STEP Support Programme

STEP 2 Miscellaneous Questions

1 **2011 S2 Q2**

Write down the cubes of the integers 1, 2, ... , 10.

The positive integers x, y and z, where $x < y$, satisfy

$$x^3 + y^3 = kz^3 , \qquad (*)$$

where k is a given positive integer.

(i) In the case $x + y = k$, show that

$$z^3 = k^2 - 3kx + 3x^2 .$$

Deduce that $(4z^3 - k^2)/3$ is a perfect square and that $\frac{1}{4}k^2 \leqslant z^3 < k^2$.

Use these results to find a solution of $(*)$ when $k = 20$.

(ii) By considering the case $x + y = z^2$, find two solutions of $(*)$ when $k = 19$.

2 **2010 S2 Q6**

Each edge of the tetrahedron $ABCD$ has unit length. The face ABC is horizontal, and P is the point in ABC that is vertically below D.

(i) Find the length of PD.

(ii) Show that the cosine of the angle between adjacent faces of the tetrahedron is $1/3$.

(iii) Find the radius of the largest sphere that can fit inside the tetrahedron.

3 **2009 S2 Q1**

Two curves have equations $x^4 + y^4 = u$ and $xy = v$, where u and v are positive constants. State the equations of the lines of symmetry of each curve.

The curves intersect at the distinct points A, B, C and D (taken anticlockwise from A). The coordinates of A are (α, β), where $\alpha > \beta > 0$. Write down, in terms of α and β, the coordinates of B, C and D.

Show that the quadrilateral $ABCD$ is a rectangle and find its area in terms of u and v only. Verify that, for the case $u = 81$ and $v = 4$, the area is 14.

4 **2009 S2 Q4**

The polynomial $\mathrm{p}(x)$ is of degree 9 and $\mathrm{p}(x) - 1$ is exactly divisible by $(x-1)^5$.

(i) Find the value of $\mathrm{p}(1)$.

(ii) Show that $\mathrm{p}'(x)$ is exactly divisible by $(x-1)^4$.

(iii) Given also that $\mathrm{p}(x) + 1$ is exactly divisible by $(x+1)^5$, find $\mathrm{p}(x)$.

5 **2009 S2 Q6**

The Fibonacci sequence F_1, F_2, F_3, ... is defined by $F_1 = 1$, $F_2 = 1$ and

$$F_{n+1} = F_n + F_{n-1} \qquad (n \geqslant 2).$$

Write down the values of F_3, F_4, ..., F_{10}.

Let $S = \displaystyle\sum_{i=1}^{\infty} \frac{1}{F_i}$.

(i) Show that $\dfrac{1}{F_i} > \dfrac{1}{2F_{i-1}}$ for $i \geqslant 4$ and deduce that $S > 3$.

Show also that $S < 3\frac{2}{3}$.

(ii) Show further that $3.2 < S < 3.5$.

6 2008 S2 Q1

A sequence of points (x_1, y_1), (x_2, y_2), ... in the cartesian plane is generated by first choosing (x_1, y_1) then applying the rule, for $n = 1, 2, \ldots$,

$$(x_{n+1}, y_{n+1}) = (x_n^2 - y_n^2 + a,\ 2x_n y_n + b + 2)\,,$$

where a and b are given real constants.

(i) In the case $a = 1$ and $b = -1$, find the values of (x_1, y_1) for which the sequence is constant.

(ii) Given that $(x_1, y_1) = (-1, 1)$, find the values of a and b for which the sequence has period 2.

7 2007 S2 Q1

In this question, you are not required to justify the accuracy of the approximations.

(i) Write down the binomial expansion of $\left(1 + \dfrac{k}{100}\right)^{\frac{1}{2}}$ in ascending powers of k, up to and including the k^3 term.

(a) Use the value $k = 8$ to find an approximation to five decimal places for $\sqrt{3}$.

(b) By choosing a suitable integer value of k, find an approximation to five decimal places for $\sqrt{6}$.

(ii) By considering the first two terms of the binomial expansion of $\left(1 + \dfrac{k}{1000}\right)^{\frac{1}{3}}$, show that $\dfrac{3029}{2100}$ is an approximation to $\sqrt[3]{3}$.

8 2005 S2 Q5

The angle A of triangle ABC is a right angle and the sides BC, CA and AB are of lengths a, b and c, respectively. Each side of the triangle is tangent to the circle S_1 which is of radius r. Show that $2r = b + c - a$.

Each vertex of the triangle lies on the circle S_2. The ratio of the area of the region between S_1 and the triangle to the area of S_2 is denoted by R. Show that

$$\pi R = -(\pi - 1)q^2 + 2\pi q - (\pi + 1)\,,$$

where $q = \dfrac{b + c}{a}$. Deduce that

$$R \leqslant \frac{1}{\pi(\pi - 1)}\,.$$

9 **2011 S2 Q7** The two sequences a_0, a_1, a_2, ... and b_0, b_1, b_2, ... have general terms

$$a_n = \lambda^n + \mu^n \quad \text{and} \quad b_n = \lambda^n - \mu^n\,,$$

respectively, where $\lambda = 1 + \sqrt{2}$ and $\mu = 1 - \sqrt{2}$.

(i) Show that $\displaystyle\sum_{r=0}^{n} b_r = -\sqrt{2} + \frac{1}{\sqrt{2}}\, a_{n+1}$, and give a corresponding result for $\displaystyle\sum_{r=0}^{n} a_r$.

(ii) Show that, if n is odd,

$$\sum_{m=0}^{2n}\left(\sum_{r=0}^{m} a_r\right) = \tfrac{1}{2} b_{n+1}^2\,,$$

and give a corresponding result when n is even.

(iii) Show that, if n is even,

$$\left(\sum_{r=0}^{n} a_r\right)^2 - \sum_{r=0}^{n} a_{2r+1} = 2\,,$$

and give a corresponding result when n is odd.