

## STEP Support Programme

### Assignment 9

#### Warm-up

- 1 (i) The isosceles triangle  $\triangle ABC$  has  $AB = BC$ . A line is drawn connecting  $B$  to the midpoint  $M$  of  $AC$ . By considering congruent triangles show that this line is perpendicular to  $AC$ , that it bisects the angle at  $B$ , and that  $\angle BAC = \angle BCA$ .

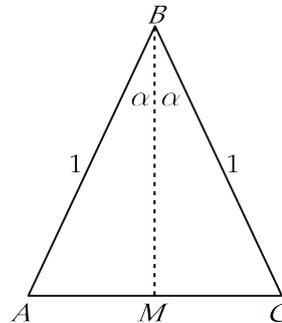
- (ii) The triangle  $ABC$  has side lengths  $BC = a$ ,  $CA = b$  and  $AB = c$ . Given that the area of a triangle can be expressed as  $\frac{1}{2}$  base  $\times$  height, show (by taking the base to be  $AC$ ) that

$$\text{Area} = \frac{1}{2}ba \sin C.$$

Show also that  $\text{Area} = \frac{1}{2}bc \sin A$  and deduce the *sine rule*

$$\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin B}{b}.$$

- (iii)  $\triangle ABC$  is shown below, with  $AM = MC$ ,  $AB = BC = 1$  and  $\angle ABM = \angle CBM = \alpha$ .



Show that  $AC = 2 \sin \alpha$ , and (by using the sine rule) that  $\sin(2\alpha) = 2 \sin \alpha \cos \alpha$ .



## Preparation

- 2 (i) Solve the equation  $5x^2 + 3x = 0$
- (ii) Consider the graph of  $y = x^3 - 12x + 1$ .
- Find the  $x$  coordinates of the turning points.
  - By considering the shape of the graph, state which of the turning points is the maximum and which is the minimum.
  - Find where the graph intersects the  $y$ -axis.
  - Find the  $y$  coordinates of the turning points and sketch the graph.
  - How many real roots of the equation  $x^3 - 12x + 1 = 0$  are there? **We are not asking you to find the values of the roots!**
- (iii) Sketch the graph of each of the following cubics, labelling the coordinates of the turning points and the  $y$  intercepts. Do not attempt to find the intercepts with the  $x$ -axis; do not use a graphical calculator; do not plot points (other than turning points and the  $y$ -intercept).
- $y = x^3 + 3x^2 + 1$
  - $y = 2x^3 + 6x^2 - 3$
  - $y = 4x^3 + 6x^2 - 3$
  - $y = x^3 - 6x^2 + 2$
  - $y = 2x^3 - 3x^2 + 2$
  - $y = x^3 - 12x^2 - 6$

Note that the scale of the graph is not important, just make sure that the positions of the turning points and  $y$  intercept are in the right sort of place — i.e. the point (2,5) should be **above** the  $x$  axis.

- (iv) Which of the above graphs have three  $x$  intercepts?

Consider the locations of the turning points in relation to the  $x$ -axis (that is above or below the axis). What is the difference between the graphs with three  $x$  intercepts and the graphs with one  $x$  intercept?



## The STEP question

3 Sketch the curve

$$f(x) = x^3 + Ax^2 + B$$

first in the case  $A > 0$  and  $B > 0$ , and then in the case  $A < 0$  and  $B > 0$ .

Show that the equation

$$x^3 + ax^2 + b = 0,$$

where  $a$  and  $b$  are real, will have three distinct real roots if

$$27b^2 + 4a^3b < 0,$$

but will have fewer than three if

$$27b^2 + 4a^3b > 0.$$

## Discussion

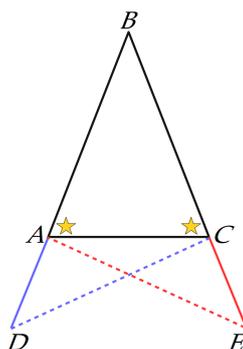
The STEP question differs from the Preparation in that the coefficients are *parameters* rather than known constants. You can still proceed in exactly the same way.

For the second part, you may find it helpful to factorise  $27b^2 + 4a^3b$ . No serious calculations are required: the results follow from appropriate sketches.



## Warm down

- 4 The isosceles triangle  $\triangle ABC$  has  $AB = BC$ . The lines  $BA$  and  $BC$  are extended by equal lengths to  $D$  and  $E$  respectively. The task is to show that the angles  $BAC$  and  $BCA$  (the “base” angles of the isosceles triangle) are equal. You can only use congruence established with SAS and the fact that angles in a triangle are supplementary — they sum to  $180^\circ$ .
- (i) Show that angles  $BCD$  and  $BAE$  are equal (by showing that two triangles are congruent).
- (ii) Show that angles  $DCA$  and  $EAC$  are equal.
- (iii) Deduce that angles  $BAC$  and  $BCA$  are equal.



Euclid was a Greek mathematician who lived roughly from the middle of the fourth century BC to the middle of the third century BC (about 100 years after Plato). In his treatise *The Elements*, he builds up what is now called Euclidean geometry in a logical order, by means of a series of propositions each one relying only on previous propositions.

Euclid’s *Elements* were used as a basis for teaching geometry for 23 centuries. The logical structure of Euclid’s *Elements* is the model for the teaching of other subjects, notably university level mathematical analysis which lays the foundations of calculus.

Euclid’s Proposition 5 says that the base angles of an isosceles triangle are equal. Euclid’s proof was a bit harder than the one suggested in question 1(i), because he had not by that point (i.e. in propositions 1 to 4) shown that two triangles with corresponding sides of the same length are congruent (what we call SSS). However, he had shown (though perhaps not satisfactorily) that the SAS condition is enough for two triangles to be congruent and this is what he used in the proof below.

Because Proposition 5 is the first proposition for which the proof is not straightforward, it is called the Pons Asinorum (bridge of donkeys) — presumably meaning a bridge to the later propositions that separates mathematical donkeys from more fleet-footed mathematicians.

