STEP Support Programme

2023 STEP 3 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the OCR website. These are the general comments for the STEP 2023 exam from the Examiner's report:

The total entry was a marginal increase on that of 2022 (by just over 1%). Two questions were attempted by more than 90% of candidates, another two by 80%, and another two by about two thirds. The least popular questions were attempted by more than a sixth of candidates. All the questions were perfectly answered by at least three candidates (but mostly more than this), with one being perfectly answered by eighty candidates. Very nearly 90% of candidates attempted no more than 7 questions.

One general comment regarding all the questions is that candidates need to make sure that they read the question carefully, paying particular attention to command words such as "hence" and "show that".

Please send any corrections, comments or suggestions to step@maths.org.

Question 1

1 The distinct points $P(2ap,\, ap^2)$ and $Q(2aq,\, aq^2)$ lie on the curve $x^2 = 4ay$, where $a > 0$.

(i) Given that

$$(p+q)^2 = p^2q^2 + 6pq + 5\,, \qquad (*)$$

show that the line through P and Q is a tangent to the circle with centre $(0, 3a)$ and radius $2a$.

(ii) Show that, for any given value of p with $p^2 \neq 1$, there are two distinct real values of q that satisfy equation $(*)$.

Let these values be q_1 and q_2. Find expressions, in terms of p, for $q_1 + q_2$ and q_1q_2.

(iii) Show that, for any given value of p with $p^2 \neq 1$, there is a triangle with one vertex at P such that all three vertices lie on the curve $x^2 = 4ay$ and all three sides are tangents to the circle with centre $(0,\, 3a)$ and radius $2a$.

Examiner's report

Although this was a very popular question, being attempted by nearly 93% of the candidature, it was very narrowly beaten into second place by question 5. It was the third most successfully attempted with a mean score of approximately 9.5 out of 20.

Most candidates found the equation of the line, and of the circle and then solved simultaneously in part **(i)** to find common points, rather than using the perpendicular distance from a line formula; some using the distance formula misquoted it, with a common error being failure to include the modulus signs. Then they generally applied use of the discriminant, but with varying success.

In part **(ii)**, most candidates successfully expressed the given expression as a quadratic in q, obtained the determinant and the two required expressions using Vieta's formulas, but failed to fully demonstrate the inequality.

Attempts at part **(iii)** were frequently inelegant and involved repeating work from previous parts of the question, rather than using the results of part **(i)** and **(ii)**.

Solution

(i) The line through P and Q has gradient $\dfrac{aq^2 - ap^2}{2aq - 2ap} = \dfrac{p+q}{2}$ (since P and Q are distinct we know $p \neq q$). The equation of the line is therefore:

$$\begin{aligned} y - ap^2 &= \frac{p+q}{2}(x - 2ap) \\ y - ap^2 &= \left(\frac{p+q}{2}\right)x - ap(p+q) \\ y - ap^2 &= \left(\frac{p+q}{2}\right)x - ap^2 - apq \\ y &= \left(\frac{p+q}{2}\right)x - apq \end{aligned}$$

The equation of the circle is $x^2 + (y - 3a)^2 = 4a^2 \implies x^2 + y^2 - 6ay + 5a^2 = 0$.

If the line is a tangent to the circle then there is a repeated root when they intersect. Substituting the expression for y into the equation for the circle gives:

$$\begin{aligned} x^2 + \left[\left(\frac{p+q}{2}\right)x - apq\right]^2 - 6a\left[\left(\frac{p+q}{2}\right)x - apq\right] + 5a^2 &= 0 \\ \left[1 + \left(\frac{p+q}{2}\right)^2\right]x^2 - \big[apq(p+q) + 3a(p+q)\big]x + a^2p^2q^2 + 6a^2pq + 5a^2 &= 0 \\ \left[1 + \left(\frac{p+q}{2}\right)^2\right]x^2 - a(p+q)(pq+3)x + a^2\big[p^2q^2 + 6pq + 5\big] &= 0 \\ \left[1 + \left(\frac{p+q}{2}\right)^2\right]x^2 - a(p+q)(pq+3)x + a^2(p+q)^2 &= 0 \end{aligned}$$

where the last line uses the equation $(*)$ given in the question.

The discriminant of this quadratic is given by:

$$\begin{aligned} \Delta &= \big[a(p+q)(pq+3)\big]^2 - 4\left[1 + \left(\frac{p+q}{2}\right)^2\right]a^2(p+q)^2 \\ \Delta &= a^2(p+q)^2\left((pq+3)^3 - 4\left[1 + \left(\frac{p+q}{2}\right)^2\right]\right) \\ \Delta &= a^2(p+q)^2\Big((pq+3)^3 - 4 - (p+q)^2\Big) \\ \Delta &= a^2(p+q)^2\Big(p^2q^2 + 6pq + 9 - 4 - (p+q)^2\Big) \\ \Delta &= a^2(p+q)^2\Big(p^2q^2 + 6pq + 5 - (p+q)^2\Big) \end{aligned}$$

and by using $(*)$ we have $\Delta = 0$ and so the line through P and Q is tangent to the circle.

(ii) Rearranging $(*)$ to give a quadratic in q gives:

$$\begin{aligned} p^2 + 2pq + q^2 &= p^2q^2 + 6pq + 5 \\ \implies (p^2-1)q^2 + 4pq + 5 - p^2 &= 0 \end{aligned}$$

Note that the condition $p^2 \neq 1$ ensures that this is a quadratic. The discriminant is given by:

$$\begin{aligned} 16p^2 - 4(p^2-1)(5-p^2) &= 16p^2 - 4(6p^2 - 5 - p^4) \\ &= 4p^4 - 8p^2 + 20 \\ &= 4\big[p^4 - 2p^2 + 5\big] \\ &= 4\big[(p^2-1)^2 + 4\big] \end{aligned}$$

which is always strictly positive. Therefore there are always two distinct values of q that satisfy $(*)$ for any value of p where $p^2 \neq 1$.

If we have a quadratic equation $ax^2 + bx + c = 0$ then we know the sum of the roots is equal to $-\frac{b}{c}$ and the product of the roots is equal to $\frac{c}{a}$. These are know as Vieta's formulae.

Using the quadratic in q this gives:

$$\begin{aligned} q_1 + q_2 &= -\frac{4p}{p^2-1} \\ q_1q_2 &= \frac{5-p^2}{p^2-1} \end{aligned}$$

(iii) If Q_1 and Q_2 are the points corresponding to parameters q_1 and q_2 then we know that the lines PQ_1 and PQ_2 are both tangents to the circle. We also know that P, Q_1 and Q_2 all lie on the line $x^2 = 4ay$.

To show that the line Q_1Q_2 is also a tangent we can show that q_1 and q_2 satisfies the equation $(*)$, i.e we have:

$$\begin{aligned} (q_1+q_2)^2 &= (q_1q_2)^2 + 6q_1q_2 + 5 \\ \left(\frac{4p}{p^2-1}\right)^2 &= \left(\frac{5-p^2}{p^2-1}\right)^2 + 6\left(\frac{5-p^2}{p^2-1}\right) + 5 \\ \frac{16p^2}{(p^2-1)^2} &= \frac{(5-p^2)^2 + 6(5-p^2)(p^2-1) + 5(p^2-1)^2}{(p^2-1)^2} \\ \frac{16p^2}{(p^2-1)^2} &= \frac{p^4 - 10p^2 + 25 - 6p^4 + 36p^2 - 30 + 5p^4 - 10p^2 + 5}{(p^2-1)^2} \\ \frac{16p^2}{(p^2-1)^2} &= \frac{(\cancel{p^4} - \cancel{6p^4} + \cancel{5p^4}) + (36p^2 - 10p^2 - 10p^2) + (\cancel{25} - \cancel{30} + \cancel{5})}{(p^2-1)^2} \checkmark \end{aligned}$$

and so q_1 and q_2 satisfy $(*)$ and the triangle with vertices P, Q_1 and Q_2 satisfies the requirements of the question.

Question 2

2 The polar curves C_1 and C_2 are defined for $0 \leqslant \theta \leqslant \pi$ by

$$r = k(1 + \sin\theta)$$

$$r = k + \cos\theta$$

respectively, where k is a constant greater than 1.

(i) Sketch the curves on the same diagram. Show that if $\theta = \alpha$ at the point where the curves intersect, $\tan\alpha = \dfrac{1}{k}$.

(ii) The region A is defined by the inequalities

$$0 \leqslant \theta \leqslant \alpha \quad \text{and} \quad r \leqslant k(1 + \sin\theta)\,.$$

Show that the area of A can be written as

$$\frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1 - \cos\alpha)\,.$$

(iii) The region B is defined by the inequalities

$$\alpha \leqslant \theta \leqslant \pi \quad \text{and} \quad r \leqslant k + \cos\theta\,.$$

Find an expression in terms of k and α for the area of B.

(iv) The total area of regions A and B is denoted by R. The area of the region enclosed by C_1 and the lines $\theta = 0$ and $\theta = \pi$ is denoted by S. The area of the region enclosed by C_2 and the lines $\theta = 0$ and $\theta = \pi$ is denoted by T.

Show that as $k \to \infty$,

$$\frac{R}{T} \to 1$$

and find the limit of

$$\frac{R}{S}$$

as $k \to \infty$.

Examiner's report

This was the fifth most popular question on the paper, being attempted by about two thirds of the candidates, just a few more than question 8. However, it was the most successfully attempted with a mean score of over 12/20. Many candidates produced excellent responses to this question, and a number scored a perfect 20/20.

The two curves were generally well sketched in part **(i)**, with the commonest fault being failure to obtain values for r at the points where $\theta = 0, \frac{1}{2}\pi$ and π.

The derivation of the required result in **(i)** for the point of intersection as well as the result for the area of A in part **(ii)** were generally well done.

Similarly, in part **(iii)** the area of B was well attempted, although algebraic errors were more common here with the required result not being given in the question, unlike the area of A in part **(ii)**.

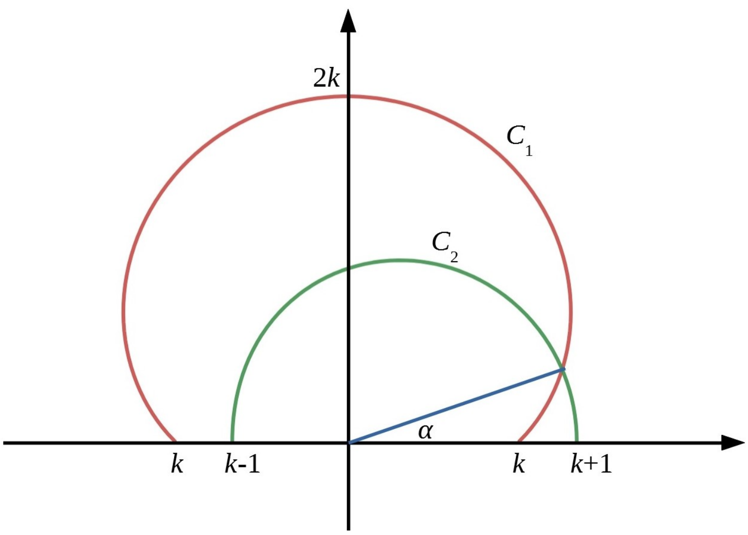
Part **(iv)** was found to be the most difficult part of the question, though marks for finding expressions for S and T were generally obtained. After this, the first challenge was to notice that $\alpha \to 0$, and it was important to justify this observation using the expression $\tan\alpha = \frac{1}{k}$ from part **(i)**. A small number of candidates made heuristic arguments that $\alpha \to 0$, using their sketches - however, this was not acceptable for the mark without further justification. The next challenge was to compute the limits rigorously, and candidates found this to be the most challenging aspect of the question. Common mistakes included not noticing that α depended on k, and substituting $\alpha = 0$ prematurely, which, for example, led to the erroneous conclusion that $k\sin\alpha \to 0$.

Solution

(i) For C_1 we have $r = k(1 + \sin\theta)$. When $\theta = 0$ orπ the radius is equal to k and when $\theta = \frac{1}{2}\pi$ the radius is equal to $2k$, with the radius at first increasing and then decreasing.

For $C_2 : r = k + \cos\theta$ we have $r = k + 1$ when $\theta = 0$ and then decreases as θ increases until we have $r = k - 1$ when $\theta = \pi$. Since we know $k > 1$ the radius does not become negative.

This gives us two curves like those below:



Where the curves intersect we have:

$$\begin{aligned} k(1+\sin\theta) &= k+\cos\theta \\ k\sin\theta &= \cos\theta \\ \tan\theta &= \frac{1}{k} \end{aligned}$$

So if the value of θ where they meet is α we have $\tan\alpha = \dfrac{1}{k}$.

(ii) The area of region A is given by

$$\begin{aligned} \tfrac{1}{2}\int r^2\,\mathrm{d}\theta &= \tfrac{1}{2}\int_0^\alpha k^2(1+\sin\theta)^2\,\mathrm{d}\theta \\ &= \tfrac{1}{2}k^2\int_0^\alpha 1+2\sin\theta+\sin^2\theta\,\mathrm{d}\theta \\ &= \tfrac{1}{2}k^2\int_0^\alpha 1+2\sin\theta+\tfrac{1}{2}(1-\cos 2\theta)\,\mathrm{d}\theta \\ &= \tfrac{1}{2}k^2\Big[\tfrac{3}{2}\theta-2\cos\theta-\tfrac{1}{4}\sin 2\theta\Big]_0^\alpha \\ &= \tfrac{1}{2}k^2\Big[\left(\tfrac{3}{2}\alpha-2\cos\alpha-\tfrac{1}{4}\sin 2\alpha\right)-(-2)\Big] \\ &= \tfrac{3}{4}k^2\alpha-k^2\cos\alpha-\tfrac{1}{8}k^2\sin 2\alpha+k^2 \\ &= \frac{k^2}{4}(3\alpha-\sin\alpha\cos\alpha)+k^2(1-\cos\alpha) \end{aligned}$$

(iii) The area of the region B is given by:

$$\begin{aligned} \tfrac{1}{2}\int r^2\,\mathrm{d}\theta &= \tfrac{1}{2}\int_\alpha^\pi (k+\cos\theta)^2\,\mathrm{d}\theta \\ &= \tfrac{1}{2}\int_\alpha^\pi k^2+2k\cos\theta+\cos^2\theta\,\mathrm{d}\theta \\ &= \tfrac{1}{2}\int_\alpha^\pi k^2+2k\cos\theta+\tfrac{1}{2}(\cos 2\theta+1)\,\mathrm{d}\theta \\ &= \tfrac{1}{2}\Big[k^2\theta+2k\sin\theta+\tfrac{1}{4}\sin 2\theta+\tfrac{1}{2}\theta\Big]_\alpha^\pi \\ &= \tfrac{1}{2}\Big[\left(k^2\pi+\tfrac{1}{2}\pi\right)-\left(k^2\alpha+2k\sin\alpha+\tfrac{1}{4}\sin 2\alpha+\tfrac{1}{2}\alpha\right)\Big] \\ &= \tfrac{1}{4}\Big[2k^2\pi+\pi-2k^2\alpha-4k\sin\alpha-\sin\alpha\cos\alpha-\alpha\Big] \end{aligned}$$

(iv) S can be found by substituting $\alpha=\pi$ into the expression for the area of region A, so we have:

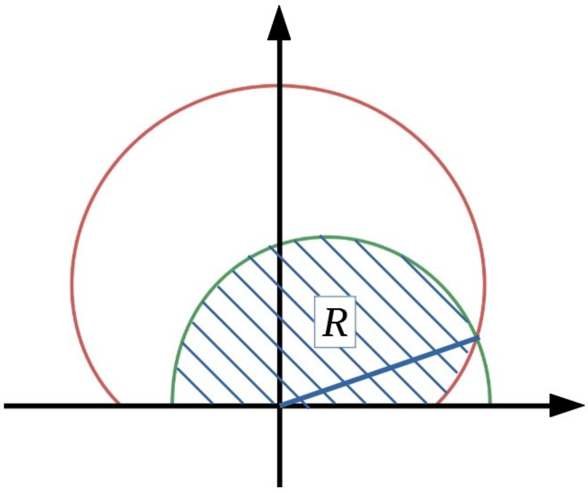
$$S = \frac{k^2}{4}(3\pi-0)+k^2(1--1) = \frac{k^2}{4}(3\pi+8)$$

T can be found by substitution $\alpha = 0$ into the expression for the area of region B, and so we have:

$$T = \tfrac{1}{4}\Big[2k^2\pi+\pi\Big] = \frac{k^2\pi}{2}+\frac{\pi}{4}$$

R is the sum of the two areas in part **(ii)** and **(iii)**. The given form for the area of region A looks perhaps a little strange — it's worth asking yourself why it might have been given in this form? Also, we haven't yet used everything from part **(i)**, so perhaps that might also be useful.

R is the region shown below:



We are interested in the limit as $k \to \infty$. As $k \to \infty$ we have $\tan\alpha = \frac{1}{k}$, and so α is small. If we use $\alpha \approx \sin\alpha \approx 0$ and $\cos\alpha \approx 1$ then we get that the region of A is approximately 0, which doesn't seem quite right (from the diagram it looks are if the area will be getting larger as k increases!).

Instead let $\alpha \approx \sin\alpha \approx \tan\alpha = \frac{1}{k}$ and $\cos\alpha \approx 1 - \frac{1}{2k^2}$. This gives the area of A as approximately:

$$\begin{aligned}\frac{k^2}{4}(3\alpha - \sin\alpha\cos\alpha) + k^2(1-\cos\alpha) &\approx \frac{k^2}{4}\left(\frac{3}{k} - \frac{1}{k}\left(1 - \frac{1}{2k^2}\right)\right) + k^2\left(\frac{1}{2k^2}\right)\\ &\approx \frac{k}{2} + \frac{1}{2} + \frac{1}{8k}\end{aligned}$$

Similarly the area of B is approximately:

$$\begin{aligned}\tfrac{1}{4}\left[2k^2\pi + \pi - 2k^2\alpha - 4k\sin\alpha - \sin\alpha\cos\alpha - \alpha\right] &\approx \tfrac{1}{4}\left[2k^2\pi + \pi - 4 - \frac{1}{k}\left(1 - \frac{1}{2k^2}\right) - \frac{1}{k}\right]\\ &\approx \frac{k^2\pi}{2} + \frac{\pi}{4} - 1 - \frac{2}{k} + \frac{1}{2k^2}\end{aligned}$$

Summing these two areas gives:

$$R = \frac{k^2\pi}{2} + \mathcal{O}(k)$$

where the symbol $\mathcal{O}(k)$ means terms of order k and smaller. As $k \to \infty$ we have $k^2 >> k$ and so we can ignore the smaller terms.

We have:

$$\frac{R}{T} \approx \frac{\frac{k^2\pi}{2}}{\frac{k^2\pi}{2} + \frac{\pi}{4}}$$

and so as $k \to \infty$ we have $\dfrac{R}{T} \to 1$.

From the diagram above it makes sense that R will approximately the same as T, as difference between the shaded area and the area contained under C_2 becomes proportionally insignificant as k increases. You can see numerically how the areas change on this Desmos page.

In a similar way we have:

$$\frac{R}{S} \approx \frac{\frac{k^2\pi}{2}}{\frac{k^2}{4}(3\pi + 8)} = \frac{2\pi}{3\pi + 8}$$

so we have $\dfrac{R}{S} \to \dfrac{2\pi}{3\pi + 8}$.

Question 3

3 **(i)** Show that, if a and b are complex numbers, with $b \neq 0$, and s is a positive real number, then the points in the Argand diagram representing the complex numbers $a + sb\mathrm{i}$, $a - sb\mathrm{i}$ and $a + b$ form an isosceles triangle.

Given three points which form an isosceles triangle in the Argand diagram, explain with the aid of a diagram how to determine the values of a, b and s so that the vertices of the triangle represent complex numbers $a + sb\mathrm{i}$, $a - sb\mathrm{i}$ and $a + b$.

(ii) Show that, if the roots of the equation $z^3 + pz + q = 0$, where p and q are complex numbers, are represented in the Argand diagram by the vertices of an isosceles triangle, then there is a non-zero real number s such that

$$\frac{p^3}{q^2} = \frac{27(3s^2 - 1)^3}{4(9s^2 + 1)^2} \, .$$

(iii) Sketch the graph $y = \dfrac{(3x - 1)^3}{(9x + 1)^2}$, identifying any stationary points.

(iv) Show that if the roots of the equation $z^3 + pz + q = 0$ are represented in the Argand diagram by the vertices of an isosceles triangle then $\dfrac{p^3}{q^2}$ is a real number and $\dfrac{p^3}{q^2} > -\frac{27}{4}$.

Examiner's report

This was the least popular of the Pure questions, being attempted by just under 45% of the candidates. Furthermore, it was not well answered yielding a mean score of 6/20.

Many incorrectly treated a and b as real numbers in part **(i)** which rendered the question very simple. On the other hand, there were some that correctly simplified their working by 'ignoring' a in each number and then translating the triangle by a after. The second part of **(i)** was often well-answered.

Most attempts at part **(ii)** were decent. Students that attempted it recognised that the roots should be written using a, b, s from part **(i)** and wrote down the sum/product of roots formulae for p and q. A few did not write down the equation for the coefficient of z^2, and without this it was not possible for them to earn further credit for simplifying p and q. Sign errors in q were not uncommon.

In part **(iii)**, there was a large variety among the sketches seen. Only a few candidates specified the leading order behaviour at infinity. A fair number of candidates did not reflect the nature of the point of inflection in their drawing. Some did not specify intercepts. Pretty nearly all recognised the asymptote at $-\frac{1}{9}$.

In part **(iv)**, the majority that had successfully drawn the sketch in part **(iii)** managed to successfully satisfy the logic, although some failed to obtain the reality of the expression even though this was explicitly required.

Solution

(i) There are various was of doing the first part. Don't forget that a and b are complex, but s is real.

The distance between $a + b$ and $a + sb\text{i}$ is given by:

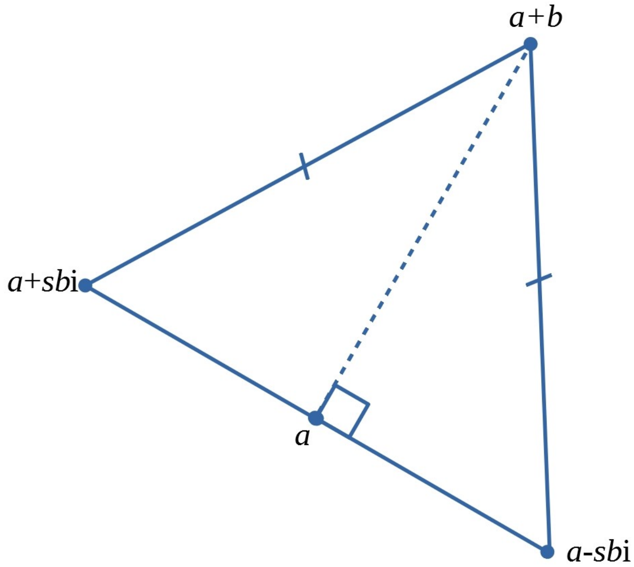
$$\begin{aligned}|(a+b)-(a+sb\text{i})| &= |b - sb\text{i}|\\ &= |b||1-s\text{i}|\\ &= |b|\sqrt{1+s^2}\end{aligned}$$

Similarly the distance between $a + b$ and $a - sb\text{i}$ is given by:

$$\begin{aligned}|(a+b)-(a-sb\text{i})| &= |b + sb\text{i}|\\ &= |b||1+s\text{i}|\\ &= |b|\sqrt{1+s^2}\end{aligned}$$

Therefore two of the sides of the triangle have the same length, so it is isosceles.

For the second part, consider an isosceles triangle. Call the side which is a different length to the equal sides the "base". We know that the line joining the vertex between the equal sides and the midpoint of the base is perpendicular to the base.



We can let the midpoint of the base be a, and then let the vector between the midpoint of the base and the opposite vertex be represented by the complex number b then the vertex between the equal sides will have position $a + b$.

We also know that bi is perpendicular to b, so the other two vertices will be on the line $a+\lambda b\mathrm{i}$, and they will be equidistant from a in opposite directions, so will have positions $a \pm sb\mathrm{i}$.

In essence part (i) is asking you to show that a triangle is isosceles if and only if you can represent the three vertices with the complex numbers $a + sb\mathrm{i}$, $a - sb\mathrm{i}$ and $a + b$. The request was split up into two parts with the hint to use a diagram for the "only if" direction to make this part a little more approachable.

(ii) We know that the vertices of the triangle can be written as $a + b, a \pm sb\mathrm{i}$ and so we can write the equation as:

$$\begin{aligned}
z^3 + pz + q &= (z - a - b)(z - a - sb\mathrm{i})(z - a + sb\mathrm{i}) \\
&= \Big((z - a) - b\Big)\Big((z - a) - sb\mathrm{i}\Big)\Big((z - a) + sb\mathrm{i}\Big) \\
&= \Big((z - a) - b\Big)\Big((z - a)^2 + s^2b^2\Big) \\
&= (z - a)^3 - b(z - a)^2 + s^2b^2(z - a) - s^2b^3 \\
&= z^3 - 3az^2 + 3a^2z - a^3 - bz^2 + 2abz - ba^2 + s^2b^2z - as^2b^2 - s^2b^3 \\
&= z^3 - (3a + b)z^2 + (3a^2 + 2ab + s^2b^2)z - (a^3 + a^2b + as^2b^2 + s^2b^3)
\end{aligned}$$

Equating coefficients of z^2 gives $3a = b$. Substituting $b = -3a$ gives:

$$\begin{aligned}
p &= 3a^2 - 6a^2 + 9s^2a^2 = 3a^2(3s^2 - 1) \\
q &= -(a^3 - 3a^3 + 9a^3s^2 - 27a^3s^2) = 2a^3(9s^2 + 1)
\end{aligned}$$

this gives:

$$\begin{aligned}
\frac{p^3}{q^2} &= \frac{3^3a^6(3s^2 - 1)^3}{2^2a^6(9s^2 + 1)^2} \\
&= \frac{27(3s^2 - 1)^3}{4(9s^2 + 1)^2}
\end{aligned}$$

(iii) Differentiating $y = \dfrac{(3x-1)^3}{(9x+1)^2}$ gives:

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{(9x+1)^2 \times 9(3x-1)^2 - 18(9x+1)(3x-1)^3}{(9x+1)^4} \\ &= \frac{9(3x-1)^2(9x+1)\Big[(9x+1) - 2(3x-1)\Big]}{(9x+1)^4} \\ &= \frac{9(3x-1)^2\big[3x+3\big]}{(9x+1)^3}\end{aligned}$$

So the stationary points are where $x = \frac{1}{3}$ and $x = -1$. Substituting back into the equation of the curve gives the points $(-1,-1)$ and $\left(\frac{1}{3},0\right)$.

There is a vertical asymptote when $x = -\frac{1}{9}$, and it crosses the x axis when $y = -1$.

We have $y = \dfrac{27x^3 - 27x^2 + 9x - 1}{81x^2 + 18x + 1}$.

To work out the behaviour as $x \pm \infty$ we need to rewrite the fraction by dividing the numerator by the denominator, but we don't actually need to complete the division to get what we need!

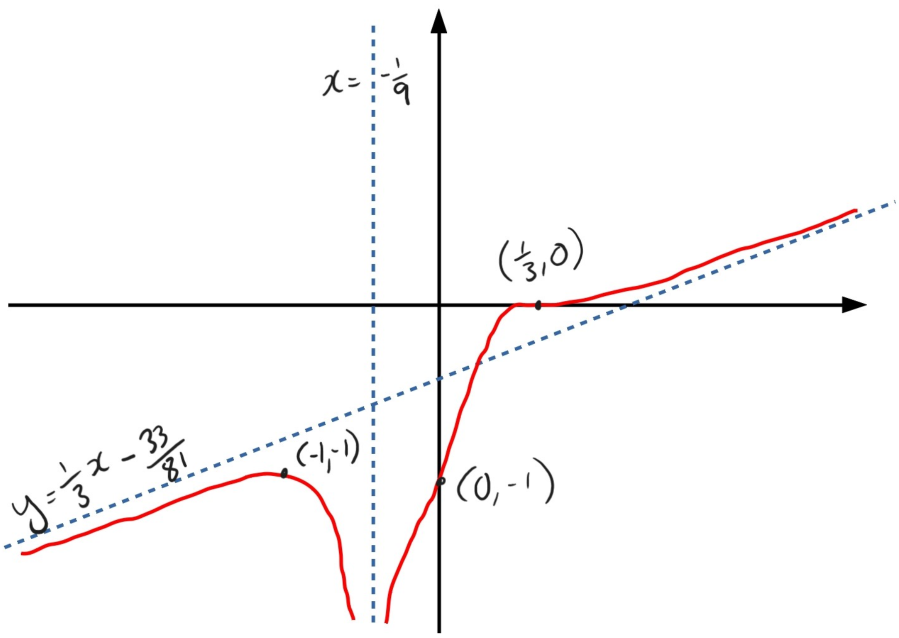
We have:

$$27x^3 - 27x^2 + 9x - 1 = (81x^2 + 18x + 1)\left(\tfrac{1}{3}x - \tfrac{33}{81}\right) + ax + b$$

where the coefficients of the linear factor were found by equating coefficients of x^3 and x^2. This gives:

$$y = \frac{27x^3 - 27x^2 + 9x - 1}{81x^2 + 18x + 1} = \tfrac{1}{3}x - \tfrac{33}{81} + \frac{ax+b}{81x^2 + 18x + 1}$$

and so as $x \to \infty$ we have an asymptote of $y = \frac{1}{3}x - \frac{33}{81}$. Note that we did not have to actually find a and b.



(iv) Remember that s is real. From part **(ii)** we know that if the roots can be represented by vertices of an isosceles triangle then we have:

$$\frac{p^3}{q^2} = \frac{27(3s^2-1)^3}{4(9s^2+1)^2}$$

and since s is real, we have $\dfrac{p^3}{q^2}$ is real.

We know that $s^2 > 0$ (it can't be equal to 0 as then we would not have a triangle as two vertices would be in the same place). Considering the graph from part **(iii)** with $x > 0$ we have $\dfrac{(3s^2-1)^3}{(9s^2+1)^2} > -1$. This means we have $\dfrac{p^3}{q^2} > -\dfrac{27}{4}$.

Question 4

4 Let n be a positive integer. The polynomial p is defined by the identity

$$\mathrm{p}(\cos\theta) \equiv \cos\big((2n+1)\theta\big) + 1\,.$$

(i) Show that

$$\cos\big((2n+1)\theta\big) = \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta(\cos^2\theta - 1)^r\,.$$

(ii) By considering the expansion of $(1+t)^{2n+1}$ for suitable values of t, show that the coefficient of x^{2n+1} in the polynomial $\mathrm{p}(x)$ is 2^{2n}.

(iii) Show that the coefficient of x^{2n-1} in the polynomial $\mathrm{p}(x)$ is $-(2n+1)2^{2n-2}$.

(iv) It is given that there exists a polynomial q such that

$$\mathrm{p}(x) = (x+1)\left[\mathrm{q}(x)\right]^2$$

and the coefficient of x^n in $\mathrm{q}(x)$ is greater than 0.

Write down the coefficient of x^n in the polynomial $\mathrm{q}(x)$ and, for $n \geqslant 2$, show that the coefficient of x^{n-2} in the polynomial $\mathrm{q}(x)$ is

$$2^{n-2}(1-n)\,.$$

Examiner's report

This was only marginally more popular than question 3 and was the least successfully attempted question on the paper with a mean score of 5/20.

In the vast majority of cases, there was no substantially correct attempt except in part **(i)**. Those that used de Moivre's theorem, expanded binomially, equated real parts and replaced $\sin^{2r}\theta$ by $(\cos^2\theta - 1)^r$, generally scored well in this part, but marks could not be credited where mathematical steps were glossed over when the question stated, 'Show that'. Some attempted use of proof by induction, but their conclusions were not supported by their mathematical argument.

Many attempting part **(ii)** wrote down the coefficient required but made no further progress, or made the expansion and went no further. There were some that did substitute +1 and -1, and solved this part.

Part **(iii)** was solved generally by those that had made the substitutions in **(ii)** and saw that differentiation might be useful.

Part **(iv)** could be answered using the given results from the previous parts of the question, however this part was almost exclusively only attempted by candidates that had had a reasonable level of success on the previous three parts. Those candidates that set out a careful and organised solution were more successful in part **(iv)**.

Solution

(i) By De Moivre's theorem we have:

$$\cos\big((2n+1)\theta\big) + \mathrm{i}\sin\big((2n+1)\theta\big) = (\cos\theta + \mathrm{i}\sin\theta)^{(2n+1)}$$

and equating real parts:

$$\begin{aligned}\cos\big((2n+1)\theta\big) &= \cos^{2n+1}\theta - \binom{2n+1}{2}\cos^{2n-1}\sin^2\theta + \binom{2n+1}{4}\cos^{2n-3}\sin^4\theta - \cdots \\ &= \sum_{r=0}^{n}(-1)^r\binom{2n+1}{2r}\cos^{2n+1-2r}\theta\sin^{2r}\theta \\ &= \sum_{r=0}^{n}\binom{2n+1}{2r}\cos^{2n+1-2r}\theta(-1)^r(1-\cos^2\theta)^r \\ &= \sum_{r=0}^{n}\binom{2n+1}{2r}\cos^{2n+1-2r}\theta(\cos^2\theta-1)^r\end{aligned}$$

(ii) Looking at the coefficient of $\cos^{2n+1}\theta$ in the expression for $\cos\big((2n+1)\theta\big)$ found in part **(i)**, there is a $\cos^{2n+1}$ terms for each value of r, as each term has the form $\cos^{2n+1-2r}\left(\cos^2\theta - 1\right)^r = \cos^{2n+1-2r}\left(\cos^{2r}\theta + \cdots\right)$.

This means that the coefficient of $\cos^{2n+1}\theta$ in $\mathrm{p}(\cos\theta)$ is given by $\displaystyle\sum_{r=0}^{n}\binom{2n+1}{2r}$.

We are told to consider $(1+t)^{2n+1}$ which is:

$$(1+t)^{2n+1} = \sum_{i=0}^{2n+1}\binom{2n+1}{i}t^i$$

We are also told to consider this "for suitable values of t". Substituting $t=1$ gives:

$$2^{2n+1} = \sum_{i=0}^{2n+1}\binom{2n+1}{i} \qquad (*)$$

This looks a little like the expression we want, but we don't want to include any of the odd values if i. If instead we take $t=-1$ we have:

$$0 = \sum_{i=0}^{2n+1}\binom{2n+1}{i}(-1)^i \qquad (\dagger)$$

If we add together the series $(*)$ and $(\dagger)$ then the odd terms in i will cancel and we have:

$$2^{2n+1} = 2\sum_{\text{even } i}\binom{2n+1}{i} = 2\sum_{r=0}^{n}\binom{2n+1}{2r}$$

Using a substitution of $i=2r$. Dividing by 2 gives the required result.

(iii) Picking out the coefficient of $\cos^{2n-1}\theta$ is a little more difficult. We have:

$$\cos\big((2n+1)\theta\big) = \sum_{r=0}^{n} \binom{2n+1}{2r} \cos^{2n+1-2r}\theta(\cos^2\theta - 1)^r$$

The $r = 0$ value cannot return a contribution to the coefficient of $\cos^{2n-1}\theta$ as this only contains a $\cos^{2n+1}\theta$ term. Working through the first few positive values of r the contributions to the coefficient of $\cos^{2n-1}\theta$ are:

$$\begin{aligned}
r = 1 &\to \binom{2n+1}{2} \cos^{2n-1}\theta(\cos^2\theta - 1)^1 \to -\binom{2n+1}{2} \\
r = 2 &\to \binom{2n+1}{4} \cos^{2n-3}\theta(\cos^2\theta - 1)^2 \\
&= \binom{2n+1}{4} \cos^{2n-3}\theta(\cos^4\theta - 2\cos^2\theta + 1) \to -2\binom{2n+1}{4} \\
r = 3 &\to \binom{2n+1}{6} \cos^{2n-5}\theta(\cos^2\theta - 1)^3 \\
&= \binom{2n+1}{6} \cos^{2n-5}\theta(\cos^6\theta - 3\cos^4\theta + 3\cos^2\theta - 1) \to -3\binom{2n+1}{6}
\end{aligned}$$

So in general the coefficient of $\cos^{2n-1}\theta$ is given by $\displaystyle\sum_{r=0}^{n} -r\binom{2n+1}{2r}$. Note that we can start our sum with $r = 0$ as this will be a zero contribution!

As in part **(i)** we have:

$$(1+t)^{2n+1} = \sum_{i=0}^{2n+1} \binom{2n+1}{i} t^i$$

Differentiation with respect to t gives us the result:

$$(2n+1)(1+t)^{2n} = \sum_{i=0}^{2n+1} i\binom{2n+1}{i} t^{i-1}$$

Like before, we want to pick out the terms where i is even. Substituting $x = 1$ and $x = -1$ gives:

$$(2n+1)2^{2n} = \sum_{i=0}^{2n+1} i\binom{2n+1}{i} \qquad (1)$$

$$0 = \sum_{i=0}^{2n+1} (-1)^{i-1} i\binom{2n+1}{i} \qquad (2)$$

We want to find $\displaystyle\sum_{r=0}^{n} -r\binom{2n+1}{2r}$. Combining the above series using (2) − (1) gives

$$-(2n+1)2^{2n} = 2\sum_{\text{even } i} -i\binom{2n+1}{i}$$

Using a substitution of $2r = i$ then gives:

$$2\sum_{r=0}^{n} -2r\binom{2n+1}{2r} = -(2n+1)2^{2n}$$
$$\implies \sum_{r=0}^{n} -r\binom{2n+1}{2r} = -(2n+1)2^{2n-2}$$

and so the coefficient of x^{2n-1} is $-(2n+1)2^{2n-2}$

Note that it is very easy to loose a factor of 2 or make other mistakes, which is one of the reasons why the answer is given in this part so that you can still attempt the last part even if this part goes wrong. However that does mean that you have to show enough working to support your answer fully!

(iv) Let $\mathrm{q}(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots$, where $a_n > 0$.

We then have:

$$\begin{aligned}\mathrm{p}(x) &= (x+1)\big[a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots\big]^2 \\ &= a_n^2 x^{2n+1} + \big(a_n^2 + 2a_n a_{n-1}\big)x^{2n} + \big(a_{n-1}^2 + 2a_n a_{n-2} + 2a_n a_{n-1}\big)x^{2n-1} + \cdots\end{aligned}$$

The question tells you to “Write down” the coefficient of x^n which tells you that you probably don’t need to do any working out. From part **(i)** we have $a_n^2 = 2^{2n}$ and so $a_n = 2^n$.
Note that the condition that $a_n > 0$ means you can discard the -2^n possibility.

In the expansion of $\mathrm{p}(\cos\theta)$ there are only odd powers of $\cos\theta$ which means we need the coefficient of x^{2n} in the expression for $\mathrm{p}(x)$ to be equal to zero. From the expansion for $\mathrm{p}(x)$ above we have $a_n^2 + 2a_n a_{n-1} = 0$ and as $a_n > 0$ we have $a_{n-1} = -\frac{1}{2}a_n = -2^{n-1}$.

Equating the coefficient of x^{2n-1} with the value given in part **(iii)** we have:

$$\begin{aligned}-(2n+1)2^{2n-2} &= a_{n-1}^2 + 2a_n a_{n-2} + 2a_n a_{n-1} \\ -(2n+1)2^{2n-2} &= (-2^{n-1})^2 + 2(2^n)a_{n-2} + 2(2^n)(-2^{n-1}) \\ -(2n+1)2^{2n-2} &= 2^{2n-2} + 2^{n+1}a_{n-2} - 2^{2n} \\ -(2n+1)2^{n-2} &= 2^{n-2} + 2a_{n-2} - 2^n \qquad \big(\text{dividing by } 2^n\big) \\ \implies 2a_{n-2} &= 2^n - 2^{n-2} - (2n+1)2^{n-2} \\ 2a_{n-2} &= 2^{n-2}\big(4 - 1 - (2n+1)\big) \\ 2a_{n-2} &= 2^{n-2}\big(2 - 2n\big) \\ a_{n-2} &= 2^{n-2}\big(1 - n\big)\end{aligned}$$

A perhaps more natural thing to divide by is 2^{n+1} so that you make the coefficient of a_{n-2} equal to 1, but I decided not to do that as I thought there was a danger I might end up with a sign error. In the end this worked out quite nicely as I could divide by 2 very simply at the end of the question!

Question 5

5 **(i)** Show that if

$$\frac{1}{x} + \frac{2}{y} = \frac{2}{7},$$

then $(2x - 7)(y - 7) = 49$.

By considering the factors of 49, find all the pairs of positive integers x and y such that

$$\frac{1}{x} + \frac{2}{y} = \frac{2}{7}.$$

(ii) Let p and q be prime numbers such that

$$p^2 + pq + q^2 = n^2$$

where n is a positive integer. Show that

$$(p + q + n)(p + q - n) = pq$$

and hence explain why $p + q = n + 1$.

Hence find the possible values of p and q.

(iii) Let p and q be positive and

$$p^3 + q^3 + 3pq^2 = n^3.$$

Show that $p + q - n < p$ and $p + q - n < q$.

Show that there are no prime numbers p and q such that $p^3 + q^3 + 3pq^2$ is the cube of an integer.

Examiner's report

Whilst this was the most popular question, it was only the sixth most successful with a mean mark of a little under 9/20. Many of the candidates made substantial attempts at parts **(i)** and **(ii)** but found it more challenging to make progress with part **(iii)**. Two common general errors were lack of precision when handling inequalities, and working backwards from a required result without demonstrating that the logic was reversible.

In part **(i)**, a small number of candidates rearranged the first equation to remove denominators then wrote the required result without adequate intermediate steps of working. There were a very small number of arithmetic errors when finding the pairs of x and y.

In **(ii)** many candidates commented that as p and q were prime then the only possible factors of pq were $1, p, q$ and pq and went on to test each of these as possible values for $p + q + n$. Many candidates were able to form a relevant equation involving p and q and whilst most factorised it, similarly to part **(i)**, a small number attempted alternative approaches. The most successful of these was to write p as a function of q and rewrite the improper fraction to see that $q-2$ must divide 3. A small number of candidates spotted that p and q were the solutions to the quadratic equation $t^2-(n+1)t+(2n+1)=0$ but from here few were able to fully justify that the only solutions for (p,q) came from $n=7$.

The first two results of part **(iii)** caused much confusion. Relatively few candidates realised at the start of their attempts that these were equivalent to $q<n$ and $p<n$. Those who did recognise this completed part **(iii)** with relative ease. For the second part, a pleasing number of candidates realised that $(p+q)^3$ would be a useful expression to consider and those who did usually managed to get to the difference of two cubes expression necessary to make progress. Some candidates were unsure where to go next but a good number realised the importance of the printed inequalities and correctly deduced that $p+q-n$ must be 1 or 3. From here candidates often managed to rule out one case but ruling out both successfully was relatively rare.

Solution

(i) Multiplying throughout by $7xy$ gives:

$$\begin{aligned} 2xy &= 7y+14x \\ 2xy-14x-7y &= 0 \\ (2x-7)(y-7)-49 &= 0 \\ (2x-7)(y-7) &= 49 \end{aligned}$$

The only two ways that 49 can be factorised are 1×49 and 7×7. These lead to the results:

$$\begin{aligned} 2x-7=1 \text{ and } y-7=49 &\implies x=4, y=56 \\ 2x-7=7 \text{ and } y-7=7 &\implies x=7, y=14 \\ 2x-7=49 \text{ and } y-7=1 &\implies x=28, y=8 \end{aligned}$$

(ii) Starting from the given statement we have:

$$\begin{aligned} p^2+pq+q^2 &= n^2 \\ p^2+pq+q^2-n^2 &= 0 \\ p^2+2pq+q^2-n^2 &= pq \\ (p+q)^2-n^2 &= pq \\ (p+q+n)(p+q-n) &= pq \end{aligned}$$

where the last step uses difference of two squares.

We are told that n is a positive integer so we know that $p+q+n>p+q-n$. The only possible factors of pq are $1, p, q$ and pq. We know that $p+q-n\neq pq$ as $p+q+n>p+q-n$.

If we take $p+q-n=p$ then we would need $p+q+n=q \implies q=-n$, which is not possible as q and n are both positive integers. Similarly we cannot have $p+q-n=q$.

Therefore the only possible option is $p + q - n = 1 \implies p + q = n + 1$ (and also we have $p + q + n = pq$). Substituting the first of these into the second gives:

$$\begin{aligned}
p + q + n &= pq \\
p + q + (p + q - 1) &= pq \\
pq - 2p - 2q + 1 &= 0 \\
(p-2)(q-2) - 3 &= 0 \\
(p-2)(q-2) &= 3
\end{aligned}$$

The only factors of 3 are 1 and 3, so we have either $p - 2 = 1, q - 2 = 3$ or $p - 2 = 3, q - 2 = 1$ so we either have $p = 3, q = 5$ or $p = 5, q = 3$.

(iii) Note that in this part we don't know that p and q are prime until the end of the question.

In this part we have powers of three, and in the previous part the difference of two squares was useful. It might be worth considering the difference of two cubes:

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Rearranging the given equation we have:

$$\begin{aligned}
p^3 + q^3 + 3pq^2 &= n^3 \\
p^3 + 3p^2q + 3pq^2 + q^3 - 3p^2q &= n^3 \\
(p+q)^3 - 3p^2q &= n^3 \\
(p+q)^3 - n^3 &= 3p^2q \\
(p+q-n)\Big[(p+q)^2 + (p+q)n + n^2\Big] &= 3p^2q \qquad (*)
\end{aligned}$$

We have quite a few possible factors to consider for $p + q - n$ (12 in fact!), so it might be worth trying to narrow this down a bit. In fact this is what the first request in this part is asking us to do!

Considering the original equation $p^3 + q^3 + 3pq^3 = n^3$, and the fact that p, q, n are all positive we have $p^3 < n^3 \implies p < n$, and similarly $q < n$.

We then have $p + q - n < p + n - n \implies p + q - n < p$ and similarly $p + q - n < q$.

From $(*)$ we know that $p + q - n$ is a factor of $3pq^2$, and if p and q are integers this means we must have $p + q - n$ equal to either 1 or 3.

Substituting $p + q - n = 1$ into $(p+q)^3 - n^3 = 3p^2q$ gives:

$$\begin{aligned}
(n+1)^3 - n^3 &= 3p^2q \\
3n^2 + 3n + 1 &= 3p^2q
\end{aligned}$$

There are no integer solutions of n for this equation as the LHS is not a multiple of 3 but the RHS is.

Substituting $p+q-n=3$ into $(p+q)^3-n^3=3p^2q$ gives:

$$\begin{aligned}(n+3)^3-n^3&=3p^2q\\ 9n^2+27n+27&=3p^2q\\ 3n^2+9n+9&=p^2q\\ 3\left(n^2+3n+3\right)&=p^2q\end{aligned}$$

Therefore with p or q (or both!) must be equal to a multiple of 3.

For the last part of this question we are considering p and q prime again. If p and q are prime and at least one of them must be a multiple of 3 then either p or q must be equal to 3.

If $p=3$ then substituting this into $p+q-n=3$ gives $q-n=0$, which is not possible as $q<n$. Similarly if $q=3$ we have $p-n=0$ which is also not possible. Hence there are no prime values of p and q such that $p^3+q^3-3pq^2$ is the cube of an integer.

Question 6

6 **(i)** By considering the Maclaurin series for e^x, show that for all real x,

$$\cosh^2 x \geqslant 1+x^2.$$

Hence show that the function f, defined for all real x by $\mathrm{f}(x)=\tan^{-1}x-\tanh x$, is an increasing function.

Sketch the graph $y=\mathrm{f}(x)$.

(ii) Function g is defined for all real x by $\mathrm{g}(x)=\tan^{-1}x-\frac{1}{2}\pi\tanh x$.

(a) Show that g has at least two stationary points.

(b) Show, by considering its derivative, that $(1+x^2)\sinh x - x\cosh x$ is non-negative for $x \geqslant 0$.

(c) Show that $\dfrac{\cosh^2 x}{1+x^2}$ is an increasing function for $x \geqslant 0$.

(d) Hence or otherwise show that g has exactly two stationary points.

(e) Sketch the graph $y=\mathrm{g}(x)$.

Examiner's report

The fourth most popular question being attempted by just over three quarters of the candidates, it was the second most successful with a mean of just over half marks. The best responses involved clear algebra and working, with the given results fully justified. Many candidates picked up marks by accurate differentiation, whilst the best candidates were able to sketch graphs showing all the main features and could carefully justify results. Many parts of the question asked candidates to show a given result, which meant that candidates needed to ensure they showed sufficient working before reaching the given result. In part **(i)** and part **(ii)(b)** candidates were required to use a specified method; candidates who did not use this method did not gain all the marks.

There were some good answers to part **(ii)**, but many candidates failed to show a stationary point of inflection at the origin, possibly as they assumed that they had shown the graph was strictly increasing rather than increasing. Failing to show the asymptote limits was another common mistake.

Part **(ii) (a)** was found to be the hardest. Many candidates did not justify their results (such as the behaviour of $\mathrm{g}(x)$ or $\mathrm{g}'(x)$ as x tends to infinity). Some candidates drew graphs to help justify their result, but these generally did not explain why their graphs looked as they did. However, part **(ii)(b)** was generally done well, as was **(c)** by those that attempted it.

Many candidates failed to gain the marks in part **(d)**, mainly through failing to consider the symmetry of $\dfrac{\cosh^2 x}{1+x^2}$.

Candidates found the graph in part **(e)** easier to sketch than the one in part **(i)**. The most common mistake here was to have the graph reflected, with $\mathrm{g}(x)$ positive when x is positive, incorrectly.

Solution

(i) We are told to consider the Maclaurin series for e^x, so we better do that! We have:

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Using this with the definition of $\cosh x$ gives:

$$\begin{aligned}\cosh x &= \frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})\\ &= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots\right)\right]\\ &= \frac{1}{2}\left(2 + \frac{2x^2}{2!} + \frac{2x^4}{4!} + \cdots\right)\\ &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\end{aligned}$$

So we have $\cosh x \geqslant 1 + \frac{1}{2}x^2$. Since both sides are greater than 1 we can square this to give:

$$\begin{aligned}\cosh^2 x &\geqslant \left(1 + \frac{1}{2}x^2\right)^2\\ \cosh^2 x &\geqslant 1 + 2 \times \frac{1}{2}x^2 + \frac{1}{4}x^4\\ \cosh^2 x &\geqslant 1 + x^2\end{aligned}$$

To show that $\mathrm{f}(x) = \tan^{-1} x - \tanh x$ is an increasing function consider the derivative. You can just quote the derivatives of the two terms if you know them, but they are not too hard to derive.

$$\begin{aligned} y &= \tanh x \\ y &= \frac{\sinh x}{\cosh x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{sech}^2 x \end{aligned}$$

$$\begin{aligned} y &= \tan^{-1} x \\ \tan y &= x \\ \sec^2 y \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{\sec^2 y} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1+\tan^2 y} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1+x^2} \end{aligned}$$

Therefore we have $\mathrm{f}'(x) = \dfrac{1}{1+x^2} - \mathrm{sech}^2 x$. Rearranging over a common denominator gives:

$$\mathrm{f}'(x) = \frac{\cosh^2 x - (1+x^2)}{(1+x^2)\cosh^2 x}$$

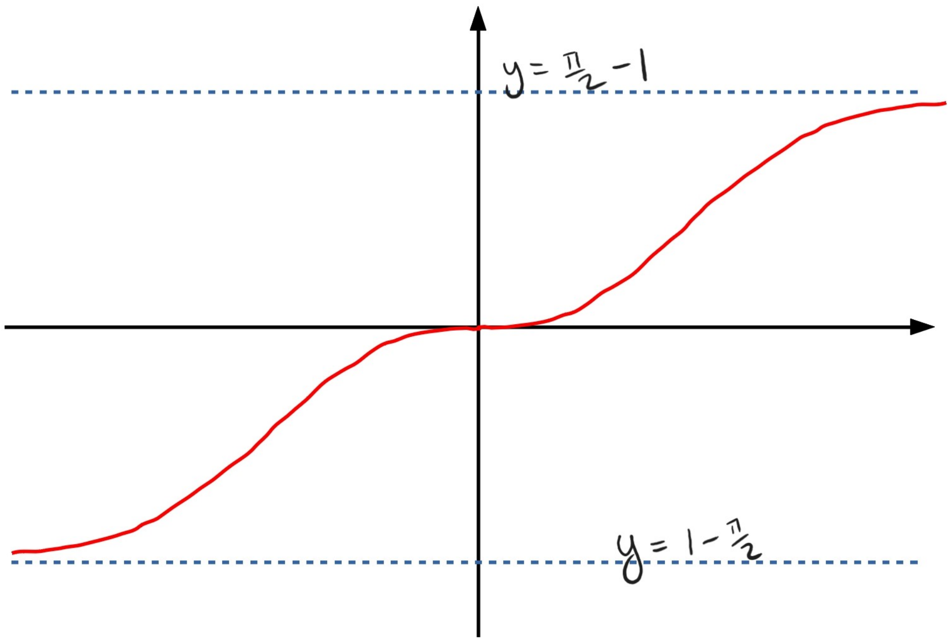
The denominator is always positive, and we have $\cosh^2 x \geqslant 1 + x^2$ and so we have $\mathrm{f}'(x) \geqslant 0$ and the function is increasing.

Note that when $x = 0$ we have $\mathrm{f}'(x) = 0$, so the function is not strictly increasing.

To sketch the graph first note that we have $\mathrm{f}(0) = \mathrm{f}'(0)$ so there is a stationary point of inflection at the origin. We also know that $\tan^{-1} x$ and $\tanh x$ are both odd functions, so we have $\mathrm{f}(-x) = -\mathrm{f}(x)$ (and so we have rotational symmetry about the origin).

As $x \to +\infty$ we have $\tan^{-1} x \to \frac{\pi}{2}$ and $\tanh x \to 1$. Using the fact that $\mathrm{f}(x)$ is odd this means that as $x \to \pm\infty$ we have $\mathrm{f}(x) \to \pm\left(\frac{\pi}{2} - 1\right)$.

This gives us enough information to sketch the graph:



(i) **(a)** We have:

$$\begin{aligned} g'(x) &= \frac{1}{1+x^2} - \frac{\pi}{2}\text{sech}^2 x \\ &= \frac{2\cosh^2 x - \pi(1+x^2)}{2(1+x^2)\cosh^2 x} \end{aligned}$$

When $x = 0$ we have $g'(x) = \dfrac{2-\pi}{2}$, i.e. we have $g'(0) < 0$. As $x \to \infty$ we have $2\cosh^2 x > \pi(1+x^2)$, and the denominator is positive, so for large enough x, $g'(x)$ will be positive. Since $g(x)$ is a continuous function there must be at least one positive value of x with $g'(x) = 0$ and so at least one stationary point for positive x.

Since $g(x)$ is an odd function there must also be at least one stationary point for negative x, and so $g(x)$ must have at least 2 stationary points.

(b) Let $h(x) = (1+x^2)\sinh x - x\cosh x$. Differentiating gives:

$$\begin{aligned} h'(x) &= (2x)\sinh x + (1+x^2)\cosh x - \cosh x - x\sinh x \\ &= x\sinh x + x^2\cosh x \end{aligned}$$

When $x \geqslant 0$ we have $\sinh x \geqslant 0$ and $\cosh x \geqslant 1$, and so $h'(x) \geqslant 0$.

When $x = 0$ we have $h(x) = 0$. Since we have $h(0) = 0$ and $h'(x) \geqslant 0$ for $x \geqslant 0$ then we have $h(x) \geqslant 0$ for $x \geqslant 0$.

(c) Let $\text{p}(x) = \dfrac{\cosh^2 x}{1+x^2}$. Differentiating gives:

$$\begin{aligned}\text{p}'(x) &= \frac{(1+x^2)\times 2\cosh x \sinh x - 2x\cosh^2 x}{(1+x^2)^2} \\ &= \frac{2\cosh x\left[(1+x^2)\sinh x - x\cosh x\right]}{(1+x^2)^2}\end{aligned}$$

By part **(b)** we have $(1+x^2)\sinh x - x\cosh x \geqslant 0$ for $x \geqslant 0$. We also have $\cosh x \geqslant 1$ and $(1+x^2)^2 \geqslant 1$, so we have $\text{p}'(x) \geqslant 0$ for $x \geqslant 0$ and $\text{p}(x)$ is increasing for $x \geqslant 0$.

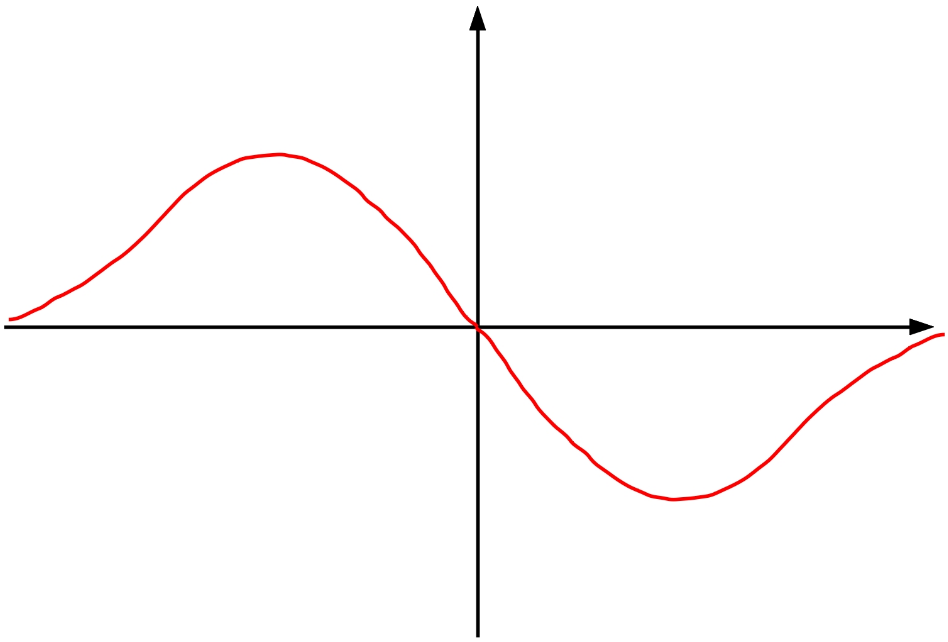
(d) We can write $\text{g}'(x)$ as:

$$\begin{aligned}\text{g}'(x) &= \frac{1}{1+x^2} - \frac{\pi}{2}\text{sech}^2 x \\ &= \text{sech}^2 x\left[\frac{\cosh^2 x}{(1+x^2)} - \frac{\pi}{2}\right]\end{aligned}$$

By part **(c)** we know that $\dfrac{\cosh^2 x}{(1+x^2)}$ is an increasing function for $x \geqslant 0$, and so there can only be one point where $\dfrac{\cosh^2 x}{(1+x^2)} = \dfrac{\pi}{2}$, and so only one stationary point for $\text{g}(x)$ for $x \geqslant 0$. Therefore there are exactly two stationary points for $\text{g}(x)$, one when $x > 0$ and one where $x < 0$.

(e) We have $\text{g}(0) = 0$ and $\text{g}'(0) < 0$ from previous work. We also know that there are exactly two stationary points and that there is rotational symmetry about the origin (as $\text{g}(x)$ is an odd function).

As $x \to \pm\infty$ we have $\text{g}(x) \to \mp 0$. This gives us enough to sketch the graph, as show below.



Question 7

7 **(i)** Let f be a continuous function defined for $0 \leqslant x \leqslant 1$. Show that

$$\int_0^1 \mathrm{f}(\sqrt{x})\,\mathrm{d}x = 2\int_0^1 x\mathrm{f}(x)\,\mathrm{d}x\,.$$

(ii) Let g be a continuous function defined for $0 \leqslant x \leqslant 1$ such that

$$\int_0^1 \big(\mathrm{g}(x)\big)^2\,\mathrm{d}x = \int_0^1 \mathrm{g}(\sqrt{x})\,\mathrm{d}x - \frac{1}{3}\,.$$

Show that $\displaystyle\int_0^1 \big(\mathrm{g}(x) - x\big)^2\,\mathrm{d}x = 0$ and explain why $\mathrm{g}(x) = x$ for $0 \leqslant x \leqslant 1$.

(iii) Let h be a continuous function defined for $0 \leqslant x \leqslant 1$ with derivative h' such that

$$\int_0^1 \big(\mathrm{h}'(x)\big)^2\,\mathrm{d}x = 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\,\mathrm{d}x - \frac{1}{3}\,.$$

Given that $\mathrm{h}(0) = 0$, find h.

(iv) Let k be a continuous function defined for $0 \leqslant x \leqslant 1$ and a be a real number, such that

$$\int_0^1 \mathrm{e}^{ax}\big(\mathrm{k}(x)\big)^2\,\mathrm{d}x = 2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4}\,.$$

Show that a must be equal to 2 and find k.

Examiner's report

This was the third most popular question and was only marginally less successfully attempted than questions 1 and 11 with over 9.4/20 the mean mark. It was generally answered well by candidates, with many candidates earning more than half the marks for this question and many candidates earning full, or close to full, marks.

The majority of candidates successfully earned full credit in part **(i)**.

They also did well on part **(ii)** though quite a number did not use $(\mathrm{g}(x) - x)^2 \geqslant 0$ when justifying why $\mathrm{g}(x) = x$, or incorrectly stated that $(\mathrm{g}(x) - x)^2 > 0$.

In part **(iii)**, candidates who integrated $2\int_0^1 x\mathrm{h}'(x)\,\mathrm{d}x$ by parts generally went on to earn full, or close to full, marks for this part. A number of candidates began by writing down the equation from **(ii)** with $\mathrm{h}(x)$ in place of $\mathrm{g}(x)$. In some cases, candidates successfully 'worked backwards', cancelling down each side to verify their initial equality, however less successful attempts simply assumed the initial equation, without justification.

Many candidates observed that part **(iv)** could be solved by considering the integral $\int_0^1 \left(e^{\frac{1}{2}ax}k(x) - e^{-\frac{1}{2}ax}\right)^2 \, dx$. Sadly, many candidates failed to factorise the resulting quadratic in $\frac{1}{a}$, and another common error was simply to set the quadratic equal to zero without valid justification. Many, too, tried (unsuccessfully) to solve using integration by parts.

Solution

(i) Let $u = \sqrt{x}$. We have:

$$\frac{du}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} \implies \frac{dx}{du} = 2\sqrt{x}$$

$$\begin{aligned}\int_0^1 f\left(\sqrt{x}\right) \, dx &= \int_0^1 f(u) \times 2u \, du \\ &= 2\int_0^1 x f(x) \, dx\end{aligned}$$

(ii) We have:

$$\begin{aligned}\int_0^1 \left(g(x) - x\right)^2 dx &= \int_0^1 \left(g(x)\right)^2 dx - 2\int_0^1 x g(x) \, dx + \int_0^1 x^2 \, dx \\ &= \int_0^1 g(\sqrt{x}) \, dx - \frac{1}{3} - 2\int_0^1 x g(x) \, dx + \left[\frac{1}{3}x^3\right]_0^1 \\ &= \int_0^1 g(\sqrt{x}) \, dx - \frac{1}{3} - \int_0^1 g(\sqrt{x}) \, dx + \frac{1}{3} \\ &= 0\end{aligned}$$

We know that $\left(g(x) - x\right)^2 \geqslant 0$, and so the area under the graph of $y = \left(g(x) - x\right)^2$ must be greater than or equal to zero.

However we also know that the area under the graph between $x = 0$ and $x = 1$ is equal to zero, which can only be the case if $\left(g(x) - x\right)^2 = 0$ in the range $0 \leqslant x \leqslant 1$. Therefore we have $g(x) = x$ for $0 \leqslant x \leqslant 1$.

(iii) It might be a good idea to try the same sort of idea as in the previous part. We have:

$$\int_0^1 \left(h'(x) - x\right)^2 dx = \int_0^1 \left(h'(x)\right)^2 dx - 2\int_0^1 x h'(x) \, dx + \int_0^1 x^2 \, dx$$

We are given that:

$$\int_0^1 \left(h'(x)\right)^2 dx = 2h(1) - 2\int_0^1 h(x) \, dx - \frac{1}{3}$$

and from before we know that $\int_0^1 x^2 \, dx = \frac{1}{3}$. Using integration by parts we have:

$$\int_0^1 x h'(x) \, dx = \left[x h(x)\right]_0^1 - \int_0^1 h(x) \, dx = h(1) - \int_0^1 h(x) \, dx$$

Putting these parts together we have:

$$\begin{aligned}\int_0^1 \left(\mathrm{h}'(x) - x\right)^2 \mathrm{d}x &= \int_0^1 \left(\mathrm{h}'(x)\right)^2 \mathrm{d}x - 2\int_0^1 x\mathrm{h}'(x)\,\mathrm{d}x + \int_0^1 x^2\,\mathrm{d}x \\ &= 2\mathrm{h}(1) - 2\int_0^1 \mathrm{h}(x)\,\mathrm{d}x - \frac{1}{3} - 2\left[\mathrm{h}(1) - \int_0^1 \mathrm{h}(x)\,\mathrm{d}x\right] + \frac{1}{3} \\ &= 0\end{aligned}$$

In the same way as before this means that we have $\mathrm{h}'(x) = x$ for $0 \leqslant x \leqslant 1$, and since $\mathrm{h}(0) = 0$ integrating gives $\mathrm{h}(x) = \frac{1}{2}x^2$ for $0 \leqslant x \leqslant 1$.

(iv) It feels as if a squaring trick might be useful again. Considering $\left(\mathrm{e}^{\frac{ax}{2}}\mathrm{k}(x) - \mathrm{q}(x)\right)^2$ we have:

$$\left(\mathrm{e}^{\frac{ax}{2}}\mathrm{k}(x) - \mathrm{q}(x)\right)^2 = \mathrm{e}^{ax}\left(\mathrm{k}(x)\right)^2 - 2\mathrm{e}^{\frac{ax}{2}}\mathrm{q}(x)\mathrm{k}(x) + \left(\mathrm{q}(x)\right)^2$$

We would quite like the cross term to look something like $2\mathrm{k}(x)$, so that the given result might be useful, so take $\mathrm{e}^{\frac{ax}{2}}\mathrm{q}(x) = 1$, i.e. consider $\left(\mathrm{e}^{\frac{ax}{2}}\mathrm{k}(x) - \mathrm{e}^{\frac{-ax}{2}}\right)^2$. We have:

$$\begin{aligned}&\int_0^1 \left(\mathrm{e}^{\frac{ax}{2}}\mathrm{k}(x) - \mathrm{e}^{\frac{-ax}{2}}\right)^2 \mathrm{d}x \\ =&\int_0^1 \mathrm{e}^{ax}\left(\mathrm{k}(x)\right)^2 \mathrm{d}x - 2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \int_0^1 \mathrm{e}^{-ax}\,\mathrm{d}x \\ =&2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - 2\int_0^1 \mathrm{k}(x)\,\mathrm{d}x + \left[-\frac{1}{a}\mathrm{e}^{-ax}\right]_0^1 \\ =&\frac{\mathrm{e}^{-a}}{a} - \frac{1}{a^2} - \frac{1}{4} - \frac{\mathrm{e}^{-a}}{a} + \frac{1}{a} \\ =&-\frac{4 + a^2 - 2a}{4a} \\ =&-\frac{(a-2)^2}{4a}\end{aligned}$$

We know that $\int_0^1 \left(\mathrm{e}^{\frac{ax}{2}}\mathrm{k}(x) - \mathrm{e}^{\frac{-ax}{2}}\right)^2 \mathrm{d}x \geqslant 0$, but also that $-\frac{(a-2)^2}{4a} \leqslant 0$. Hence they must both be equal to zero and we have $a = 2$.

We also have $\mathrm{e}^{\frac{ax}{2}}\mathrm{k}(x) - \mathrm{e}^{\frac{-ax}{2}} = 0$ for $0 \leqslant x \leqslant 1$, and so $\mathrm{k}(x) = \mathrm{e}^{-ax} = \mathrm{e}^{-2x}$ for $0 \leqslant x \leqslant 1$.

Question 8

8 If

$$y = \begin{cases} \mathrm{k}_1(x) & x \leqslant b \\ \mathrm{k}_2(x) & x \geqslant b \end{cases}$$

with $k_1(b) = k_2(b)$, then y is said to be *continuously differentiable* at $x = b$ if $k_1'(b) = k_2'(b)$.

(i) Let $\mathrm{f}(x) = x\mathrm{e}^{-x}$. Verify that, for all real x, $y = \mathrm{f}(x)$ is a solution to the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$$

and that $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Show that $\mathrm{f}'(x) \geqslant 0$ for $x \leqslant 1$.

(ii) You are given the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + y = 0$$

where $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$. Let

$$y = \begin{cases} \mathrm{g}_1(x) & x \leqslant 1 \\ \mathrm{g}_2(x) & x \geqslant 1 \end{cases}$$

be a solution of the differential equation which is continuously differentiable at $x = 1$.

Write down an expression for $\mathrm{g}_1(x)$ and find an expression for $\mathrm{g}_2(x)$.

(iii) State the geometrical relationship between the curves $y = \mathrm{g}_1(x)$ and $y = \mathrm{g}_2(x)$.

(iv) Prove that if $y = \mathrm{k}(x)$ is a solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0$$

in the interval $r \leqslant x \leqslant s$, where p and q are constants, then, in a suitable interval which you should state, $y = \mathrm{k}(c - x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - p\frac{\mathrm{d}y}{\mathrm{d}x} + qy = 0\,.$$

(v) You are given the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\left|\frac{\mathrm{d}y}{\mathrm{d}x}\right| + 2y = 0$$

where $y = 0$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$ when $x = 0$.

Let $\mathrm{h}(x) = \mathrm{e}^{-x}\sin x$. Show that $\mathrm{h}'\big(\frac{1}{4}\pi\big) = 0$.

It is given that $y = \mathrm{h}(x)$ satisfies the differential equation in the interval $-\frac{3}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$ and that $\mathrm{h}'(x) \geqslant 0$ in this interval.

In a solution to the differential equation which is continuously differentiable at $(n+\frac{1}{4})\pi$ for all $n \in \mathbb{Z}$, find y in terms of x in the intervals

(a) $\frac{1}{4}\pi \leqslant x \leqslant \frac{5}{4}\pi$,

(b) $\frac{5}{4}\pi \leqslant x \leqslant \frac{9}{4}\pi$.

Examiner's report

This was only a little less popular than question 2 and was only answered with moderate success having a mean score of 7.3/20.

Part **(i)** was mostly well answered, although some candidates lost marks by not being thorough in demonstrating the inequality, or by using a characteristic equation and failing to verify their solution as required.

Part **(ii)** was found difficult, for although g_1 was almost always stated, many struggled to find g_2, often just flipping the sign of g_1. Even when a general solution was found, many candidates used the boundary conditions of **(i)** instead of appreciating the sign change of the derivative at $x = 1$.

Part **(iii)** was done extremely poorly, even by candidates who had the right functions and the algebraic relationship between them.

Candidates could often pick up marks on part **(iv)**, even if they were less successful with the rest of the question, though notation of what derivatives were being taken was often ambiguous.

In attempting to answer part **(v)**, many candidates knew they needed to use the result of part **(iv)** for part **(a)**, although a significant number lost marks for giving no explanation or working. Part **(b)** proved much harder than part **(a)**, since few candidates realized they had to match their function at $\frac{5}{4}\pi$, not at $-\frac{1}{4}\pi$ again. Some candidates fully solved both **(a)** and **(b)** directly via the characteristic equation which led to a very lengthy solution.

Solution

This looks like a very long question (I can't think of another example where the STEP question goes onto a second page!). However there is a lot of information given and the question has been spaced out to help increase readability. Don't be put off by a long question!

The stem of the question is introducing the idea of a function being *continuously differentiable*. The condition $k_1(b) = k_2(b)$ ensures that the function itself is continuous, i.e. there are no jumps in the graph. The condition $k_1'(b) = k_2'(b)$ means that the function is smooth as you pass over $x = b$, i.e. there are no sharp "kinks" in the graph (such as the one at $x = 0$ for $y = |x|$ — this function would not be continuously differentiable).

(i) Verify means "stick it in and show it works". We have

$$\begin{aligned} y &= x\mathrm{e}^{-x} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \mathrm{e}^{-x} - x\mathrm{e}^{-x} \\ \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= -\mathrm{e}^{-x} - \mathrm{e}^{-x} + x\mathrm{e}^{-x} \\ &= x\mathrm{e}^{-x} - 2\mathrm{e}^{-x} \end{aligned}$$

This gives us:

$$\begin{aligned} &\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y \\ =&\left(x\mathrm{e}^{-x} - 2\mathrm{e}^{-x}\right) + 2\left(\mathrm{e}^{-x} - x\mathrm{e}^{-x}\right) + x\mathrm{e}^{-x} \\ =&(x - 2x + x)\mathrm{e}^{-x} + (-2 + 2)\mathrm{e}^{-x} = 0 \end{aligned}$$

So the function satisfies the differential equation. We also have:

$$x = 0 \implies y = 0\mathrm{e}^0 = 0$$
$$x = 0 \implies \frac{\mathrm{d}y}{\mathrm{d}x} + \mathrm{e}^0 - 0\mathrm{e}^0 = 1$$

and so the function also satisfies the initial conditions.

We have $\mathrm{f}'(x) = (1 - x)\mathrm{e}^{-x}$. We know that $\mathrm{e}^{-x} > 0$, so if $1 - x \geqslant 0$ we have $\mathrm{f}'(x) \geqslant 0$. Therefore we have $\mathrm{f}'(x) \geqslant 0$ for $x \leqslant 1$.

There is probably more detail here than was needed!

(ii) From part **(i)** we know that when $\frac{\mathrm{d}y}{\mathrm{d}x}$ is non-negative then the function from the previous part solves the equation. We have $\mathrm{g}_1(x) = x\mathrm{e}^{-x}$ which satisfies $\frac{\mathrm{d}y}{\mathrm{d}x} \geqslant 0$ for $x \leqslant 1$.

Note that the question says "Write down an expression for $\mathrm{g}_1(x)$". This implies that there is no working necessary!

For $\mathrm{g}_2(x)$ we need to find a solution to $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0$.

We also need y to be continuously differentiable at $x = 1$, so we need $\mathrm{g}_1(1) = \mathrm{g}_2(1)$ and $\mathrm{g}_1'(1) = \mathrm{g}_2'(1)$.

It is likely that $g_2(x)$ will have a similar form to $g_1(x)$, so try using $y = (ax+b)e^{kx}$. This then gives:

$$\begin{aligned}
&\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y \\
=& \left[kae^{kx} + k^2(ax+b)e^{kx} + kae^{kx}\right] - 2\left[ae^{kx} + k(ax+b)e^{kx}\right] + (ax+b)e^{kx} \\
=& x\left[k^2a - 2ka + a\right]e^{kx} + \left[2ka + k^2b - 2a - 2kb + b\right]e^{kx} \\
=& ax(k-1)^2e^{kx} + 2a(k-1)e^{kx} + b(k-1)^2e^{kx}
\end{aligned}$$

Therefore if we take $k = 1$ we have a solution to the differential equation with $g_2(x) = (ax+b)e^x$.

Using $g_1(1) = g_2(1)$ and $g_1'(1) = g_2'(1)$ gives:

$$\begin{aligned}
g_1(1) = g_2(1) &\implies e^{-1} = (a+b)e^1 \\
g_1'(1) = g_2'(1) &\implies 0 = (2a+b)e^1
\end{aligned}$$

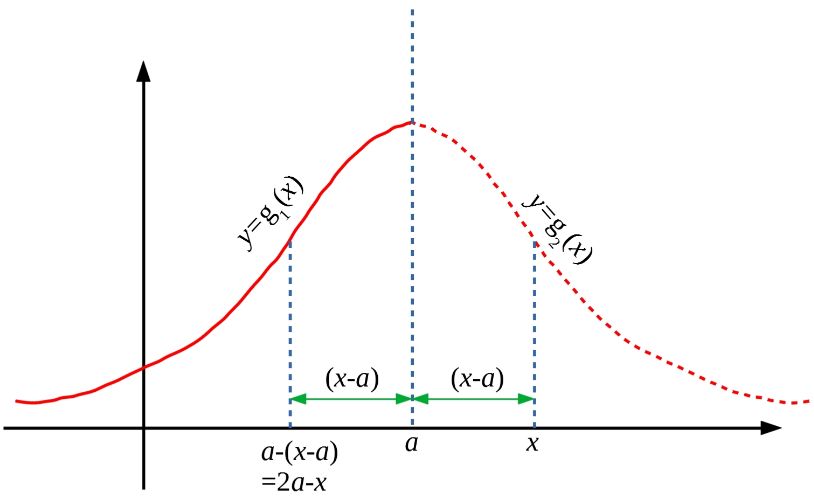
The second equation gives $b = -2a$ and so the first equation becomes:

$$\begin{aligned}
a + b &= e^{-2} \\
\implies a &= -e^{-2}
\end{aligned}$$

and so we have $g_2(x) = (-e^{-2}x + 2e^{-2})e^x = (2-x)e^{x-2}$.

(iii) This question says "state" so it shouldn't need a lot of work. We have $g_1(x) = xe^{-x}$ and $g_2(x) = (2-x)e^{-(2-x)}$. This means that $y = g_2(x)$ is a reflection in the line $x = 1$ of $y = g_1(x)$.

In general if function $g_2(x)$ is the reflection of $g_1(x)$ in the line $x = a$ then we have the relationship $g_2(x) = g_1(2a - x)$. The diagram below can be used to demonstrate why this is true. The opposite also holds, i.e. if we have $g_2(x) = g_1(2a-x)$ then function $g_2(x)$ is the reflection of $g_1(x)$ in the line $x = a$.



(iv) A little bit of care is needed here, as k is a function, not a constant as it usually is! In fact if you look closely you can see that this k is not in italics (constants are usually italicised and functions usually not!).

The definition $y = \mathrm{k}(c - x)$ means that the function k is being evaluated at $c - x$.

Let $y = k(c - x)$ then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{k}'(c - x)$ (using the chain rule) and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{k}''(c - x)$.

Substituting into the differential equation gives:

$$\begin{aligned}&\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - p\frac{\mathrm{d}y}{\mathrm{d}x} + qy\\ =&\mathrm{k}''(c-x) - \big[-p\mathrm{k}'(c-x)\big] + q\mathrm{k}(c-x)\\ =&\mathrm{k}''(c-x) + p\mathrm{k}'(c-x) + q\mathrm{k}(c-x)\end{aligned}$$

This is then equal to 0 as long as $r \leqslant c - x \leqslant s$, which can be rearranged to $c - s \leqslant x \leqslant c - r$.

(v) We have $\mathrm{h}(x) = \mathrm{e}^{-x}\sin x$. Differentiating gives:

$$\begin{aligned}\mathrm{h}'(x) &= -\mathrm{e}^{-x}\sin x + \mathrm{e}^{-x}\cos x\\ \mathrm{h}'\left(\frac{\pi}{4}\right) &= -\mathrm{e}^{-\frac{\pi}{4}}\sin\tfrac{\pi}{4} + \mathrm{e}^{-\frac{\pi}{4}}\cos\tfrac{\pi}{4}\\ &= -\frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\pi}{4}} + \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\pi}{4}}\\ &= 0\end{aligned}$$

We are told that $y = \mathrm{h}(x)$ satisfies the differential equation in the interval $-\frac{3}{4}\pi \leqslant x \leqslant \frac{1}{4}\pi$ and that $\mathrm{h}'(x) \geqslant 0$ in this interval, so in the relevant interval the differential equation is actually:

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + 2y = 0$$

(a) Using the ideas in part **(iv)**, if we reflect the function $\mathrm{h}(x)$ in the line $x = \frac{1}{4}\pi$ then we will get a solution which satisfies the differential equation in the interval $\frac{1}{4}\pi \leqslant x \leqslant \frac{5}{4}\pi$.

The function $y = \mathrm{e}^{-\left(\frac{1}{2}\pi - x\right)}\sin\left(\frac{1}{2}\pi - x\right)$ is the reflection of $\mathrm{h}(x)$ in the line $x = \frac{1}{4}\pi$, and we we substitute $c = \frac{1}{2}\pi$ into the range $c - s \leqslant x \leqslant c - r$ from part **(iv)** we get $\frac{1}{4}\pi \leqslant x \leqslant \frac{5}{4}\pi$. We can simplify the function a little bit to get $\mathrm{h}_2(x) = \mathrm{e}^{-\left(\frac{1}{2}\pi - x\right)}\cos x$

We also need this function to be continuously differentiable. Since we know that $\mathrm{h}'\left(\frac{1}{4}\pi\right) = 0$ we know that the gradient over the "boundary" is 0, and so reflecting in this line will not lead to a pointy graph, and the join between the two sections will be smooth (i.e. it is continuously differentiable).

(b) We can repeat the same process, but this time we reflect the function from part **(v)(a)** in the line $x = \frac{5}{4}\pi$. This means that the function here will be $\mathrm{h}_3(x) = \mathrm{h}_2\left(\frac{5}{2}\pi - x\right)$. This gives:

$$\begin{aligned}y &= \mathrm{e}^{-\left(\frac{1}{2}\pi - \left[\frac{5}{2}\pi - x\right]\right)}\cos\left[\tfrac{5}{2}\pi - x\right]\\ &= \mathrm{e}^{2\pi - x}\sin x\end{aligned}$$

We also need to check that we have zero gradient at the point $x = \frac{5}{4}\pi$. Differentiating $\mathrm{h}_3(x)$ gives:

$$\begin{aligned} \mathrm{h}_3'(x) &= -\mathrm{e}^{2\pi - x}\sin x + \mathrm{e}^{2\pi - x}\cos x \\ \mathrm{h}_3'\left(\tfrac{5}{4}\pi\right) &= -\mathrm{e}^{2\pi - \frac{5}{4}\pi}\sin\tfrac{5}{4}\pi + \mathrm{e}^{2\pi - \frac{5}{4}\pi}\cos\tfrac{5}{4}\pi \\ &= \mathrm{e}^{-\frac{3}{4}\pi}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) \\ &= 0 \end{aligned}$$

Therefore the solution is continuously differentiable at $x = \frac{5}{4}\pi$.

Question 9

9 Two particles, A of mass m and B of mass M, are fixed to the ends of a light inextensible string AB of length r and lie on a smooth horizontal plane. The origin of coordinates and the x- and y-axes are in the plane.

Initially, A is at $(0, 0)$ and B is $(r, 0)$. B is at rest and A is given an instantaneous velocity of magnitude u in the positive y direction.

At a time t after this, A has position (x, y) and B has position (X, Y). You may assume that, in the subsequent motion, the string remains taut.

(i) Explain by means of a diagram why

$$X = x + r\cos\theta$$
$$Y = y - r\sin\theta$$

where θ is the angle *clockwise* from the positive x-axis of the vector $\overrightarrow{AB}$.

(ii) Find expressions for $\dot{X}$, $\dot{Y}$, $\ddot{X}$ and $\ddot{Y}$ in terms of $\ddot{x}$, $\ddot{y}$, $\dot{x}$, $\dot{y}$, r, $\ddot{\theta}$, $\dot{\theta}$ and θ, as appropriate.

Assume that the tension T in the string is the only force acting on either particle.

(iii) Show that

$$\ddot{x}\sin\theta + \ddot{y}\cos\theta = 0$$
$$\ddot{X}\sin\theta + \ddot{Y}\cos\theta = 0$$

and hence that $\theta = \dfrac{ut}{r}$.

(iv) Show that

$$m\ddot{x} + M\ddot{X} = 0$$
$$m\ddot{y} + M\ddot{Y} = 0$$

and find $my + MY$ in terms of t and m, M, u, r as appropriate.

(v) Show that

$$y = \frac{1}{m+M}\left(mut + Mr\sin\left(\frac{ut}{r}\right)\right).$$

(vi) Show that, if $M > m$, then the y component of the velocity of particle A will be negative at some time in the subsequent motion.

Examiner's report

This question only just beat question 11 to be the least popular question on the paper. Although its mean score was only 6.9/20, it was more successfully attempted than two of the Pure questions and the other Mechanics question.

Parts **(i)** and **(ii)** were well answered by a good number of candidates, if candidates once set up the problem and then got going. The diagrams were done well, and the derivatives didn't pose much of a problem for most, although there were some errors in the second derivatives and applying the chain rule, with candidates forgetting to multiply by $\dot{\theta}$ to produce $\ddot{\theta}$.

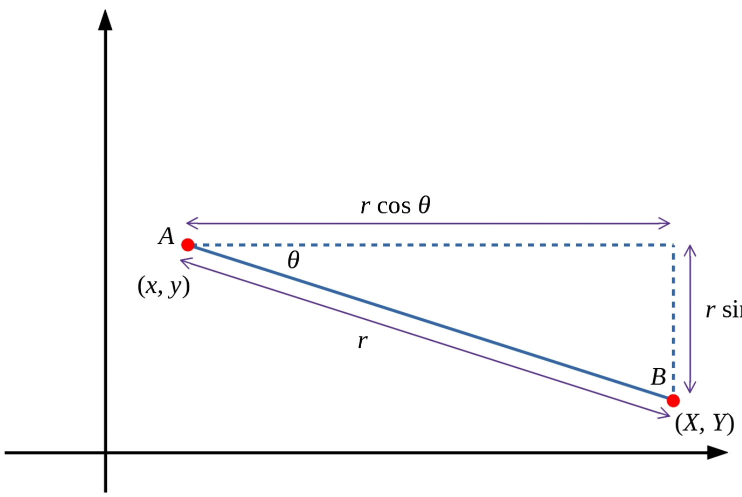
Part **(iii)** was more mixed in terms of good responses. Those who did this by resolving forces horizontally and vertically set up the remainder of the question well, but some seemed to struggle with the first part and just assumed the equations were true. The biggest problem for a large number of candidates here was applying boundary conditions when integrating and justifying the choice of boundary conditions. Parts **(iv)** and **(v)** were answered fairly well.

Part **(vi)** lead to a lot of marks lost, as they were required to justify the velocity being negative by finding a suitable time to ensure it happens, which most did not do.

Solution

This is another question which looks rather long, and it has six separate parts!

(i) The question asks for a diagram, so:



From this you can see that the x coordinate of B is given by $X = x + r\cos\theta$ and the y coordinate of B is given by $Y = y - r\sin\theta$.

(ii) Differentiating the expression for X gives:

$$\dot{X} = \dot{x} - r\dot{\theta}\sin\theta$$
$$\ddot{X} = \ddot{x} - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta$$

and differentiating the expression for Y gives:

$$\dot{Y} = \dot{y} - r\dot{\theta}\cos\theta$$
$$\ddot{Y} = \ddot{y} - r\ddot{\theta}\cos\theta + r\dot{\theta}^2\sin\theta$$

(iii) Using "F=ma" horizontally for particle A gives:

$$m\ddot{x} = T\cos\theta \qquad (1)$$

and vertically for A gives:

$$m\ddot{y} = -T\sin\theta \qquad (2)$$

Consider $\sin\theta \times (1) + \cos\theta \times (2)$:

$$m\ddot{x}\sin\theta + m\ddot{y}\cos\theta = T\cos\theta\sin\theta - T\cos\theta\sin\theta = 0$$

Therefore we have $\ddot{x}\sin\theta + \ddot{y}\cos\theta = 0$.

Similarly we can use "$F = ma$" on particle B to get:

$$M\ddot{X} = -T\cos\theta \qquad \text{horizontal} \qquad (3)$$
$$M\ddot{Y} = T\sin\theta \qquad \text{vertical} \qquad (4)$$

Then $\sin\theta \times (3) + \cos\theta \times (4)$ gives:

$$m\ddot{X}\sin\theta + m\ddot{Y}\cos\theta = -T\cos\theta\sin\theta + T\cos\theta\sin\theta = 0$$

Therefore we have $\ddot{X}\sin\theta + \ddot{Y}\cos\theta = 0$.

Using $\ddot{X}\sin\theta + \ddot{Y}\cos\theta = 0$ and substituting for $\ddot{X}$ and $\ddot{Y}$ we have:

$$\left(\ddot{x} - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta\right)\sin\theta + \left(\ddot{y} - r\ddot{\theta}\cos\theta + r\dot{\theta}^2\sin\theta\right)\cos\theta = 0$$
$$\cancel{\ddot{x}\sin\theta + \ddot{y}\cos\theta} - r\ddot{\theta}\left(\sin^2\theta + \cos^2\theta\right) - \cancel{r\dot{\theta}^2\cos\theta\sin\theta} + \cancel{r\dot{\theta}^2\sin\theta\cos\theta} = 0$$
$$0 - r\ddot{\theta} + 0 = 0$$

and so we have $r\ddot{\theta} = 0$. Dividing by r and integrating with respect to time gives $\theta = kt + c$, and as initially we have $\theta = 0$ we have $\theta = kt$.

The initial velocity of A is u in the positive y direction, so this gives $r\dot{\theta} = u$ initially, and so $k = \dfrac{u}{r}$ and we have $\theta = \dfrac{ut}{r}$.

(iv) From equations (1) and (3) from the previous part we have $m\ddot{x} + M\ddot{X} = 0$, and from equations (2) and (4) we have $m\ddot{y} + M\ddot{Y} = 0$.

Integrating $m\ddot{y} + M\ddot{Y} = 0$ we have $m\dot{y} + M\dot{Y} = c$. Using the initial conditions we have $\dot{y} = u$ and so $m\dot{y} + M\dot{Y} = mu$.

Integrating a second time and using the fact that both particles have y co-ordinates of 0 we have $my + MY = mut$.

(iv) Substituting for Y (from part **(i)**) in this last equation gives:

$$\begin{aligned} my + MY &= mut \\ my + M\big(y - r\sin\theta\big) &= mut \\ \big(m + M\big)y &= mut + Mr\sin\theta \\ y &= \frac{1}{m+M}\big(mut + Mr\sin\theta\big) \\ y &= \frac{1}{m+M}\left(mut + Mr\sin\left(\frac{ut}{r}\right)\right) \end{aligned}$$

(vi) The "y component of the velocity of particle A" is $\dot{y}$, which is given by:

$$\begin{aligned} \dot{y} &= \frac{1}{m+M}\left(mu + Mr \times \frac{u}{r}\cos\left(\frac{ut}{r}\right)\right) \\ \dot{y} &= \frac{1}{m+M}\left(mu + Mu\cos\left(\frac{ut}{r}\right)\right) \end{aligned}$$

The minimum value of $\cos\theta$ is -1, which occurs when $t = \dfrac{\pi r}{u}$ (as well as many other values of t). This means we have:

$$\dot{y}_{\text{min}} = \frac{1}{m+M}\left(mu - Mu\right)$$

and so if $M > m$ we have:

$$\dot{y}_{\text{min}} = \frac{(m-M)u}{m+M} < 0$$

as required.

Question 10

10 A thin uniform beam AB has mass $3m$ and length $2h$. End A rests on rough horizontal ground and the beam makes an angle of 2β to the vertical, supported by a light inextensible string attached to end B. The coefficient of friction between the beam and the ground at A is μ.

The string passes over a small frictionless pulley fixed to a point C which is a distance $2h$ vertically above A. A particle of mass km, where $k < 3$, is attached to the other end of the string and hangs freely.

(i) Given that the system is in equilibrium, find an expression for k in terms of β and show that $k^2 \leqslant \dfrac{9\mu^2}{\mu^2+1}$.

(ii) A particle of mass m is now fixed to the beam at a distance xh from A, where $0 \leqslant x \leqslant 2$. Given that $k = 2$, and that the system is in equilibrium, show that

$$\frac{F^2}{N^2} = \frac{x^2+6x+5}{4(x+2)^2},$$

where F is the frictional force and N is the normal reaction on the beam at A.

By considering $\frac{1}{3} - \dfrac{F^2}{N^2}$, or otherwise, find the minimum value of μ for which the beam can be in equilibrium whatever the value of x.

Examiner's report

The most popular of the Applied questions, it was also the least successfully attempted, and was only slightly better attempted than question 4. If a candidate found this question difficult, it tended to be from the start, failing to draw a correct diagram of what was going on. If they did set up the problem correctly then finding different angles in terms of the given angle β proved problematic. This meant that often sin was found instead of cos and vice versa, within attempted solutions.

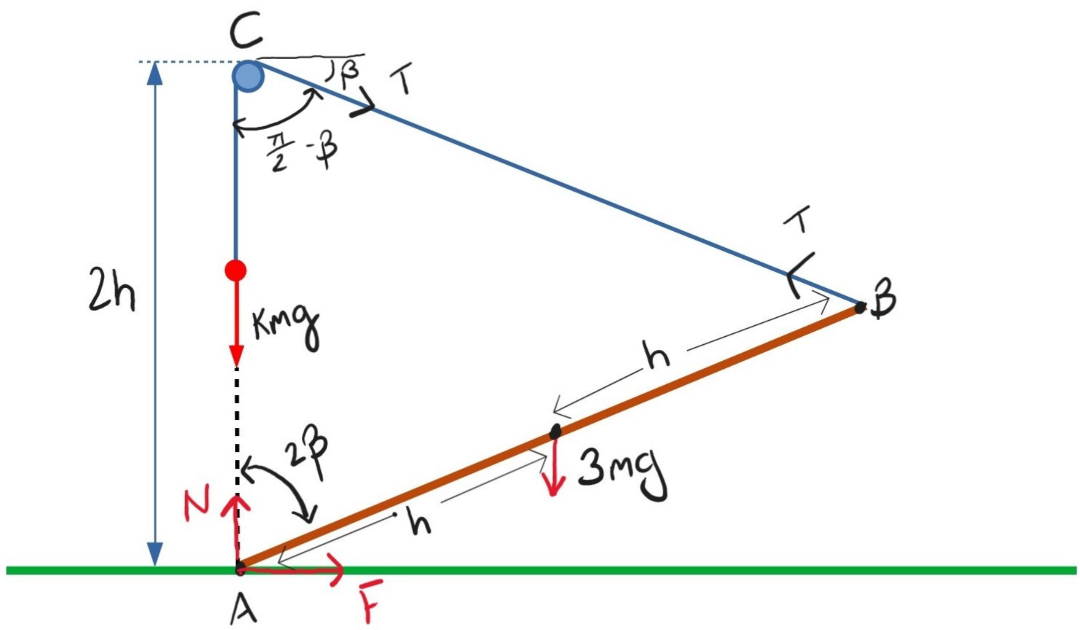
Many successfully took moments about A and resolved forces vertically and horizontally, but most were unable to use these to produce the inequality required in part **(i)**.

Those few that did well in part **(i)** generally also did well in part **(ii)**, with only one adjustment to the diagram needed, and the resulting algebra was worked through with little issue. The hint for the final part was used well by candidates, but only a few managed to turn the resulting square into a minimum value for μ.

Solution

The first thing to do read the whole question! In part **(ii)** it gives letters for the frictional and normal reaction forces, so it actually makes sense to use these letters from the start.

Next step is to draw a diagram. Note that the angle 2β is measured from the horizontal which is slightly unusual.



The triangle ABC is an isosceles triangle, so we have $\angle ACB = \angle ABC = \frac{\pi}{2} - \beta$. The string BC makes an angle of β with the horizontal.

(i) Often it's a good idea to start by writing down some equations and numbering them so we can refer to them later.

The pulley is frictionless so we have:

$$T = kmg \tag{1}$$

Resolving horizontally for forces on the beam:

$$F = T\cos\beta \tag{2}$$

Resolving vertically for forces on the beam:

$$3mg = N + T\sin\beta \tag{3}$$

Taking moments for the beam about point A gives:

$$3mgh\sin 2\beta = 2hT\cos\beta \tag{4}$$

Substituting $T = kmg$ into equation (4) gives:

$$\begin{aligned} 3mgh\sin 2\beta &= 2hkmg\cos\beta \\ 3\sin 2\beta &= 2k\cos\beta \\ 6\sin\beta\cos\beta &= 2k\cos\beta \\ \implies k &= 3\sin\beta \end{aligned} \tag{5}$$

For the inequality we know that we have $F \leqslant \mu N$.

Using equation (4) we have:

$$\begin{aligned} N &= 3mg - T\sin\beta \\ &= 3mg - kmg\sin\beta \\ &= 3mg - 3mg\sin^2\beta \qquad \text{using (5)} \\ &= 3mg\cos^2\beta \end{aligned}$$

so this gives:

$$\begin{aligned} F &\leqslant \mu N \\ T\cos\beta &\leqslant 3\mu mg\cos^2\beta \\ kmg\cos\beta &\leqslant 3\mu mg\cos^2\beta \\ k &\leqslant 3\mu\cos\beta \qquad \text{as } mg\cos\beta > 0 \end{aligned}$$

As both sides are positive we can square the inequality to get:

$$\begin{aligned} k^2 &\leqslant 9\mu^2\cos^2\beta \\ k^2 &\leqslant 9\mu^2 - 9\mu^2\sin^2\beta \\ k^2 &\leqslant 9\mu^2 - \mu^2k^2 \\ (1+\mu^2)k^2 &\leqslant 9\mu^2 \\ k^2 &\leqslant \frac{9\mu^2}{1+\mu^2} \end{aligned}$$

(ii) We now have an extra mass added to the beam at some point between A and B. This will affect equations (3) and (4) from part **(i)**.

Equations (1) and (2) gives $F = kmg\cos\beta = 2mg\cos\beta$ (as we are now told that $k = 2$).

Resolving vertically for the beam now gives:

$$3mg + mg = N + 2mg\sin\beta \tag{6}$$

The moment equation becomes:

$$\begin{aligned} 3mgh\sin 2\beta + mgxh\sin 2\beta &= 2Th\cos\beta \\ 3mgh\sin 2\beta + mgxh\sin 2\beta &= 4mgh\cos\beta \\ 3\sin 2\beta + x\sin 2\beta &= 4\cos\beta \\ 3\sin\beta + x\sin\beta &= 2 \end{aligned} \tag{7}$$

Where the last line uses $\sin 2\beta = 2\sin\beta\cos\beta$.

Equation (7) gives $\sin\beta = \frac{2}{3+x}$. Substituting this into equation (6) gives:

$$\begin{aligned} 4mg &= N + 2mg \times \frac{2}{3+x} \\ \implies N &= 4mg\left(1 - \frac{1}{3+x}\right) \\ N &= 4mg \times \frac{2+x}{3+x} \end{aligned}$$

Using the expressions for F and N we have:

$$\begin{aligned} \frac{F^2}{N^2} &= \frac{4m^2g^2\cos^2\beta}{16m^2g^2} \times \frac{(3+x)^2}{(2+x)^2} \\ &= \frac{(3+x)^2}{4(2+x)^2}\cos^2\beta \\ &= \frac{(3+x)^2}{4(2+x)^2}(1-\sin^2\beta) \\ &= \frac{(3+x)^2}{4(2+x)^2}\left(1 - \frac{4}{(3+x)^2}\right) \\ &= \frac{1}{4(2+x)^2}\left((3+x)^2 - 4\right) \\ &= \frac{x^2+6x+5}{4(2+x)^2} \end{aligned}$$

For the last part you are given a suggested method, and it is often a good idea to use that! We have:

$$\begin{aligned} \frac{1}{3} - \frac{F^2}{N^2} &= \frac{1}{3} - \frac{x^2+6x+5}{4(2+x)^2} \\ &= \frac{4(2+x)^2 - 3(x^2+6x+5)}{12(2+x)^2} \\ &= \frac{4x^2+16x+16-3x^2-18x-15}{12(2+x)^2} \\ &= \frac{x^2-2x+1}{12(2+x)^2} \\ &= \frac{(x-1)^2}{12(2+x)^2} \geqslant 0 \end{aligned}$$

Therefore we have $\frac{F^2}{N^2} \leqslant \frac{1}{3}$ or equivalently $F \leqslant \frac{1}{\sqrt{3}}N$. If we want the system to be in equilibrium for all values of x then we need $\mu \geqslant \frac{1}{\sqrt{3}}$.

Question 11

11 Show that

$$\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k = (x+1)\mathrm{e}^x - 1 \,.$$

In the remainder of this question, n is a fixed positive integer.

(i) Random variable Y has a Poisson distribution with mean n. One observation of Y is taken. Random variable D is defined as follows. If the observed value of Y is zero then $D = 0$. If the observed value of Y is k, where $k \geqslant 1$, then a fair k-sided die (with sides numbered 1 to k) is rolled once and D is the number shown on the die.

(a) Write down $\mathrm{P}(D = 0)$.

(b) Show, from the definition of the expectation of a random variable, that

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} \left[d \sum_{k=d}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \right) \right].$$

Show further that

$$\mathrm{E}(D) = \sum_{k=1}^{\infty} \left(\frac{1}{k} \cdot \frac{n^k}{k!} \mathrm{e}^{-n} \sum_{d=1}^{k} d \right).$$

(c) Show that $\mathrm{E}(D) = \frac{1}{2}(n + 1 - \mathrm{e}^{-n})$.

(ii) Random variables $X_1, X_2, \ldots, X_n$ all have Poisson distributions. For each $k \in \{1, 2, \ldots, n\}$, the mean of X_k is k.

A fair n-sided die, with sides numbered 1 to n, is rolled. When k is the number shown, one observation of X_k is recorded. Let Z be the number recorded.

(a) Find $\mathrm{P}(Z = 0)$.

(b) Show that $\mathrm{E}(Z) > \mathrm{E}(D)$.

Examiner's report

Although it was unpopular, those that attempted this scored better than on other Applied questions and, indeed, five of the Pure questions. Most candidates recognised that the expression on the left in the stem could be separated into two sums, one of which would produce $e^x - 1$ while the other would produce xe^x.

A small number of candidates falsely seemed to assume that since

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x$$

it immediately follows that

$$\sum_{k=0}^{\infty} (k+1)\frac{x^k}{k!} = (x+1)e^x$$

Some candidates chose to write out the sums without using summation notation, which made some parts of the justification more difficult to express clearly.

In part **(i)** most candidates were able to work out the value of $P(D = 0)$ and a large number were able to give some justification of the first formula for $E(D)$. Many however did not manage to justify fully how all parts of the required expression were deduced and in a question in which the answer to be reached is known, it is important that solutions clearly express the steps that are involved. In particular, many candidates simply stated that the sum over values of k ran from d to infinity without any comment that this is because there had to be at least d sides on the die. Similarly, many candidates failed to express the clear reasoning required to show the second form of $E(D)$.

Candidates were generally successful in using the formula for $E(D)$ to complete part **(i)(c)** and the majority recognised the significance of the result in the stem to the work here.

Many of the candidates who attempted part **(ii)** were able to make good progress, although there were some who failed to understand the sequence in which the events take place in this second situation. Most were able to find an initial expression for the value of $P(Z = 0)$ and the majority recognised that this was a sum of a geometric series. However, several candidates calculated the sum to infinity instead of the required sum of n terms and some of those who correctly calculated the sum of n terms then made errors when dealing with the powers in the simplification.

Many of the candidates who attempted to calculate $E(Z)$ were able to reach a correct form, but relatively few recognised that changing the order of summation (as in part **(i)**) would again help to simplify the expression. Those who found the correct expression were generally able to justify that $E(Z) > E(D)$, although some did not comment on the fact that the exponential term must be positive as part of their justification.

Solution For the stem request we have:

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{k+1}{k!} x^k &= \sum_{k=1}^{\infty} \frac{k}{k!} x^k + \sum_{k=1}^{\infty} \frac{1}{k!} x^k \\
&= \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^k + (\mathrm{e}^x - 1) \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} x^{i+1} + (\mathrm{e}^x - 1) \\
&= x\mathrm{e}^x + (\mathrm{e}^x - 1) \\
&= (x+1)\mathrm{e}^x - 1
\end{aligned}$$

(i) **(a)** Note that this part says "write down" so no working necessary!

We have $\mathrm{P}(D = 0) = \mathrm{P}(Y = 0) = \mathrm{e}^{-n}$.

(b) We have:

$$\mathrm{E}(D) = \sum_{d=0}^{\infty} d\mathrm{P}(D = d) = \sum_{d=1}^{\infty} d\mathrm{P}(D = d)$$

Consider $\mathrm{P}(D = 3)$. This could happen if $Y = 3$, and a "three sided die"[1] is rolled and the number 3 comes up or if $Y = 4$ and a 3 appears on the four sided die etc. If we want to have $D = d$ then we need to have $Y \geqslant d$ so that "d" appears on one of the faces of the die.

Using this idea we have:

$$\begin{aligned}
\mathrm{P}(D = d) &= \sum_{k=d}^{\infty} \mathrm{P}(D = d | Y = k) \times \mathrm{P}(Y = k) \\
&= \sum_{k=d}^{\infty} \frac{1}{k} \times \frac{n^k \mathrm{e}^{-n}}{k!}
\end{aligned}$$

So we have:

$$\mathrm{E}(D) = \sum_{d=1}^{\infty} d \left[\sum_{k=d}^{\infty} \frac{1}{k} \times \frac{n^k \mathrm{e}^{-n}}{k!} \right]$$

as required.

For the second part we want to swap the order of summations over, but it's a little tricky as d appears both as an index and as a limit of the other sum. If we consider pairs of indices of the form (d, k) then we have:

$$\begin{aligned}
\sum_{d=1}^{\infty} \left[\sum_{k=d}^{\infty} \right] &= (1,1) + (1,2) + (1,3) + (1,4) + \cdots \\
&\qquad + (2,2) + (2,3) + (2,4) + \cdots + (3,3) + (3,4) + \cdots \\
&= \Big[(1,1)\Big] + \Big[(1,2) + (2,2)\Big] + \Big[(1,3) + (2,3) + (3,3)\Big] \\
&\qquad + \Big[(1,4) + (2,4) + (3,4) + (4,4)\Big] + \cdots \\
&= \sum_{k=1}^{\infty} \left[\sum_{d=1}^{k} \right]
\end{aligned}$$

[1] Perhaps "spinner with 3 sections" might have been easier to visualise.

The expression for the expectation can then be written as:

$$\begin{aligned}
\mathrm{E}(D) &= \sum_{d=1}^{\infty} d\left[\sum_{k=d}^{\infty} \frac{1}{k} \times \frac{n^k \mathrm{e}^{-n}}{k!}\right] \\
&= \sum_{k=1}^{\infty} \left[\sum_{d=1}^{k} d \times \frac{1}{k} \times \frac{n^k \mathrm{e}^{-n}}{k!}\right] \\
&= \sum_{k=1}^{\infty} \left[\frac{1}{k} \times \frac{n^k \mathrm{e}^{-n}}{k!} \sum_{d=1}^{k} d\right]
\end{aligned}$$

(c) Using the fact that the sum of the first n integers is given by $\frac{1}{2}n(n+1)$ we have:

$$\begin{aligned}
\mathrm{E}(D) &= \sum_{k=1}^{\infty} \left[\frac{1}{k} \times \frac{n^k \mathrm{e}^{-n}}{k!} \sum_{d=1}^{k} d\right] \\
&= \sum_{k=1}^{\infty} \left[\frac{1}{k} \times \frac{n^k \mathrm{e}^{-n}}{k!} \times \tfrac{1}{2}k(k+1)\right] \\
&= \tfrac{1}{2}\mathrm{e}^{-n} \sum_{k=1}^{\infty} \frac{(k+1)n^k}{k!} \\
&= \tfrac{1}{2}\mathrm{e}^{-n}\left[(n+1)\mathrm{e}^{n} - 1\right] \quad \text{(using the stem result)} \\
&= \tfrac{1}{2}\left[(n+1) - \mathrm{e}^{-n}\right]
\end{aligned}$$

(ii) In this situation we only have one die (which is an n sided die, where n is a fixed positive integer).

(a) To get $Z = 0$ we first roll the die to choose one of the n Poisson distributions and then see whether that is equal to zero or not. Let random variable Q be the number on the die. We have:

$$\begin{aligned}
\mathrm{P}(Z=0) &= \mathrm{P}(Q=1)\mathrm{P}(X_1=0) + \mathrm{P}(Q=2)\mathrm{P}(X_2=0) \\
&\qquad + \mathrm{P}(Q=3)\mathrm{P}(X_3=0) + \cdots + \mathrm{P}(Q=n)\mathrm{P}(X_n=0) \\
&= \frac{1}{n} \times \mathrm{e}^{-1} + \frac{1}{n} \times \mathrm{e}^{-2} + \frac{1}{n} \times \mathrm{e}^{-3} + \cdots + \frac{1}{n} \times \mathrm{e}^{-n} \\
&= \frac{e^{-1}}{n}\left[1 + \mathrm{e}^{-1} + \mathrm{e}^{-2} + \cdots + \mathrm{e}^{-(n-1)}\right] \\
&= \frac{e^{-1}}{n} \times \frac{1 - \mathrm{e}^{-n}}{1 - \mathrm{e}^{-1}} \quad \text{(using the sum of a GP)} \\
&= \frac{1}{n} \times \frac{1 - \mathrm{e}^{-n}}{e - 1}
\end{aligned}$$

(b) For the expectation of Z we have:

$$\begin{aligned}
\mathrm{E}(Z) &= \sum_{i=1}^{\infty} i\mathrm{P}(Z=i) \\
&= \sum_{i=1}^{\infty} i\left[\sum_{k=1}^{n} \frac{1}{n} \times \mathrm{P}(X_k = i)\right] \\
&= \frac{1}{n}\sum_{k=1}^{n}\left[\sum_{i=1}^{\infty} i\mathrm{P}(X_k = i)\right] \\
&= \frac{1}{n}\sum_{k=1}^{n}[\mathrm{E}(X_k)] \\
&= \frac{1}{n}\sum_{k=1}^{n} k \\
&= \frac{1}{n} \times \frac{1}{2}n(n+1) \\
&= \frac{1}{2}(n+1)
\end{aligned}$$

Note that for this compound sum the limits are independent of the indices so we can just change the order without difficulty. Note further that we have used that fact that the expectation of X_k is equal to k (i.e. the mean of the distribution!). It is possible to prove that this is true, or you can quote and use it.

We are asked to show that $\mathrm{E}(Z) > \mathrm{E}(D)$. We have:

$$\begin{aligned}
\mathrm{E}(Z) - \mathrm{E}(D) &= \frac{1}{2}(n+1) - \frac{1}{2}(n+1-\mathrm{e}^{-n}) \\
&= \frac{1}{2}\mathrm{e}^{-n} > 0
\end{aligned}$$

and so we have $\mathrm{E}(Z) > \mathrm{E}(D)$.

Question 12

12 A drawer contains n pairs of socks. The two socks in each pair are indistinguishable, but each pair of socks is a different colour from all the others. A set of $2k$ socks, where k is an integer with $2k \leqslant n$, is selected at random from this drawer: that is, every possible set of $2k$ socks is equally likely to be selected.

(i) Find the probability that, among the socks selected, there is no pair of socks.

(ii) Let $X_{n,k}$ be the random variable whose value is the number of pairs of socks found amongst those selected. Show that

$$\mathrm{P}(X_{n,k} = r) = \frac{\binom{n}{r}\binom{n-r}{2(k-r)}2^{2(k-r)}}{\binom{2n}{2k}}$$

for $0 \leqslant r \leqslant k$.

(iii) Show that

$$r\mathrm{P}(X_{n,k} = r) = \frac{k(2k-1)}{2n-1}\mathrm{P}(X_{n-1,k-1} = r-1)\,,$$

for $1 \leqslant r \leqslant k$, and hence find $\mathrm{E}(X_{n,k})$.

Examiner's report

This was only a little more popular than questions 9 and 11, and it was the seventh most successfully attempted with a mean score of 7.4/20 . While there were a number of attempts that did not manage to make any significant progress, those who were able to analyse the situation were able to make very good progress.

When finding the probability in part **(i)**, the most common methods employed were to count the number of ways in which the outcome could be achieved or to create a product of individual probabilities by considering the socks being taken one at a time.

Answers to part **(ii)** were generally well done, with most candidates providing a good explanation of the role of each part of the required formula.

For the final part, most candidates chose to use algebra to show the given result, and this was generally done successfully, although a small number of candidates did not show fully the steps that were being taken. Almost all candidates who reached this point were able to apply the result shown to the formula for the expectation and most realised that several factors could be moved outside the sum. Only a small number did not realise that the summation that remained was the sum of the probabilities of all possible outcomes for that random variable.

Solution

(i) Each sock has to belong to a different pair from all the socks before it. This means that the first sock can be any sock, but the next one has to be one of the $2n-2$ socks which does not include the partner of the first sock and so on. Once r socks have been taken with no repeats then the next sock has to be one of the $2n-2r$ socks which have not had their partner taken, and there are a total of $2n-r$ sock left in the drawer.

The probability that you select $2k$ socks from the initial $2n$ socks without selecting a pair is given by:

$$\begin{aligned}&\frac{2n}{2n} \times \frac{2n-2}{2n-1} \times \frac{2n-4}{2n-2} \times \frac{2n-6}{2n-3} \times \cdots \times \frac{2n-2(2k-1)}{2n-(2k-1)}\\ =&\frac{2^{2k} \times n(n-1)(n-2)\cdots(n-2k+1)}{2n(2n-1)(2n-2)\cdots(2n-2k+1)}\\ =&2^{2k} \times \frac{n!}{(n-2k)!} \times \frac{(2n-2k)!}{(2n)!}\\ =&2^{2k} \times \frac{n!}{(2k)!(n-2k)!} \times \frac{(2k)!(2n-2k)!}{(2n)!}\\ =&2^{2k}\frac{{}^{n}C_{2k}}{{}^{2n}C_{2k}}\end{aligned}$$

You could use the $\binom{n}{r}$ notation, but I think this is a little nicer if you are going to have a fraction involved.

Alternate:

Given the Instead you can use a “number of ways” argument. The total number of ways to pick $2k$ socks from the total in the drawer is ${}^{2n}C_{2k}$. If you want to pick a set of $2k$ socks each of which belongs to a different pair then there are ${}^{n}C_{2k}$ ways of picking the pairs which will contribute a sock each, and in each pair you have 2 choices as to which sock you take.

Therefore there are $2^{2k} \times {}^{n}C_{2k}$ ways of picking $2k$ socks to that they each belong to a different pair. This then gives the probability as $2^{2k}\frac{{}^{n}C_{2k}}{{}^{2n}C_{2k}}$ as before.

(ii) If $X_{n,k} = r$ then $2r$ socks will be from r pairs of socks and $2k-2r$ will be socks without partners.

The number of ways of picking r pairs is $\binom{n}{r}$ and the number of ways of picking $2k-2r$ singleton socks from the rest is $2^{2k-2r}\binom{n-r}{2k-2r}$ (where the factor of 2^{2k-2r} comes from the fact that we have 2 choices of sock from each pair).

This time I am using the other form of the binomial coefficient as I think in this case it is a little clearer, though it means that I felt I needed extra spaces between lines of working on the next page.

Dividing by the total number of ways of picking $2k$ socks from the $2n$ socks gives:

$$\mathrm{P}(X_{n.k} = r) = \frac{2^{2k-2r}\binom{n}{r}\binom{n-r}{2k-2r}}{\binom{2n}{2k}}$$

$$= \frac{2^{2(k-r)}\binom{n}{r}\binom{n-r}{2(k-r)}}{\binom{2n}{2k}}$$

(iii) We have:

$$r\mathrm{P}(X_{n.k} = r)$$

$$= r \times \frac{2^{2(k-r)}\binom{n}{r}\binom{n-r}{2(k-r)}}{\binom{2n}{2k}}$$

$$= r2^{2(k-r)} \times \frac{n!}{r!(n-r)!} \times \binom{n-r}{2(k-r)} \times \frac{(2k)!(2n-2k)!}{(2n)!}$$

$$= \cancel{r}2^{2(k-r)} \times \frac{n(n-1)!}{\cancel{r}(r-1)!(n-r)!} \times \binom{n-r}{2(k-r)} \times \frac{(2k)(2k-1)(2k-2)!(2n-2k)!}{(2n)(2n-1)(2n-2)!}$$

$$= \frac{2^{2(k-r)} \times \cancel{n}\binom{n-1}{r-1} \times \binom{n-r}{2(k-r)}}{\binom{2n-2}{2k-2}} \times \frac{\cancel{2}k(2k-1)}{(\cancel{2n})(2n-1)}$$

$$= \frac{k(2k-1)}{(2n-1)} \times \frac{2^{2[(k-1)-(r-1)]} \times \binom{n-1}{r-1} \times \binom{(n-1)-(r-1)}{2[(k-1)-(r-1)]}}{\binom{2n-2}{2k-2}}$$

$$= \frac{k(2k-1)}{(2n-1)} \times \mathrm{P}(X_{n-1,k-1} = r-1)$$

An alternative, and possibly nicer, approach is to start with the right hand side of the equality and manipulate this to get the left hand side. It's a little nicer as you are making the expressions more simple as you go. For this part in particular it is a good idea to leave lots of space to make it easier to follow (and so that you have room to correct mistakes).

The last part says "hence", so you need to use what you have just done!

$$\begin{aligned}
\mathrm{E}(X_{n,k}) &= \sum_{r=0}^{k} r\mathrm{P}(X_{n,k} = r) \\
&= \sum_{r=1}^{k} r\mathrm{P}(X_{n,k} = r) \\
&= \sum_{r=1}^{k} \frac{k(2k-1)}{(2n-1)} \times \mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{(2n-1)} \sum_{r=1}^{k} \mathrm{P}(X_{n-1,k-1} = r-1) \\
&= \frac{k(2k-1)}{(2n-1)} \sum_{j=0}^{k-1} \mathrm{P}(X_{n-1,k-1} = j) \\
&= \frac{k(2k-1)}{(2n-1)}
\end{aligned}$$

Where the last line uses the fact that $\displaystyle\sum_{j=0}^{k-1} \mathrm{P}(X_{n-1,k-1} = j) = 1$.