STEP Support Programme

Assignment 4

Warm-up

1 **(i)** *For this question, you need a *good* diagram — nice and big. You may find it helpful to use letters for the sizes of some of the angles; for example, you might want to write 'Let $\angle APO = x$'.*

Let A, B and P be points on the circumference of a circle with centre O, such that O lies inside the triangle ABP. What sort of triangle is APO?

By considering the triangles APO and BPO, prove that

$$\angle AOB = 2\angle APB$$

(where $\angle APB$ is acute and $\angle AOB \leqslant 180°$).

(ii) Prove the same result in the case where O does not lie inside the triangle ABP and $\angle APB$ is acute (and $\angle AOB \leqslant 180°$).

Preparation

2 **(i)** Factorise $x^2 - 3x - 4$. Sketch the graph $y = x^2 - 3x - 4$ and hence find the range(s) of values of x for which $x^2 - 3x - 4 > 0$.

(ii) Show that $x = 3$ is a root (solution) of the equation $x^3 - 2x^2 - 5x + 6 = 0$.

Find two more integer roots and write $x^3 - 2x^2 - 5x + 6$ as a product of three linear factors.

Sketch the graph $y = x^3 - 2x^2 - 5x + 6$. Use your sketch to find the ranges of values of x for which $x^3 - 2x^2 - 5x + 6 \leqslant 0$

(iii) Factorise $x^2 - 3x + 2$. Use your answer to help you factorise $x^2 - 3xy + 2y^2$.

Show that if $x^2 - 3xy + 2y^2 = 0$, then the point with coordinates (x, y) lies on one (or both) of two straight lines (you should give the equations of the lines). Sketch the lines on the same set of axes.

(iv) On a sketch, shade the regions of the (x,y) plane in which $x^2 - 3xy + 2y^2 \leqslant 0$.

One technique you can use to determine which regions you want is to pick a point in each region (**not** on the boundary lines) and check to see if $x^2 - 3xy + 2y^2 \leqslant 0$ is true for this point.

The STEP question

3 **(i)** Find the real values of x for which

$$x^3 - 4x^2 - x + 4 \geqslant 0.$$

(ii) Find the three lines in the (x, y) plane on which

$$x^3 - 4x^2y - xy^2 + 4y^3 = 0.$$

(iii) On a sketch shade the regions of the (x, y) plane for which

$$x^3 - 4x^2y - xy^2 + 4y^3 \geqslant 0.$$

Discussion

This STEP question has a rather typical feature: you need the first part to help with the second part; and more obviously you need the second part to help with the third part. It is unusual for a STEP question to have parts that are unrelated, though sometimes the relationship is only that they use the same kind of ideas.

Although it didn't arise (probably) in this question, the thing to be very careful of when dealing with inequalities is multiplying or dividing: if the thing you are multiplying or dividing by is not positive, or may not be positive, the inequality may be reversed.

Warm down

Here are a couple of problems, the first of which requires careful logic and the second of which (a bit of an old chestnut) requires some lateral thinking.

4 **(i)** I have 4 double-sided cards in front of me, each has a number on one side and a letter on the other. The faces that are up are showing "6", "E", "Q" and "7". I claim that if there is an even number on one side of the card then there is a vowel on the other. Which cards do you need to turn over in order to check my claim? (Justify *very briefly* your choices.)

(ii) I have two identical but non-uniform ropes. I know that if I light one end of either rope it will burn for exactly one hour. How can I time 45 minutes? (I have a lighter of some sort).