\neq X * \Big[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}\Big] \\ \lambda\Big[\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\Big] + (1-\lambda)\mathbf{z} &\neq \lambda\mathbf{x} + (1-\lambda)\Big[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}\Big] \\ \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z} &\neq \lambda(1-\lambda)\mathbf{x} + (1-\lambda)\Big[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}\Big] \\ (1-\lambda)\Big[\lambda\mathbf{y} + \mathbf{z}\Big] &\neq (1-\lambda)\Big[\lambda\mathbf{x} + \lambda\mathbf{y} + \mathbf{z} - \lambda\mathbf{z}\Big] \\ 0 &\neq (1-\lambda)\Big[\lambda\mathbf{x} - \lambda\mathbf{z}\Big] \\ 0 &\neq \lambda(1-\lambda)(\mathbf{x} - \mathbf{z}) \end{aligned}$$

Hence for $(X * Y) * Z$ and $X * (Y * Z)$ to be distinct we need $\lambda \neq 0$, $\lambda \neq 1$ and $\mathbf{x} \neq \mathbf{z}$.

(iii) We have $(X * Y) * Z = \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z}$ from part **(ii)**.

We also have:

$$\begin{aligned} (X * Z) * (Y * Z) &= \Big[\lambda\mathbf{x} + (1-\lambda)\mathbf{z}\Big] * \Big[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}\Big] \\ &= \lambda\Big[\lambda\mathbf{x} + (1-\lambda)\mathbf{z}\Big] + (1-\lambda)\Big[\lambda\mathbf{y} + (1-\lambda)\mathbf{z}\Big] \\ &= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\big(\lambda + (1-\lambda)\big)\mathbf{z} \\ &= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{z} \end{aligned}$$

which is the same as $(X * Y) * Z$.

In a similar way, you can show that $X * (Y * Z) = (X * Y) * (X * Z)$ (you need to expand each side and show that they are the same).

(iv) We have:

$$\begin{aligned}
P_1 &= \lambda\mathbf{x} + (1-\lambda)\mathbf{y}\\
P_2 &= \big(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\big) * Y\\
&= \lambda^2\mathbf{x} + \lambda(1-\lambda)\mathbf{y} + (1-\lambda)\mathbf{y}\\
&= \lambda^2\mathbf{x} + (\lambda - \lambda^2 + 1 - \lambda)\mathbf{y}\\
&= \lambda^2\mathbf{x} + (1-\lambda^2)\mathbf{y}\\
P_3 &= \big(\lambda^2\mathbf{x} + (1-\lambda^2)\mathbf{y}\big) * Y\\
&= \lambda^3\mathbf{x} + \lambda(1-\lambda^2)\mathbf{y} + (1-\lambda)\mathbf{y}\\
&= \lambda^3\mathbf{x} + (\lambda - \lambda^3 + 1 - \lambda)\mathbf{y}\\
&= \lambda^3\mathbf{x} + (1-\lambda^3)\mathbf{y}
\end{aligned}$$

It looks as if $P_n = \lambda^n\mathbf{x} + (1-\lambda^n)\,\mathbf{y}$. We know this is true when $n = 1, 2, 3$.

Assume that the conjecture is true when $n = k$, so we have $P_k = \lambda^k\mathbf{x} + \big(1-\lambda^k\big)\,\mathbf{y}$. Now consider $n = k+1$. We have:

$$\begin{aligned}
P_{k+1} &= \Big(\lambda^k\mathbf{x} + (1-\lambda^k)\mathbf{y}\Big) * Y\\
&= \lambda\Big(\lambda^k\mathbf{x} + (1-\lambda^k)\mathbf{y}\Big) + (1-\lambda)\mathbf{y}\\
&= \lambda^{k+1}\mathbf{x} + \big(\lambda - \lambda^{k+1} + 1 - \lambda\big)\mathbf{y}\\
&= \lambda^{k+1}\mathbf{x} + \big(1 - \lambda^{k+1}\big)\mathbf{y}
\end{aligned}$$

which has the required form with $n = k+1$. Hence if the conjecture is true for $n = k$ then it is true for $n = k+1$, and since it is true for $n = 1, 2, 3$ it is true for all integers $n \geqslant 1$.

Hence $P_n = \lambda^n\mathbf{x} + \big(1-\lambda^n\big)\mathbf{y}$ and the point P_n divides the line segment XY in the ratio $\lambda^n : 1 - \lambda^n$.

2008 S1 Q4

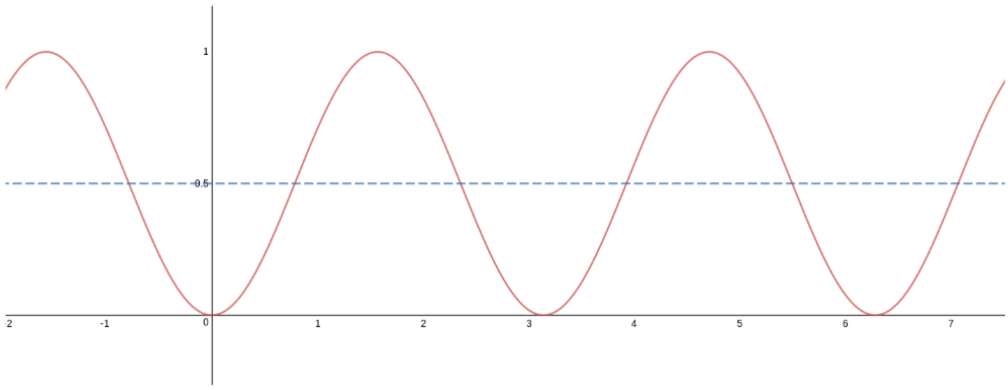
13 Preparation

(i)

$$\begin{aligned} 2\big(1-\cos^2 x\big)+3\cos x &= 0 \\ 2\cos^2 x - 3\cos x - 2 &= 0 \\ (2\cos x+1)(\cos x-2) &= 0 \\ \cos x &= -\tfrac{1}{2} \end{aligned}$$

So we have $x = \dfrac{2\pi}{3}$ and $x = \dfrac{4\pi}{3}$.

(ii) $\sin^2 x = \frac{1}{2}(1-\cos 2x)$, which we can sketch as a series of translations of $y = \sin x$ (thinking about it like this avoids overly "spiky" bits). Your picture should look like the following:



(iii) $f'(x) = -2\sin 2x\, e^{\cos 2x}$ and $f''(x) = -4\cos 2x\, e^{\cos 2x} + 4\sin^2 2x\, e^{\cos 2x}$.

(iv) $f'(x) = -\sin(\tan x)\times\sec^2 x$ and
$f''(x) = -\sin(\tan x)\times 2\sec^3 x\sin x - \cos(\tan x)\times\sec^4 x$
$= -\sec^2 x\Big(\sin(\tan x)\times 2\tan x + \sec^2 x\cos(\tan x)\Big)$.

(v)

$$\begin{aligned} \cot\alpha &= \frac{\cos\alpha}{\sin\alpha} \\ &= \frac{\sin\left(\frac{\pi}{2}-\alpha\right)}{\cos\left(\frac{\pi}{2}-\alpha\right)} \\ &= \tan\left(\tfrac{\pi}{2}-\alpha\right) \end{aligned}$$

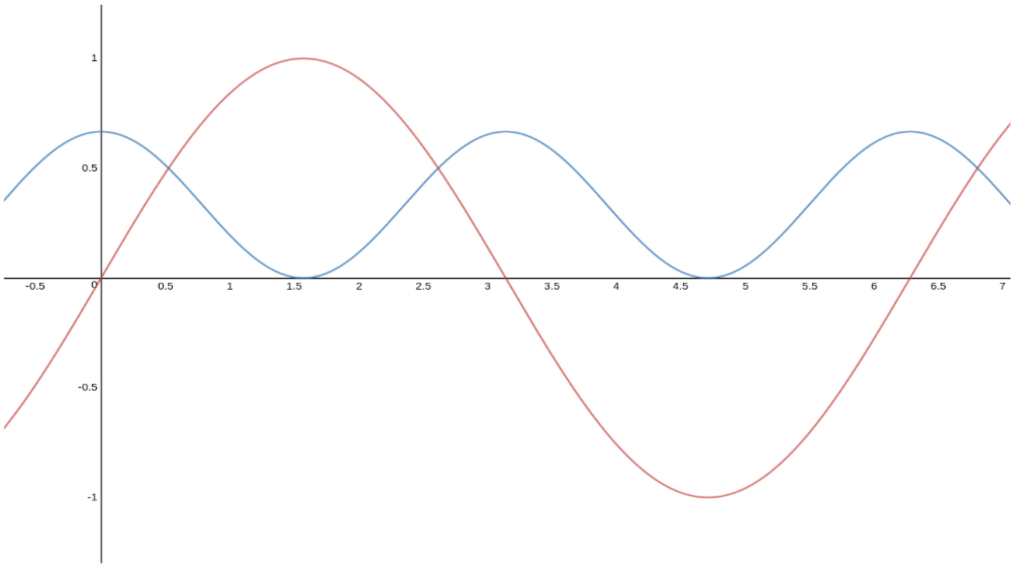
(vi)

$$\begin{aligned} \sec^2 x &= 2\tan x \\ \tan^2 x + 1 &= 2\tan x \\ \tan x - 2\tan x + 1 &= 0 \\ (\tan x - 1)^2 &= 0 \end{aligned}$$

So $\tan x = 1$ and $x = \frac{\pi}{4}$ or $x = \frac{5\pi}{4}$.

14 The STEP I question

(i) First note that $\frac{2}{3}\cos^2 x = \frac{1}{3}(\cos 2x + 1)$ which will help you get the correct "shape" of this curve. You should have a sketch looking something like the following:



For the next part $\mathrm{f}'(x) = \frac{2}{3}\cos x\, \mathrm{e}^{\frac{2}{3}\sin x}$ and $\mathrm{f}''(x) = \frac{4}{9}\cos^2 x\, \mathrm{e}^{\frac{2}{3}\sin x} - \frac{2}{3}\sin x\, \mathrm{e}^{\frac{2}{3}\sin x}$. Hence $\mathrm{f}(x)$ is convex when $\mathrm{f}''(x) \geqslant 0$, i.e.:

$$\begin{aligned} \tfrac{4}{9}\cos^2 x\, \mathrm{e}^{\frac{2}{3}\sin x} - \tfrac{2}{3}\sin x\, \mathrm{e}^{\frac{2}{3}\sin x} &\geqslant 0 \\ \tfrac{2}{3}\mathrm{e}^{\frac{2}{3}\sin x}\left(\tfrac{2}{3}\cos^2 x - \sin x\right) &\geqslant 0 \\ \tfrac{2}{3}\cos^2 x - \sin x &\geqslant 0 \qquad \text{since } \tfrac{2}{3}\mathrm{e}^{\frac{2}{3}\sin x} > 0 \end{aligned}$$

The inequality $\frac{2}{3}\cos^2 x - \sin x \geqslant 0$ corresponds to the regions of the graph where the blue curve ($y = \frac{2}{3}\cos^2 x$) is higher than the red curve ($y = \sin x$). We need to find the intersections of these two, i.e. solve $\frac{2}{3}\cos^2 x = \sin x$.

$$\begin{aligned} \tfrac{2}{3}\cos^2 x &= \sin x \\ \tfrac{2}{3}\left(1 - \sin^2 x\right) &= \sin x \\ 2 - 2\sin^2 x &= 3\sin x \\ 2\sin^2 x + 3\sin x - 2 &= 0 \\ (2\sin x - 1)(\sin x + 2) &= 0 \end{aligned}$$

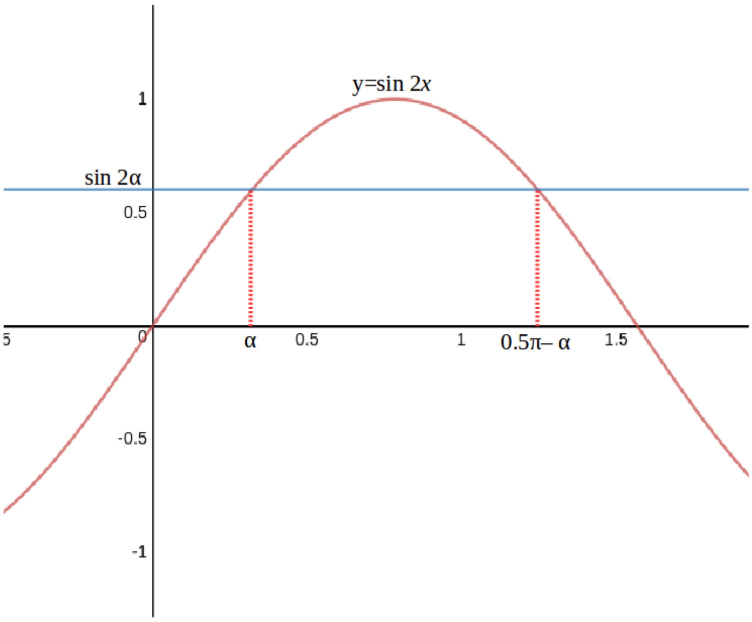
So the intersections are at $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$ and so the curve $\mathrm{f}(x)$ is convex when $0 < x < \frac{\pi}{6}$ and $\frac{5\pi}{6} < x < 2\pi$.

(ii) $\mathrm{g}'(x) = -k\sec^2 x\, \mathrm{e}^{-k\tan x}$ and $\mathrm{g}''(x) = k^2\sec^4 x\, \mathrm{e}^{-k\tan x} - 2k\sin x \sec^3 x\, \mathrm{e}^{-k\tan x}$. The second derivative can be written as $\mathrm{g}''(x) = k\sec^2 x\, \mathrm{e}^{-k\tan x}\left[k\sec^2 x - 2\sin x\sec x\right]$.

Since $k > 0$ (as $k = \sin 2\alpha$ and $0 < \alpha < \frac{1}{4}\pi$), $\sec^2 x > 0$ and $e^{-k\tan x} > 0$, $g(x)$ will be convex if and only if:

$$\begin{aligned} k\sec^2 x - 2\sin x \sec x &> 0 \\ \frac{k - 2\sin x\cos x}{\cos^2 x} &> 0 \\ k - \sin 2x &> 0 \\ \sin 2\alpha - \sin 2x &> 0 \end{aligned}$$

Since $0 < \alpha < \frac{1}{4}\pi$, $\sin 2\alpha > 0$. The graph of $y = \sin 2x$ for $0 < x < \frac{1}{2}\pi$ looks like:



From this graph you can see that $\sin 2\alpha - \sin x > 0$ when $0 < x < \alpha$ and also when $\frac{1}{2}\pi - \alpha < x < \frac{1}{2}\pi$.

2013 S1 Q7

15 Preparation

(i) There are lots of ways you can integrate this. You could use a substitution (such as $y = x^2$, $y = 1 - x^2$ or $y = \sin\theta$), you could use partial fractions, or you could use the result $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$ (which is essentially a substitution, but it is well worth knowing this standard result). The answer is $-\frac{1}{2}\ln|1 - x^2| + c$ (your answer might look a little different depending on which method you use but should be equivalent to this).

(ii) Separating the variables gives:

$$\begin{aligned}
\int y^{-2}\,\mathrm{d}y &= \int x^{-1}\,\mathrm{d}x \\
-y^{-1} &= \ln x + c \\
-\frac{1}{2} &= c \qquad \text{substituting} \quad x = 1, y = 2 \\
-\frac{1}{y} &= \ln x - \frac{1}{2} \\
-2 &= 2y\ln x - y \\
2 &= y\,(1 - 2\ln x) \\
y &= \frac{2}{1 - 2\ln x}
\end{aligned}$$

To sketch the graph, note that $x > 0$ and we cannot have $2\ln x = 1$ i.e. $x^2 \neq \mathrm{e}$. As $x \to \mathrm{e}^{\frac{1}{2}}$ from below we have $y \to \infty$ and as $x \to \mathrm{e}^{\frac{1}{2}}$ from above we have $y \to -\infty$. As $x \to 0$ we have $y \to 0$ and as $x \to \infty$ we have $y \to 0$. When $0 < x < \mathrm{e}^{\frac{1}{2}}$ y is positive and when $x > \mathrm{e}^{\frac{1}{2}}$ y is negative. You can also differentiate y and show that it is always positive.

Check your graph by visiting Desmos.

(iii) Rearranging gives $y^2 = 3x(x^2 - 1)$. For y to be real we need $3x(x^2 - 1) \geqslant 0$. The easiest way to solve this is to sketch a graph of $3x(x + 1)(x - 1)$ which will show you that y is real when $-1 \leqslant x \leqslant 0$ or $x \geqslant 1$.

16 The STEP I question

(i) Using the given substitution:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{y} + \frac{y}{x} \\
x\frac{\mathrm{d}u}{\mathrm{d}x} + u &= \frac{x}{ux} + \frac{ux}{x} \\
x\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{1}{u} \\
\int u \, \mathrm{d}u &= \int \frac{1}{x} \, \mathrm{d}x \\
\tfrac{1}{2}u^2 &= \ln x + c \\
\tfrac{1}{2} \times 2^2 &= c \qquad \text{using} \quad x = 1, y = 2 \implies u = 2 \\
\tfrac{1}{2}u^2 &= \ln x + 2 \\
u^2 &= 2\ln x + 4 \\
\frac{y^2}{x^2} &= 2\ln x + 4 \\
y^2 &= 2x^2 \ln x + 4x^2 \\
y &= x\sqrt{4 + 2\ln x}
\end{aligned}$$

(ii) Here it would be good if we could eliminate a term again. Try the substitution $y = ux^2$ ($y = ux$ will work, but does not led to as nice a differential equation).

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{x}{y} + \frac{2y}{x} \\
x^2\frac{\mathrm{d}u}{\mathrm{d}x} + 2xu &= \frac{x}{ux^2} + \frac{2ux^2}{x} \\
x^2\frac{\mathrm{d}u}{\mathrm{d}x} &= \frac{1}{ux} \\
\int u \, \mathrm{d}u &= \int x^{-3} \, \mathrm{d}x \\
\tfrac{1}{2}u^2 &= -\tfrac{1}{2}x^{-2} + c \\
\tfrac{1}{2} \times 2^2 &= -\tfrac{1}{2} + c \qquad \text{using} \quad x = 1, y = 2 \implies u = 2 \\
\tfrac{1}{2}u^2 &= -\tfrac{1}{2}x^{-2} + \tfrac{5}{2} \\
u^2 &= -\frac{1}{x^2} + 5 \\
\frac{y^2}{x^4} &= 5 - \frac{1}{x^2} \\
y^2 &= 5x^4 - x^2 \\
y &= x\sqrt{5x^2 - 1}
\end{aligned}$$

(iii) Here use $y = ux^2$ again.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x}$$

$$x^2\frac{\mathrm{d}u}{\mathrm{d}x} + 2xu = \frac{x^2}{ux^2} + \frac{2ux^2}{x}$$

$$x^2\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x^2}{ux^2}$$

$$\int u\,\mathrm{d}u = \int x^{-2}\,\mathrm{d}x$$

$$\tfrac{1}{2}u^2 = -x^{-1} + c$$

$$\tfrac{1}{2} \times 2^2 = -1 + c \qquad \text{using} \quad x = 1, y = 2 \implies u = 2$$

$$\tfrac{1}{2}u^2 = -x^{-1} + 3$$

$$u^2 = -\frac{2}{x} + 6$$

$$\frac{y^2}{x^4} = 6 - \frac{2}{x}$$

$$y^2 = 6x^4 - 2x^3$$

$$y = x\sqrt{6x^2 - 2x}$$