STEP Support Programme

2024 STEP 3 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the OCR STEP website. These are the general comments for the STEP 3 2024 exam from the Examiner's report:

The total entry was an increase on that of 2023 by more than 10%. One question was attempted by more than 98% of candidates, another two by about 80%, and another five by between 50% and 70%. The remaining four questions were attempted by between 5% and 30% of candidates, these being from Section B: Mechanics, and Section C: Probability and Statistics, though the Statistics questions were in general attempted more often and more successfully. All questions were perfectly solved by some candidates. About 84% of candidates attempted no more than 7 questions.

Please send any corrections, comments or suggestions to step@maths.cam.ac.uk.

Question 1

1 Throughout this question, N is an integer with $N \geqslant 1$ and $S_N = \displaystyle\sum_{r=1}^{N} \frac{1}{r^2}$.

You may assume that $\displaystyle\lim_{N \to \infty} S_N$ exists and is equal to $\frac{1}{6}\pi^2$.

(i) Show that

$$\frac{1}{r+1} - \frac{1}{r} + \frac{1}{r^2} = \frac{1}{r^2(r+1)}.$$

Hence show that

$$\sum_{r=1}^{N} \frac{1}{r^2(r+1)} = \sum_{r=1}^{N} \frac{1}{r^2} - 1 + \frac{1}{N+1}.$$

Show further that $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)} = \frac{1}{6}\pi^2 - 1$.

(ii) Find $\displaystyle\sum_{r=1}^{N} \frac{1}{r^2(r+1)(r+2)}$ in terms of S_N, and hence evaluate

$$\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)(r+2)}.$$

(iii) Show that

$$\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)^2} = \sum_{r=1}^{\infty} \frac{2}{r^2(r+1)} - 1.$$

Examiner's report

This was comfortably both the most popular question and the most successful, with a mean score of about 15/20. There were numerous correct methods employed to approach the partial fractions. Every part had many excellent clear responses. Generally, if candidates could do the partial fractions algorithm correctly and wrote more than the bare minimum for the limiting and telescoping operations they got almost full marks.

In part **(i)**, most could do the calculations correctly, though explanations less so.

In parts **(ii)** and **(iii)**, many candidates did not attempt the correct decomposition. Explanations of cancelling terms in the telescoping series and taking limits were frequently not clear. Particular weaknesses were treating harmonic series as if they converged, and substituting ∞ into expressions as if it were a number.

There were many clever ways of doing the last part without a full partial fraction decomposition, but probably the cleanest was as follows.

$$\begin{aligned}\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)^2}-\sum_{r=1}^{\infty}\frac{2}{r^2(r+1)} &= \sum_{r=1}^{\infty}\frac{-2r-1}{r^2(r+1)^2}\\ &= \sum_{r=1}^{\infty}\frac{r^2-(r+1)^2}{r^2(r+1)^2}\\ &= \sum_{r=1}^{\infty}\frac{1}{(r+1)^2}-\frac{1}{r^2}\\ &= -1\end{aligned}$$

Solution

See STEP 3 2018 Question 7 for a derivation of the result $\displaystyle\sum_{r=1}^{\infty}\frac{1}{r^2}=\frac{1}{6}\pi^2$

(i) We have:

$$\begin{aligned}\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2} &= \frac{r^2-r(r+1)+(r+1)}{r^2(r+1)}\\ &= \frac{r^2-r^2-r+r+1}{r^2(r+1)}\\ &= \frac{1}{r^2(r+1)}\end{aligned}$$

We then have:

$$\begin{aligned}\sum_{r=1}^{N}\frac{1}{r^2(r+1)} &= \sum_{r=1}^{N}\frac{1}{r+1}-\frac{1}{r}+\frac{1}{r^2}\\ &= \sum_{r=1}^{N}\frac{1}{r^2}+\frac{1}{N+1}+\cancel{\frac{1}{N}}+\cancel{\frac{1}{N-1}}+\cdots+\cancel{\frac{1}{2}}\\ &\qquad -\cancel{\frac{1}{N}}-\cancel{\frac{1}{N-1}}-\cdots-\cancel{\frac{1}{2}}-\frac{1}{1}\\ &= \sum_{r=1}^{N}\frac{1}{r^2}+\frac{1}{N+1}-1\end{aligned}$$

As $N\to\infty$ we have $\dfrac{1}{N+1}\to 0$ and so:

$$\sum_{r=1}^{\infty}\frac{1}{r^2(r+1)}=\frac{1}{6}\pi^2-1\,.$$

(ii) Using partial fractions we have:

$$\frac{1}{r^2(r+1)(r+2)} \equiv \frac{A}{r} + \frac{B}{r^2} + \frac{C}{r+1} + \frac{D}{r+2}$$

and so:

$$1 \equiv Ar(r+1)(r+2) + B(r+1)(r+2) + Cr^2(r+2) + Dr^2(r+1)$$

Substituting specific values of r gives:

$$\begin{aligned} r = 0 &\implies \quad 2B = 1 \implies B = \tfrac{1}{2} \\ r = -1 &\implies \quad C = 1 \\ r = -2 &\implies -4D = 1 \implies D = -\tfrac{1}{4} \end{aligned}$$

Equating coefficients of r^3 gives:

$$0 = A + C + D \implies A = -\tfrac{3}{4}$$

This means we have:

$$\begin{aligned} \sum_{r=1}^{N} \frac{1}{r^2(r+1)(r+2)} &= -\frac{3}{4}\sum_{r=1}^{N}\frac{1}{r} + \frac{1}{2}\sum_{r=1}^{N}\frac{1}{r^2} + \sum_{r=1}^{N}\frac{1}{r+1} - \frac{1}{4}\sum_{r=1}^{N}\frac{1}{r+2} \\ &= \frac{1}{2}S_N - \frac{3}{4}\sum_{r=1}^{N}\frac{1}{r} + \sum_{r=2}^{N+1}\frac{1}{r} - \frac{1}{4}\sum_{r=3}^{N+2}\frac{1}{r} \\ &= \frac{1}{2}S_N + \left(1 - \frac{3}{4} - \frac{1}{4}\right)\sum_{r=3}^{N}\frac{1}{r} \\ &\qquad + \left(\frac{1}{2} + \frac{1}{N+1}\right) - \frac{3}{4}\left(1 + \frac{1}{2}\right) - \frac{1}{4}\left(\frac{1}{N+1} + \frac{1}{N+2}\right) \\ &= \frac{1}{2}S_N - \frac{5}{8} + \frac{3}{4}\left(\frac{1}{N+1}\right) - \frac{1}{4}\left(\frac{1}{N+2}\right) \end{aligned}$$

Then taking the limit as $N \to \infty$ we have:

$$\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)(r+2)} = \frac{\pi^2}{12} - \frac{5}{8}$$

In part **(i)** I expanded out the sums in order to simply them, but in this case I decided to manipulate the limits instead. Both methods can be used in either case, but I felt with this slightly more complicated simplification the limit manipulation was a little easier.

(iii) Using partial fractions we have:

$$\frac{1}{r^2(r+1)^2} \equiv \frac{A}{r} + \frac{B}{r^2} + \frac{C}{r+1} + \frac{D}{(r+1)^2}$$

and so:

$$1 \equiv Ar(r+1)^2 + B(r+1)^2 + Cr^2(r+1) + Dr^2$$

Substituting $r = 0$ and $r = -1$ gives:

$$\begin{aligned} r = 0 &\implies B = 1 \\ r = -1 &\implies D = 1 \end{aligned}$$

Equating coefficients of r^3 and r gives:

$$\begin{aligned} r^3 &: A + C = 0 \\ r &: A + 2B = 0 \end{aligned}$$

Therefore we have $A = -2$, $C = 2$ and $B = D = 1$.

Considering the finite sum:

$$\begin{aligned} \sum_{r=1}^{N} \frac{1}{r^2(r+1)^2} &= 2\sum_{r=1}^{N} \frac{1}{r+1} - 2\sum_{r=1}^{N} \frac{1}{r} + \sum_{r=1}^{N} \frac{1}{r^2} + \sum_{r=1}^{N} \frac{1}{(r+1)^2} \\ &= 2\left(\frac{1}{N+1} - 1\right) + S_N + \sum_{r=2}^{N+1} \frac{1}{r^2} \\ &= 2\left(\frac{1}{N+1} - 1\right) + 2S_N + \frac{1}{(N+1)^2} - 1 \end{aligned}$$

Note that the terms in the $2\sum_{r=1}^{N} \frac{1}{r+1} - 2\sum_{r=1}^{N} \frac{1}{r}$ sums almost all cancel out apart from the last term of the first sum and the first term of the second sum.

So as $N \to \infty$ we have:

$$\begin{aligned} \sum_{r=1}^{\infty} \frac{1}{r^2(r+1)^2} &= 2S_N - 3 \\ &= 2 \times \left(\frac{\pi^2}{6} - 1\right) - 1 \\ &= 2\left(\sum_{r=1}^{\infty} \frac{1}{r^2(r+1)}\right) - 1 \end{aligned}$$

as required.

Question 2

2 **(i)** Solve the inequalities

(a) $\sqrt{4x^2 - 8x + 64} \leqslant |x + 8|$,

(b) $\sqrt{4x^2 - 8x + 64} \leqslant |3x - 8|$.

(ii) **(a)** Let $\mathrm{f}(x) = \sqrt{4x^2 - 8x + 64} - 2(x - 1)$.

Show, by considering $\big(\sqrt{4x^2 - 8x + 64} + 2(x-1)\big)\mathrm{f}(x)$ or otherwise, that $\mathrm{f}(x) \to 0$ as $x \to \infty$.

(b) Sketch $y = \sqrt{4x^2 - 8x + 64}$ and $y = 2(x - 1)$ on the same axes.

(iii) Find a value of m and the corresponding value of c such that the solution set of the inequality

$$\sqrt{4x^2 - 5x + 4} \leqslant |mx + c|$$

is $\{x : x \geqslant 3\}$.

(iv) Find values of p, q, m and c such that the solution set of the inequality

$$|x^2 + px + q| \leqslant mx + c$$

is $\{x : -5 \leqslant x \leqslant 1\} \cup \{x : 5 \leqslant x \leqslant 7\}$.

Examiner's report

Three quarters of the candidates attempted this question with a mean score of just under half marks.

In part **(i)**, candidates often omitted a justification that the LHS of the inequality was real and for noting that both sides are positive before squaring.

Part **(ii)(a)** was generally done quite well, although some candidates ignored the suggested method and argued that because the lead terms cancel as $x \to \infty$, $\mathrm{f}(x) \to \infty$, not earning full marks.

The sketch in **(ii)(b)** was not generally done very well. In general, sketches just need to have the same key features as the actual plot of the function. The asymptotes and symmetry about $x = 1$ were crucial here.

Part bf (iii) was done fairly well by those that attempted it, most noticing that they should choose values of m to ensure that the x^2 terms should cancel.

There were not many significant attempts on part **(iv)**. To start, it was relatively straightforward to state that as four critical values were required, the quadratic needed to cross the x-axis, but this was often missed. However, there were some very efficient and neat solutions to this part, and candidates who got on the right path initially executed it well. The most common error was failure to get the four roots attached to the correctly signed version of the quadratic. Candidates who used a diagram were generally much more successful with this.

Solution

(i) **(a)** Both sides of the inequality are positive so we can square both sides whilst preserving the direction of the inequality.

$$\begin{aligned} 4x^2 - 8x + 64 &\leqslant (x+8)^2 \\ 4x^2 - 8x + 64 &\leqslant x^2 + 16x + 64 \\ 3x^2 - 24x &\leqslant 0 \\ 3x(x-8) &\leqslant 0 \end{aligned}$$

and a quick sketch of the quadratic shows that the required solution set is $0 \leqslant x \leqslant 8$.

(b) Again both sides are positive so we have:

$$\begin{aligned} 4x^2 - 8x + 64 &\leqslant (3x-8)^2 \\ 4x^2 - 8x + 64 &\leqslant 9x^2 - 48x + 64 \\ 5x^2 - 40x &\geqslant 0 \\ 5x(x-8) &\geqslant 0 \end{aligned}$$

In this case the sketch of the quadratic gives the solution set $x \leqslant 0$ or $x \geqslant 8$.

You should justify which interval is the correct one in some way, but this can be just a rough sketch graph!

(ii) **(a)** General speaking if you are given a suggested method then it is a good idea to use it!

$$\begin{aligned} &\left(\sqrt{4x^2 - 8x + 64} + 2(x-1)\right)\mathrm{f}(x) \\ =&\left(\sqrt{4x^2 - 8x + 64} + 2(x-1)\right)\left(\sqrt{4x^2 - 8x + 64} - 2(x-1)\right) \\ =&(4x^2 - 8x + 64) - 4(x-1)^2 \\ =&4x^2 - 8x + 64 - 4x^2 + 8x - 4 \\ =&60 \end{aligned}$$

Therefore we have:

$$\mathrm{f}(x) = \frac{60}{\sqrt{4x^2 - 8x + 64} + 2(x-1)}$$

As $x \to \infty$ we have $\sqrt{4x^2 - 8x + 64} + 2(x-1) \to \infty$ and so $\mathrm{f}(x) \to 0$.

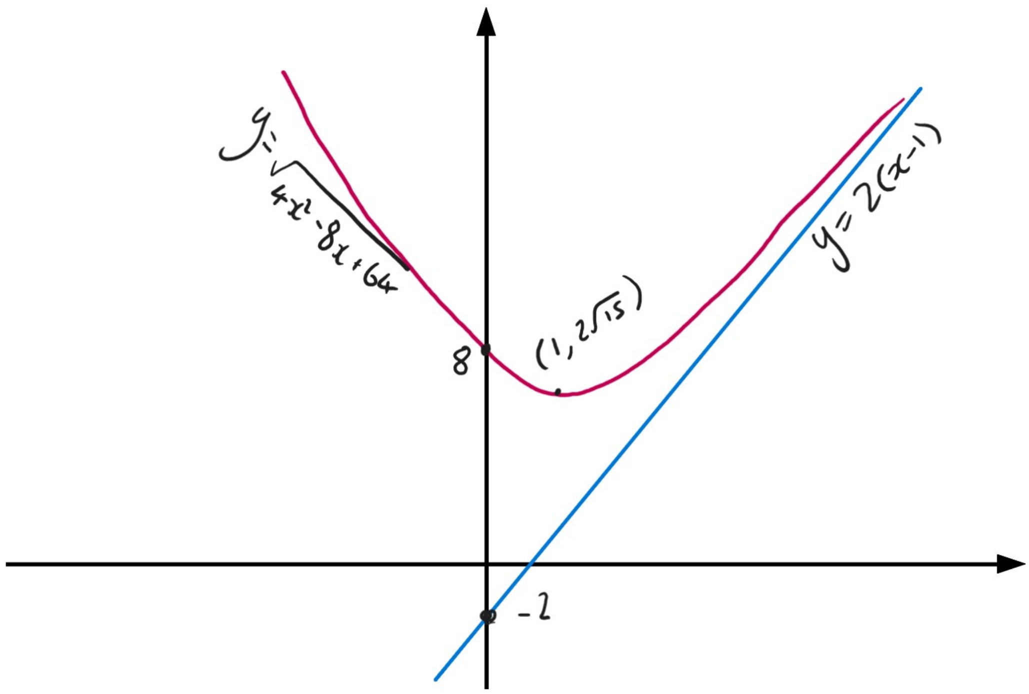
(b) From the work in the previous part we know that the two curves will tend towards each other as $x \to \infty$. From the expression we found for $\mathrm{f}(x)$ we can see that $\mathrm{f}(x) \neq 0$ for all x, so the curves do not intersect.

The equation for the first curve can be written as:

$$\begin{aligned} y &= \sqrt{4x^2 - 8x + 64} \\ &= 2\sqrt{x^2 - 2x + 16} \\ &= 2\sqrt{(x-1)^2 + 15} \end{aligned}$$

This means that the first curve has symmetry about $x = 1$, and the y value at this point (which is a minimum) is $y = 2\sqrt{15}$.

The graphs look like:



(iii) The required solution set here looks a if it comes from a linear inequality (rather than a quadratic one as we found in part **(i)**). This suggests that we want to take $m = 2$ so that the quadratic terms cancel (we could also have taken $m = -2$). As before both sides of the inequality are positive so we can square to give:

$$\begin{aligned} \sqrt{4x^2 - 5x + 4} &\leqslant |2x + c| \\ 4x^2 - 5x + 4 &\leqslant 4x^2 + 4cx + c^2 \\ (4c + 5)x &\geqslant 4 - c^2 \\ x &\geqslant \frac{4 - c^2}{4c + 5} \end{aligned}$$

This means we need:

$$\begin{aligned}\frac{4-c^2}{4c+5} &= 3\\ 4-c^2 &= 12c+15\\ c^2+12c+11 &= 0\\ (c+11)(c+1) &= 0\end{aligned}$$

which gives $c = -1$ or $c = -11$. Since we have been squaring we might have introduced spurious solutions. Both $c = -1$ and $c = -11$ satisfy $\sqrt{4x^2-5x+4} \leqslant |2x+c|$ when $x = 3$. We want the inequality to also be true when $x = 4$.

Using $x = 4$ and $c = -1$ we have:

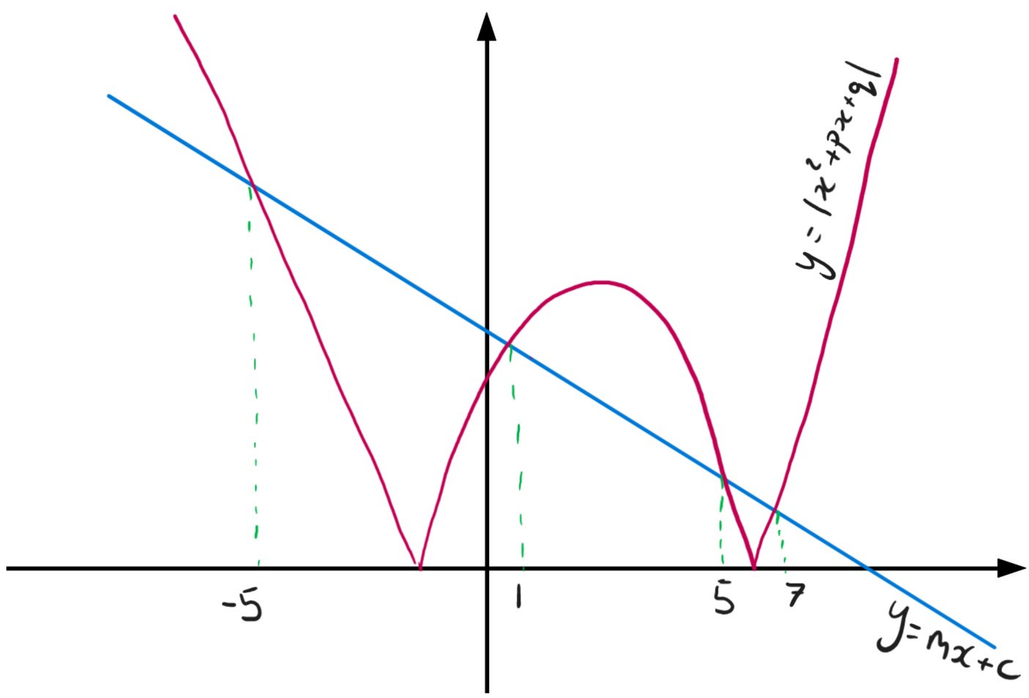
$$\begin{aligned}\sqrt{4x^2-5x+4} &\leqslant |2x+c|\\ \sqrt{48} &\leqslant |8-1|\\ \sqrt{48} &\leqslant 7 \qquad \text{which is true}\end{aligned}$$

At this point you can actually stop as you have found a possible $m(=2)$ and $c(=-1)$. For completeness we will consider the other option.

Using $x = 4$ and $c = -11$ we have:

$$\begin{aligned}\sqrt{4x^2-5x+4} &\leqslant |2x+c|\\ \sqrt{48} &\leqslant |8-11|\\ \sqrt{48} &\leqslant 3 \qquad \text{which is } \textbf{NOT} \text{ true}\end{aligned}$$

(iv) To start thinking about this question it is probably a good idea to draw a sketch! After trying a few curves and different straight lines to get a sketch which looks like the solution sets are in the correct sort of position I ended up with the following sketch:



If we consider $y = x^2 + px + q$ (i.e. no modulus sign) and $y = mx + c$ we want these two curves to intersect when $x = -5$ and $x = 7$, so we have:

$$x^2 + px + q - mx - c = (x + 5)(x - 7)$$

Therefore $p - m = -2$ and $q - c = -35$.

If now we consider $y = -(x^2 + px + q)$ (i.e. the bit of the quadratic that has "flipped" over the x axis) and $y = mx + c$ we want these curves to intersect when $x = 1$ and $x = 5$ so we have:

$$mx + c + x^2 + px + q = (x - 1)(x - 5)$$

Therefore $p + m = -6$ and $q + c = 5$.

Solving the simultaneous equations $p - m = -2$ and $p + m = -6$ gives $p = -4, m = -2$, and solving the simultaneous equations $q - c = -35$ and $q + c = 5$ gives $q = -15, c = 20$.

Therefore the solution is:

$$p = -4, q = -15, m = -2, c = 20$$

From the sketch we would expect m to be negative and c to be positive, so this is a reassuring answer!

Question 3

3 *Throughout this question, consider only* $x > 0$.

(i) Let

$$g(x) = \ln\left(1 + \frac{1}{x}\right) - \frac{x + c}{x(x+1)}$$

where $c \geqslant 0$.

(a) Show that $y = g(x)$ has positive gradient for all $x > 0$ when $c \geqslant \frac{1}{2}$.

(b) Find the values of x for which $y = g(x)$ has negative gradient when $0 \leqslant c < \frac{1}{2}$.

(ii) It is given that, for all $c > 0$, $g(x) \to -\infty$ as $x \to 0$.

Sketch, for $x > 0$, the graphs of

$$y = g(x)$$

in the cases

(a) $c = \frac{3}{4}$,

(b) $c = \frac{1}{4}$.

(iii) The function f is defined as

$$f(x) = \left(1 + \frac{1}{x}\right)^{x+c}.$$

Show that, for $x > 0$,

(a) f is a decreasing function when $c \geqslant \frac{1}{2}$;

(b) f has a turning point when $0 < c < \frac{1}{2}$;

(c) f is an increasing function when $c = 0$.

Examiner's report

The second most popular question, it was the eighth most successful with a mean score of a little under 9/20.

Whilst some candidates did not make progress with differentiating $f(x)$ in **(iii)**, most differentiated well in **(i)** and **(iii)**. However in **(i)**, sufficient justification for the positive gradient for $c \geqslant \frac{1}{2}$ was often missing in **(a)**, and some occasionally forgot that inequalities reverse when divided by a negative number in **(b)**.

In part **(ii)**, both sketch graphs were mostly drawn correctly. However, in part **(a),** many did not justify the positive gradient or asymptote for large x. In part **(b)**, whilst most found the turning point correctly, few justified the positive gradient before the turning point.

The justifications, or otherwise, in **(iii)** varied a lot in the level of detail. Forgetting to mention that $f > 0$ was a common way that candidates did not achieve full marks.

Solution

Note that there is a helpful comment at the top of the question stating that you only need to consider $x > 0$ throughout the question.

(i) **(a)** Differentiating gives:

$$\begin{aligned}
g'(x) &= \frac{1}{1+\frac{1}{x}} \times -x^{-2} - \left(\frac{x(x+1)-(x+c)(2x+1)}{x^2(x+1)^2}\right) \\
&= \frac{-1}{x^2+x} - \left(\frac{x^2+x-2x^2-(2c+1)x-c}{x^2(x+1)^2}\right) \\
&= \left(\frac{x^2+2cx+c}{x^2(x+1)^2}\right) - \frac{1}{x^2+x} \\
&= \frac{\cancel{x^2}+2cx+c-\cancel{x^2}-x}{x^2(x+1)^2} \\
&= \frac{(2c-1)x+c}{[x(x+1)]^2}
\end{aligned}$$

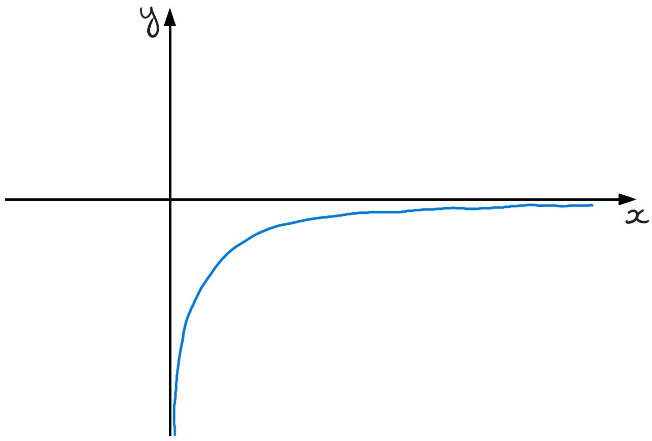
The denominator is a square, and we know that $x > 0$, so it is non-zero as well as being positive. Since we have $x > 0$ and $c \geqslant \frac{1}{2}$ we know that $(2c-1)x \geqslant 0$ and $c > 0$ therefore the numerator is also non-zero and positive. Hence $g'(x) > 0$ for $x > 0$ and $c \geqslant \frac{1}{2}$.

(b) We have $g'(x) < 0$ when $(2c-1)x + c < 0$, i.e. for $x > \dfrac{c}{1-2c}$.

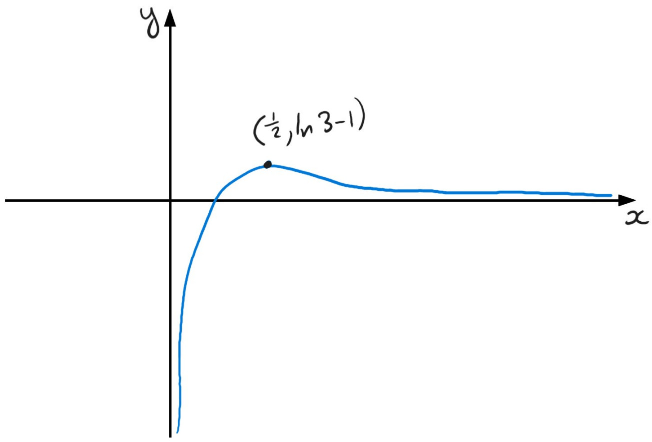
(ii) **(a)** From part **(i)(b)** when $c = \frac{3}{4} > \frac{1}{2}$ the gradient is always positive.

As $x \to \infty$ we have $\dfrac{1}{x} \to 0$ and so $\ln\left(1 + \frac{1}{x}\right) \to 0$. We also have $\dfrac{x+c}{x(x+1)} \to 0$, and so $\text{g}(x) \to 0$ as $x \to \infty$. This holds for all values of c.

The graph looks like:



(b) In this case the gradient will be negative when $x > \dfrac{c}{1-2c}$. Substituting in $c = \frac{1}{4}$ gives the result that the gradient is negative for $x > \frac{1}{2}$. This means that there is a maximum turning point when $x = \frac{1}{2}$ and $y = \ln(1+2) - \dfrac{\frac{1}{2} + \frac{1}{4}}{\frac{1}{2}\left(\frac{1}{2}+1\right)} = \ln 3 - 1$ and the graph looks like:



My graphs are perhaps a little exaggerated, but that doesn't matter (and makes it easier to see what is going on). This Desmos Page shows the graphs more clearly and you can adjust the values of c to see how the behaviour changes as c changes.

(iii) **(a)** Taking logs of both sides gives:

$$\ln \mathrm{f}(x) = (x+c)\ln\left(1+\frac{1}{x}\right)$$

Note that this looks a little bit like $\mathrm{g}(x)$ as defined earlier in the question!

Differentiating gives:

$$\begin{aligned}\frac{1}{\mathrm{f}(x)}\mathrm{f}'(x) &= \ln\left(1+\frac{1}{x}\right) + (x+c)\times\frac{1}{1+\frac{1}{x}}\times -x^{-2}\\ \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} &= \ln\left(1+\frac{1}{x}\right) - \frac{x+c}{x^2+x}\\ \frac{\mathrm{f}'(x)}{\mathrm{f}(x)} &= \mathrm{g}(x)\\ \implies \mathrm{f}'(x) &= \mathrm{f}(x)\mathrm{g}(x)\end{aligned}$$

We also have that $\mathrm{f}(x) > 0$ for $x > 0$.

From the work done in parts **(i)(a)** and **(ii)(a)** we can see that when $c \geqslant \frac{1}{2}$ we have $\mathrm{g}(x) < 0$, and so we have $\mathrm{f}'(x) < 0$ and f is a decreasing function.

(b) As shown in part **(i)(b)** and **(ii)(b)** we know that if $0 < c < \frac{1}{2}$ then there will be a point with $x > 0$ where $\mathrm{g}(x) = 0$, and so f has a turning point when $0 < c < \frac{1}{2}$.

(c) When $c = 0$ we have $\mathrm{g}'(x) = \frac{-1}{x(x+1)^2}$ which is negative for all $x > 0$. In part **(ii)(a)** we showed that $\mathrm{g}(x) \to 0$ as $x \to \infty$. Hence we must have $\mathrm{g}(x) > 0$ for all $x > 0$. Therefore $\mathrm{f}'(x) > 0$ for all $x > 0$ and so $\mathrm{f}(x)$ is an increasing function.

Question 4

4 **(i)** Show that if the acute angle between straight lines with gradients m_1 and m_2 is $45°$, then

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \pm 1.$$

The curve C has equation $4ay = x^2$ (where $a \neq 0$).

(ii) If $p \neq q$, show that the tangents to the curve C at the points with x-coordinates p and q meet at a point with x-coordinate $\frac{1}{2}(p+q)$. Find the y-coordinate of this point in terms of p and q.

Show further that any two tangents to the curve C which are at $45°$ to each other meet on the curve $(y+3a)^2 = 8a^2 + x^2$.

(iii) Show that the acute angle between any two tangents to the curve C which meet on the curve $(y+7a)^2 = 48a^2 + 3x^2$ is constant. Find this acute angle.

Examiner's report

The fourth most popular question, it was the third most successful, with a mean score of 10 marks.

Part **(i)** needed more thoroughness than many attempts displayed. Most sensibly chose to express the gradients as tangents of angles of the lines to the x-axis, but then did not define these or consider the possible cases that could arise such as which was greater, or state that the difference between the angles is $\pm 45°$ or $45°/135°$. As the result was given in the question, there was an expectation that there should be complete justification.

In part **(ii)**, most attempts at the coordinates of the point of intersection were successful, though many did not use the non-equality of p and q, and a large number got the y coordinate wrong through substituting x into the equation of the parabola. Overall, many did well with the final result of this part, employing the various results from earlier in the part and that of **(i)**.

Part **(iii)** proved challenging for most, and there was a fair amount of guesswork based on the knowledge that $30°$, $45°$ and $60°$ are angles with nice trigonometric values!

Solution

(i) If a straight line makes an angle θ with the x axis then the gradient of the line is given by $\tan\theta$. Let $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$. If the **acute** angle between the lines is $45°$ then we have $\theta_1 - \theta_2 = \pm 45°$ or $\theta_1 - \theta_2 = \pm 135°$.

The first case could be where both graphs have positive gradient, and the second could be where one has a positive gradient and one has a negative gradient. We are given no information as to which is steeper, so the lines could be "either way around", which is there the "$\pm$" part comes in.

Therefore we have $\tan(\theta_1 - \theta_2) = \pm 1$, and so:

$$\begin{aligned}\tan(\theta_1 - \theta_2) &= \pm 1\\ \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} &= \pm 1\\ \frac{m_1 - m_2}{1 + m_1 m_2} &= \pm 1\end{aligned}$$

as required.

(ii) The gradient of the curve $y = \dfrac{x^2}{4a}$ is given by $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{2a}$, and at the point where $x = p$ we have $y = \dfrac{p^2}{4a}$. The equation of the tangent where $x = p$ is therefore:

$$\begin{aligned}y - \frac{p^2}{4a} &= \frac{p}{2a}(x - p)\\ 4ay - p^2 &= 2px - 2p^2\\ 4ay &= 2px - p^2\end{aligned}$$

Similarly the equation for the tangent at the point with x coordinate q will be $4ay = 2qx - q^2$. Solving for x gives:

$$\begin{aligned}2px - p^2 &= 2qx - q^2\\ 2(p-q)x &= p^2 - q^2\\ p \neq q \implies x &= \tfrac{1}{2}(p+q)\end{aligned}$$

The y coordinate is given by:

$$\begin{aligned}4ay &= p(p+q) - p^2\\ y &= \frac{pq}{4a}\end{aligned}$$

Note that the expressions for the coordinates are symmetric in p and q, which is what you would expect. If you swap the values of p and q over the tangent would still meet in the same place. Checking for symmetry can be useful to make sure your answer is as expected and that you have not made a mistake along the way.

Let $m_1 = \dfrac{p}{2a}$ and let $m_2 = \dfrac{q}{2a}$. Substituting into the expression in part **(i)** gives:

$$\begin{aligned}\frac{\frac{p}{2a} - \frac{q}{2a}}{1 + \frac{p}{2a} \times \frac{q}{2a}} &= \pm 1\\ \frac{2a(p-q)}{4a^2 + pq} &= \pm 1\\ 4a^2(p-q)^2 &= (4a^2 + pq)^2 \qquad (*)\end{aligned}$$

This doesn't seem to be going anywhere much yet, but remember that at the point where the tangents meet we have $x = \frac{1}{2}(p+q)$ and $y = \frac{1}{4a}pq$. It would be useful to try and manipulate $(*)$ so that we can use these.

$$\begin{aligned}
4a^2(p-q)^2 &= (4a^2+pq)^2 \\
4a^2\left[(p+q)^2 - 4pq\right] &= (4a^2+pq)^2 \\
4a^2\left[4x^2 - 16ay\right] &= (4a^2+4ay)^2 \\
16a^2x^2 - 64a^3y &= 16a^4 + 32a^3y + 16a^2y^2 \\
x^2 - 4ay &= a^2 + 2ay + y^2 \\
y^2 + 6ay + a^2 &= x^2 \\
(y+3a)^2 - 8a^2 &= x^2 \\
(y+3a)^2 &= x^2 + 8a^2
\end{aligned}$$

Remember that this is a "show that" so you must show enough working to fully justify the solution!

(iii) For this part, start with the given relationship between x and y and work backwards to a relationship between p and q. We would like to be able to use some of the previous work we have done, so it would be good if we could work towards an expression like the one we used in part **(ii)**.

$$\begin{aligned}
(y+7a)^2 &= 48a^2 + 3x^2 \\
\left(\frac{pq}{4a} + 7a\right)^2 &= 48a^2 + \frac{3}{4}(p+q)^2 \\
\left(pq + 28a^2\right)^2 &= 16 \times 48a^4 + 12a^2(p+q)^2 \\
p^2q^2 + 56a^2pq + 28 \times 28a^4 &= 16 \times 48a^4 + 12a^2(p+q)^2 \\
p^2q^2 + 56a^2pq + 16 \times 49a^4 &= 16 \times 48a^4 + 12a^2(p-q)^2 + 48a^2pq \\
(pq)^2 + 8a^2pq + 16a^4 &= 12a^2(p-q)^2 \\
\left(pq + 4a\right)^2 &= 3\big(2a(p-q)\big)^2 \\
\frac{2a(p-q)}{4a+pq} &= \pm\frac{1}{\sqrt{3}} \\
\frac{\frac{p}{2a} - \frac{q}{2a}}{1 + \frac{pq}{4a}} &= \pm\frac{1}{\sqrt{3}}
\end{aligned}$$

If we let $\dfrac{p}{2a} = m_1$ and let $\dfrac{q}{2a} = m_2$ then we have:

$$\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{1}{\sqrt{3}}$$

Therefore the tangents are at an angle of $30°$ to each other.

Question 5

5 In this question, $\mathbf{M}$ and $\mathbf{N}$ are non-singular 2×2 matrices.

The *trace* of the matrix $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $\text{tr}(\mathbf{M}) = a + d$.

(i) Prove that, for any two matrices $\mathbf{M}$ and $\mathbf{N}$, $\text{tr}(\mathbf{MN}) = \text{tr}(\mathbf{NM})$ and derive an expression for $\text{tr}(\mathbf{M} + \mathbf{N})$ in terms of $\text{tr}(\mathbf{M})$ and $\text{tr}(\mathbf{N})$.

The entries in matrix $\mathbf{M}$ are functions of t and $\dfrac{\text{d}\mathbf{M}}{\text{d}t}$ denotes the matrix whose entries are the derivatives of the corresponding entries in $\mathbf{M}$.

(ii) Show that

$$\frac{1}{\det \mathbf{M}} \frac{\text{d}}{\text{d}t} (\det \mathbf{M}) = \text{tr}\left(\mathbf{M}^{-1} \frac{\text{d}\mathbf{M}}{\text{d}t}\right).$$

(iii) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{\text{d}\mathbf{M}}{\text{d}t} = \mathbf{MN} - \mathbf{NM},$$

where the entries in matrix $\mathbf{N}$ are also functions of t.

Show that $\det \mathbf{M}$, $\text{tr}(\mathbf{M})$ and $\text{tr}\big(\mathbf{M}^2\big)$ are independent of t.

In the case $\mathbf{N} = \begin{pmatrix} t & t \\ 0 & t \end{pmatrix}$, and given that $\mathbf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ when $t = 0$, find $\mathbf{M}$ as a function of t.

(iv) In this part, matrix $\mathbf{M}$ satisfies the differential equation

$$\frac{\text{d}\mathbf{M}}{\text{d}t} = \mathbf{MN},$$

where the entries in matrix $\mathbf{N}$ are again functions of t.

The trace of $\mathbf{M}$ is non-zero and independent of t. Is it necessarily true that $\text{tr}(\mathbf{N}) = 0$?

Examiner's report

This question was a little less popular than question 4 but was less successful with a mean score of under 8/20.

The first part was very well-answered with some efficiently realising that elements not on the leading diagonal did not need calculating. Sadly, some overlooked the second result required.

Part **(ii)** was well-answered too, with the same efficiency as in **(i)** being employed by some.

Part **(iii)** was less well-answered, with the non-conjugate nature of matrix multiplication often being overlooked, and in the last result treating A, B, C, and D as constants. Applying the scalar version of the chain rule to differentiate $\mathbf{M}^2$ was not an uncommon error, but those that answered this part successfully usually rewrote $\text{tr}(\mathbf{M}^2)$ in terms of $\text{tr}(\mathbf{M})$ and $\det(\mathbf{M})$.

Part **(iv)** caused the most difficulty. Only a handful attempted to provide an explicit counterexample to the statement. Some gave a counterexample that did not satisfy all the conditions on $\mathbf{M}$ and $\mathbf{N}$, and a larger number of students convinced themselves that there is no good reason for the claim to hold, but did not give a counterexample. Some students attempted to prove the claim was true. Due to this there were many more 17/20 solutions than 18 or 19/20 solutions. Only 6 candidates achieved 20/20.

Solution

(i) Let $\mathbf{N} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. We then have:

$$\mathbf{MN} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} ax+bz & ay+bw \\ cx+dz & cy+dw \end{pmatrix}$$
$$\mathbf{NM} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ax+cy & bx+dy \\ az+cw & bz+dw \end{pmatrix}$$

Therefore we have $\text{tr}\,(\mathbf{MN}) = ax + bz + cy + dw = \text{tr}\,(\mathbf{NM})$.

Note that we didn't actually need to find the top right and bottom left elements, and when answering this question it would be fine to have just put a dash or squiggle in those places. Throughout the question I have found all the elements for the sake of completeness, but in many cases when finding the trace that was not necessary.

We also have:

$$\begin{aligned}\mathbf{M}+\mathbf{N} &= \begin{pmatrix} a+x & b+y \\ c+z & d+w \end{pmatrix} \\ \implies \text{tr}\,(\mathbf{M}+\mathbf{N}) &= a+x+d+w \\ &= \text{tr}\,\mathbf{M} + \text{tr}\,\mathbf{N}\end{aligned}$$

(ii) Let $\mathbf{M} = \begin{pmatrix} \mathrm{f}(t) & \mathrm{g}(t) \\ \mathrm{h}(t) & \mathrm{k}(t) \end{pmatrix}$. This isn't necessary, you can keep working with a, b, c, d but using this notation helps me to remember that the entries are functions of t rather than constants.

We have:

$$\begin{aligned}\frac{1}{\det \mathbf{M}}\frac{\mathrm{d}}{\mathrm{d}t}(\det \mathbf{M}) &= \frac{1}{\mathrm{f}(t)\mathrm{k}(t) - \mathrm{g}(t)\mathrm{h}(t)} \times \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{f}(t)\mathrm{k}(t) - \mathrm{g}(t)\mathrm{h}(t)\right] \\ &= \frac{\mathrm{f}'(t)\mathrm{k}(t) + \mathrm{f}(t)\mathrm{k}'(t) - \mathrm{g}'(t)\mathrm{h}(t) - \mathrm{g}(t)\mathrm{h}'(t)}{\mathrm{f}(t)\mathrm{k}(t) - \mathrm{g}(t)\mathrm{h}(t)}\end{aligned}$$

and

$$\begin{aligned}\operatorname{tr}\left(\mathbf{M}^{-1}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right) &= \operatorname{tr}\left[\frac{1}{\det\mathbf{M}}\begin{pmatrix}\mathrm{k}(t) & -\mathrm{g}(t)\\ -\mathrm{h}(t) & \mathrm{f}(t)\end{pmatrix}\begin{pmatrix}\mathrm{f}'(t) & \mathrm{g}'(t)\\ \mathrm{h}'(t) & \mathrm{k}'(t)\end{pmatrix}\right]\\ &= \operatorname{tr}\left[\frac{1}{\det\mathbf{M}}\begin{pmatrix}\mathrm{k}(t)\mathrm{f}'(t)-\mathrm{g}(t)\mathrm{h}'(t) & \mathrm{k}(t)\mathrm{g}'(t)-\mathrm{g}(t)\mathrm{k}'(t)\\ -\mathrm{h}(t)\mathrm{f}'(t)+\mathrm{f}(t)\mathrm{h}'(t) & -\mathrm{h}(t)\mathrm{g}'(t)+\mathrm{f}(t)\mathrm{k}'(t)\end{pmatrix}\right]\\ &= \frac{\mathrm{k}(t)\mathrm{f}'(t)-\mathrm{g}(t)\mathrm{h}'(t)-\mathrm{h}(t)\mathrm{g}'(t)+\mathrm{f}(t)\mathrm{k}'(t)}{\mathrm{f}(t)\mathrm{k}(t)-\mathrm{g}(t)\mathrm{h}(t)}\\ &= \frac{1}{\det\mathbf{M}}\frac{\mathrm{d}}{\mathrm{d}t}(\det\mathbf{M})\end{aligned}$$

as required.

(iii) Looking back to the result we have just shown, we can now substitute for $\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}$ to get:

$$\begin{aligned}\operatorname{tr}\left(\mathbf{M}^{-1}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right) &= \operatorname{tr}\left[\mathbf{M}^{-1}(\mathbf{MN}-\mathbf{NM})\right]\\ &= \operatorname{tr}\left[\mathbf{M}^{-1}\mathbf{MN}-\mathbf{M}^{-1}\mathbf{NM}\right]\\ &= \operatorname{tr}(\mathbf{N})-\operatorname{tr}(\mathbf{NMM^{-1}})\\ &= 0\end{aligned}$$

Therefore we have

$$\frac{1}{\det\mathbf{M}}\frac{\mathrm{d}}{\mathrm{d}t}(\det\mathbf{M}) = 0$$

and so $\det\mathbf{M}$ is independent of t.

We have:

$$\frac{\mathrm{d}\left[\operatorname{tr}(\mathbf{M})\right]}{\mathrm{d}t} = \mathrm{f}'(t)+\mathrm{k}'(t) = \operatorname{tr}\left[\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right]$$

and so:

$$\begin{aligned}\frac{\mathrm{d}\left[\operatorname{tr}(\mathbf{M})\right]}{\mathrm{d}t} &= \operatorname{tr}\left[\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t}\right]\\ &= \operatorname{tr}\left[\mathbf{MN}-\mathbf{NM}\right]\\ &= \operatorname{tr}(\mathbf{MN})-\operatorname{tr}(\mathbf{NM}) = 0\end{aligned}$$

Therefore we have $\operatorname{tr}(\mathbf{M})$ is independent of t.

Considering $\mathbf{M}^2$ gives:

$$\begin{aligned}\mathbf{M}^2 &= \begin{pmatrix}\mathrm{f}(t) & \mathrm{g}(t)\\ \mathrm{h}(t) & \mathrm{k}(t)\end{pmatrix}\begin{pmatrix}\mathrm{f}(t) & \mathrm{g}(t)\\ \mathrm{h}(t) & \mathrm{k}(t)\end{pmatrix}\\ &= \begin{pmatrix}[\mathrm{f}(t)]^2+\mathrm{g}(t)\mathrm{h}(t) & \mathrm{f}(t)\mathrm{g}(t)+\mathrm{g}(t)\mathrm{k}(t)\\ \mathrm{h}(t)\mathrm{f}(t)+\mathrm{k}(t)\mathrm{h}(t) & \mathrm{h}(t)\mathrm{g}(t)+[\mathrm{k}(t)]^2\end{pmatrix}\end{aligned}$$

Therefore:

$$\begin{aligned}\operatorname{tr}\left(\mathbf{M}^2\right) &= [\mathrm{f}(t)]^2+\mathrm{g}(t)\mathrm{h}(t)+\mathrm{h}(t)\mathrm{g}(t)+[\mathrm{k}(t)]^2\\ &= \left[\mathrm{f}(t)+\mathrm{k}(t)\right]^2+2\mathrm{g}(t)\mathrm{h}(t)-2\mathrm{f}(t)\mathrm{k}(t)\\ &= (\operatorname{tr}\mathbf{M})^2-2\det\mathbf{M}\end{aligned}$$

Differentiation gives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left[\operatorname{tr}\left(\mathbf{M}^2\right)\right] &= \frac{\mathrm{d}}{\mathrm{d}t}\left[(\operatorname{tr}\mathbf{M})^2 - 2\det\mathbf{M}\right] \\ &= 2\operatorname{tr}(\mathbf{M})\ \frac{\mathrm{d}}{\mathrm{d}t}\left[\operatorname{tr}(\mathbf{M})\right] - 2\frac{\mathrm{d}}{\mathrm{d}t}(\det\mathbf{M}) \\ &= 0 \quad \text{as } \operatorname{tr}(\mathbf{M}) \text{ and } \det\mathbf{M} \text{ are independent of t}\end{aligned}$$

Hence $\operatorname{tr}\left(\mathbf{M}^2\right)$ is independent of t.

Substituting the given form of $\mathbf{N}$ into the differential equation gives:

$$\begin{aligned}\frac{\mathrm{d}\mathbf{M}}{\mathrm{d}t} &= \mathbf{MN} - \mathbf{NM} \\ \begin{pmatrix}\mathrm{f}'(t) & \mathrm{g}'(t)\\ \mathrm{h}'(t) & \mathrm{k}'(t)\end{pmatrix} &= \begin{pmatrix}\mathrm{f}(t) & \mathrm{g}(t)\\ \mathrm{h}(t) & \mathrm{k}(t)\end{pmatrix}\begin{pmatrix}t & t\\ 0 & t\end{pmatrix} - \begin{pmatrix}t & t\\ 0 & t\end{pmatrix}\begin{pmatrix}\mathrm{f}(t) & \mathrm{g}(t)\\ \mathrm{h}(t) & \mathrm{k}(t)\end{pmatrix} \\ \begin{pmatrix}\mathrm{f}'(t) & \mathrm{g}'(t)\\ \mathrm{h}'(t) & \mathrm{k}'(t)\end{pmatrix} &= \begin{pmatrix}t\mathrm{f}(t) & t\mathrm{f}(t)+t\mathrm{g}(t)\\ t\mathrm{h}(t) & t\mathrm{h}(t)+t\mathrm{k}(t)\end{pmatrix} - \begin{pmatrix}t\mathrm{f}(t)+t\mathrm{h}(t) & t\mathrm{g}(t)+t\mathrm{k}(t)\\ t\mathrm{h}(t) & t\mathrm{k}(t)\end{pmatrix} \\ &= \begin{pmatrix}-t\mathrm{h}(t) & t\mathrm{f}(t)-t\mathrm{k}(t)\\ 0 & t\mathrm{h}(t)\end{pmatrix}\end{aligned}$$

From this we can see that $\mathrm{h}'(t) = 0$ and so $\mathrm{h}(t)$ is constant and we have $\mathrm{h}(t) = C$ from the initial condition on $\mathbf{M}$.

We now have $\mathrm{f}'(t) = -Ct$, and so $\mathrm{f}(t) = -\frac{C}{2}t^2 + A$. Also $\mathrm{k}'(t) = Ct$ which gives $\mathrm{k}(t) = \frac{C}{2}t^2 + D$.

Substituting these into the expression for the top right element gives:

$$\begin{aligned}\mathrm{g}'(t) &= t\mathrm{f}(t) - t\mathrm{k}(t) \\ &= -\frac{C}{2}t^3 + At - \frac{C}{2}t^3 - Dt \\ &= (A-D)t - Ct^3 \\ \implies \mathrm{g}(t) &= \tfrac{1}{2}(A-D)t^2 - \tfrac{1}{4}Ct^4 + B\end{aligned}$$

Therefore we have:

$$\mathrm{M} = \begin{pmatrix}-\frac{1}{2}Ct^2 + A & \frac{1}{2}(A-D)t^2 - \frac{1}{4}Ct^4 + B\\ C & \frac{1}{2}Ct^2 + D\end{pmatrix}$$

(iv) The wording of this part makes me think that we want to find a counterexample, and the differential equation feels a little bit like an exponential one. Try setting $\mathbf{M} = \begin{pmatrix}\mathrm{e}^t & \mathrm{f}(t)\\ \mathrm{g}(t) & 1-\mathrm{e}^t\end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix}1 & \mathrm{h}(t)\\ \mathrm{k}(t) & 1\end{pmatrix}$. These matrices satisfy the conditions that the trace of $\mathbf{M}$ is non-zero and independent of t, and the trace of $\mathbf{N}$ is non-zero. If we can then find functions $\mathrm{f}, \mathrm{g}, \mathrm{h}$ and k that work then we will have a counterexample to show that is is not necessarily the case that $\operatorname{tr}\mathbf{N} = 0$.

We then have:

$$\begin{aligned}\begin{pmatrix}\mathrm{e}^t & \mathrm{f}'(t)\\ \mathrm{g}'(t) & -\mathrm{e}^t\end{pmatrix} &= \begin{pmatrix}\mathrm{e}^t & \mathrm{f}(t)\\ \mathrm{g}(t) & 1-\mathrm{e}^t\end{pmatrix}\begin{pmatrix}1 & \mathrm{h}(t)\\ \mathrm{k}(t) & 1\end{pmatrix} \\ \begin{pmatrix}\mathrm{e}^t & \mathrm{f}'(t)\\ \mathrm{g}'(t) & -\mathrm{e}^t\end{pmatrix} &= \begin{pmatrix}\mathrm{e}^t + \mathrm{f}(t)\mathrm{k}(t) & \mathrm{e}^t\mathrm{h}(t) + \mathrm{f}(t)\\ \mathrm{g}(t) + (1-\mathrm{e}^t)\mathrm{k}(t) & \mathrm{g}(t)\mathrm{h}(t) + 1 - \mathrm{e}^t\end{pmatrix} \qquad =\end{aligned}$$

From the top left entry we know that we must have $\mathrm{f}(t)\mathrm{k}(t) \equiv 0$ and from the bottom right entry we must have $\mathrm{g}(t)\mathrm{h}(t) \equiv -1$.

If we set $\mathrm{k}(t) \equiv 0$ then the bottom left entry gives $\mathrm{g}'(t) = \mathrm{g}(t)$, so try $\mathrm{g}(t) = \mathrm{e}^t$. This then gives $\mathrm{h}(t) = -\mathrm{e}^{-t}$. Substituting these then gives:

$$\begin{pmatrix} \mathrm{e}^t & \mathrm{f}'(t) \\ \mathrm{e}^t & -\mathrm{e}^t \end{pmatrix} = \begin{pmatrix} \mathrm{e}^t & \mathrm{f}(t) \\ \mathrm{e}^t & 1-\mathrm{e}^t \end{pmatrix} \begin{pmatrix} 1 & -\mathrm{e}^{-t} \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \mathrm{e}^t & \mathrm{f}'(t) \\ \mathrm{e}^t & -\mathrm{e}^t \end{pmatrix} = \begin{pmatrix} \mathrm{e}^t & -\mathrm{e}^t\mathrm{e}^{-t} + \mathrm{f}(t) \\ \mathrm{e}^t & -\mathrm{e}^t \end{pmatrix}$$

Looking at the top entry we need $\mathrm{f}'(t) = -1 + \mathrm{f}(t)$, which can be satisfied by $\mathrm{f}(t) = \mathrm{e}^t + 1$.

Therefore if we take the matrices $\mathbf{M} = \begin{pmatrix} \mathrm{e}^t & e^t + 1 \\ e^t & 1-\mathrm{e}^t \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 1 & -e^{-t} \\ 0 & 1 \end{pmatrix}$ then these satisfy the differential equation, have $\mathrm{tr}(\mathbf{M}) \neq 0$ and is independent of t, but we also have $\mathrm{tr}(\mathbf{N}) \neq 0$, so the trace of $\mathbf{N}$ does not have to be 0.

In this answer I have tried to show my thought processes - all you have to do was to write down a suitable $\mathbf{M}$ and $\mathbf{N}$ and show that these provide a counterexample. My starting point was considering $\mathrm{f}(t)\mathrm{k}(t) \equiv 0$ — if you then take $\mathrm{f}(t) \equiv 0$ you quickly find that you cannot solve the problem using this.

Question 6

6 **(i)** A particle moves in two-dimensional space. Its position is given by coordinates (x, y) which satisfy

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -x + 3y + u$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x + y + u$$

where t is the time and u is a function of time. At time $t = 0$, the particle has position $(x_0\,,\,y_0)$.

(a) By considering $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t}$, show that if the particle is at the origin $(0, 0)$ at some time $t > 0$, then it is necessary that $x_0 = y_0$.

(b) Given that $x_0 = y_0$, find a constant value of u that ensures that the particle is at the origin at a time $t = T$, where $T > 0$.

(ii) A particle whose position in three-dimensional space is given by co-ordinates (x, y, z) moves with time t such that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 4y - 5z + u$$
$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 2z + u$$
$$\frac{\mathrm{d}z}{\mathrm{d}t} = x - 2y + u$$

where u is a function of time. At time $t = 0$, the particle has position $(x_0\,,\,y_0\,,\,z_0)$.

(a) Show that, if the particle is at the origin $(0, 0, 0)$ at some time $t > 0$, it is necessary that y_0 is the mean of x_0 and z_0.

(b) Show further that, if the particle is at the origin $(0, 0, 0)$ at some time $t > 0$, it is necessary that $x_0 = y_0 = z_0$.

(c) Given that $x_0 = y_0 = z_0$, find a constant value of u that ensures that the particle is at the origin at a time $t = T$, where $T > 0$.

Examiner's report

This was the least popular of the Pure Mathematics section, and by a large margin the least successful of the whole paper.

Those candidates who were successful in part **(i)(a)** usually tackled the question by re-writing the differential equation as $\frac{\mathrm{d}(x-y)}{\mathrm{d}t} = -2(x-y)$. There were also some candidates who rewrote the equation as $\frac{\mathrm{d}x}{\mathrm{d}t} + 2x = \frac{\mathrm{d}y}{\mathrm{d}t} + 2y$ and used integrating factors effectively to solve this, although some integrated erroneously to achieve $x + 2xt = y + 2yt$. Some candidates correctly concluded that $x = y$ but did not go on to say that this implied that $x_0 = y_0$. Most of the candidates gaining no credit for this question substituted $x = y = 0$ into their differential equation and then integrated that.

In part **(i)(b)** those candidates who attempted it generally understood what was required, but some did not appreciate that the situation in this case had different initial conditions to that in part **(a)**. Some candidates used the given differential equations to find a second order differential equation in x or y, which was a valid if inefficient method.

Those attempting part **(ii)** generally performed in a similar way to part **(i)**, either gaining most of the credit available or making the same mistakes they had made in the previous parts. There were some candidates who rather cleverly spotted that they could combine the last two differential equations to show that $y = z$, and then show that $x = z$ and in so doing answer both parts **(ii)(a)** and **(b)** together.

Solution

Note that there is some work that can be reused in different parts but care must be taken as there are different initial conditions to be used in the different parts.

(i) **(a)** Considering $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t}$ gives:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}y}{\mathrm{d}t} &= (-x + 3y + u) - (x + y + u) \\ &= 2y - 2x \\ &= -2(x - y)\end{aligned}$$

If we let $z = x - y$ this equation becomes $\frac{\mathrm{d}z}{\mathrm{d}t} = -2z$, which has solution $z = A\mathrm{e}^{-2t}$. If we have the particle at the origin at some time $t = T$ then when $t = T$ we have $z = 0$. This means we have $A = 0$, and so $z = 0$ for all t.
This means that we have $x = y$ for all t and hence setting $t = 0$ gives $x_0 = y_0$.

(b) In a similar way to the previous part we have $z = Ae^{-2t}$ and if $x_0 = y_0$ we have $z_0 = 0$ and so $z \equiv 0$. Therefore we have $x = y$ for all t.

The differential equation for x then becomes:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= 2x + u \\
\frac{\mathrm{d}x}{\mathrm{d}t} - 2x &= u \\
\mathrm{e}^{-2t}\frac{\mathrm{d}x}{\mathrm{d}t} - 2\mathrm{e}^{-2t}x &= \mathrm{e}^{-2t}u \\
\frac{\mathrm{d}(\mathrm{e}^{-2t}x)}{\mathrm{d}t} &= \mathrm{e}^{-2t}u \\
\mathrm{e}^{-2t}x &= -\tfrac{1}{2}\mathrm{e}^{-2t}u + c \\
x &= -\tfrac{1}{2}u + c\mathrm{e}^{2t}
\end{aligned}$$

When $t = 0$ we have $x = x_0$, so we have $x_0 = -\frac{1}{2}u + c$. We want $x = 0$ when $t = T$, so we have:

$$0 = -\tfrac{1}{2}u + ce^{2T} \implies c = \tfrac{1}{2}ue^{-2T}$$

So we have:

$$\begin{aligned}
x_0 &= -\tfrac{1}{2}u + \tfrac{1}{2}u\mathrm{e}^{-2T} \\
2x_0\mathrm{e}^{2T} &= u - u\mathrm{e}^{2T} \\
\implies u &= \frac{2x_0\mathrm{e}^{2T}}{1 - \mathrm{e}^{2T}}
\end{aligned}$$

(ii) **(a)** We are first asked to show that y_0 is the mean of x_0 and z_0. This is equivalent to $2y_0 = x_0 + z_0$. Considering the differential equations we have:

$$\begin{aligned}
\frac{\mathrm{d}(x - 2y + z)}{\mathrm{d}t} &= (4y - 5z + u) - 2(x - 2z + u) + (x - 2y + u) \\
&= -x + 2y - z \\
\implies x - 2y + z &= A\mathrm{e}^{-t}
\end{aligned}$$

Then if $x = y = z = 0$ at some time t then we have $A = 0$ and so $x - 2y + z = 0$ for all t. Setting $t = 0$ gives $2y_0 = x_0 + z_0$ and so y_0 is the mean of x_0 and z_0.

(b) Considering $\frac{\mathrm{d}y}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t}$ gives:

$$\begin{aligned}
\frac{\mathrm{d}(y - z)}{\mathrm{d}t} &= 2(y - z) \\
\implies y - z &= B\mathrm{e}^{2t}
\end{aligned}$$

Then if the particle passes through $(0, 0, 0)$ we have $y = z$, and so $y_0 = z_0$. Using $x_0 = 2y_0 - z_0$ from the previous parts gives the result $x_0 = y_0 = z_0$ as required.

The version given in the mark scheme considers $\frac{\mathrm{d}x}{\mathrm{d}t} - \frac{\mathrm{d}z}{\mathrm{d}t}$, which needs use of $x - 2y + z = 0$ in order to reduce the differential equation to one that can be solved. This method feels a little neater.

(c) From the previous parts we have:

$$x - 2y + z = Ae^{-t}$$
$$y - z = Be^{2t}$$

Since we have $x_0 = y_0 = z_0$ we have $A = 0$ and $B = 0$, and so we have $x - 2y + z = 0$ and $y - z = 0$ for all t. Hence we have $x = y = z = 0$ for all t.
Considering the differential equation in x we have:

$$\begin{aligned} \frac{dx}{dt} &= -x + u \\ e^t \frac{dx}{dt} + e^t x &= ue^t \\ xe^t &= ue^t + c \\ x &= u + ce^{-t} \end{aligned}$$

At $t = 0$ we have $x = x_0$ and so $x_0 = u + c$. We want the particle to be at the origin at time $t = T$ which means we need $0 = u + ce^{-T}$. This gives:

$$\begin{aligned} x_0 &= u - ue^T \\ \implies u &= \frac{x_0}{1 - e^T} \end{aligned}$$

Note that we could have used any of the three equations so you could also write the answer as $u = \dfrac{y_0}{1 - e^T}$ or $u = \dfrac{z_0}{1 - e^T}$.

Alternatively parts **(a)** and **(b)** can be done at the same time.

Start by considering $\dfrac{dy}{dt} - \dfrac{dz}{dt}$ which gives $y = z$, and $y_0 = z_0$ as before.

Using $y = z$ gives:

$$\begin{aligned} \frac{dx}{dt} &= -y + u \\ \frac{dy}{dt} &= x - 2y + u \\ \implies \frac{d(x - y)}{dt} &= -(x - y) \\ x - y &= Be^{-t} \end{aligned}$$

So we have $x = y$, and so we have $x_0 = y_0$. Hence we have $x_0 = y_0 = z_0$, which also implies that $y_0 = \dfrac{x_0 + z_0}{2}$. This completes parts **(a)** and **(b)**.

For part **(c)** we have $y - z = Ae^{2t}$ and $x - y = Be^{-t}$. The initial condition $x_0 = y_0 = z_0$ gives $x = y = z$. The rest of the question follows on as before.

Question 7

7 *In this question, you need not consider issues of convergence.*

For positive integer n let

$$\mathrm{f}(n) = \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \dots$$

and

$$\mathrm{g}(n) = \frac{1}{n+1} - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} - \dots .$$

(i) Show, by considering a geometric series, that $0 < \mathrm{f}(n) < \dfrac{1}{n}$.

(ii) Show, by comparing consecutive terms, that $0 < \mathrm{g}(n) < \dfrac{1}{n+1}$.

(iii) Show, for positive integer n, that $(2n)!\,\mathrm{e} - \mathrm{f}(2n)$ and $\dfrac{(2n)!}{\mathrm{e}} + \mathrm{g}(2n)$ are both integers.

(iv) Show that if $q\mathrm{e} = \dfrac{p}{\mathrm{e}}$ for some positive integers p and q, then $q\mathrm{f}(2n) + p\mathrm{g}(2n)$ is an integer for all positive integers n.

(v) Hence show that the number e^2 is irrational.

Examiner's report

The third most popular question, this was a little less successfully attempted than question 2 with a mean score of just over 9/20.

Parts **(i)** and **(ii)** were not generally well done, as it was easy to guess the geometric series and then make unsubstantiated, or at least unjustified, claims which could not be given full marks.

In part **(ii)**, there was frequently lack of clarity regarding pairing of terms and arguments lacking in necessary detail to support the claims.

Part **(iii)** was done better, though the second result commonly saw $\frac{1}{\mathrm{e}}$ expanded as a reciprocal rather than as e^{-1}, and then, as a consequence, getting lost.

Part **(iv)**, too, was fairly well done. There was a good understanding of contradiction arguments for part **(v)**, though there was difficulty in choosing a suitable n in quite a few cases.

Solution

(i) Note that there are two sides to the inequality! We know that n is a positive integer so every term of $\mathrm{f}(n)$ is positive, and so we have $\mathrm{f}(n) > 0$ (showing this side of the inequality was an easy step to forget to do).

If $r > 1$ then we have $\dfrac{1}{n+r} < \dfrac{1}{n+1}$ and so we have:

$$\begin{aligned}\mathrm{f}(n) &< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \cdots \\ \mathrm{f}(n) &< \frac{\frac{1}{n+1}}{1-\frac{1}{n+1}} \\ \mathrm{f}(n) &< \frac{1}{(n+1)-1} \\ \mathrm{f}(n) &< \frac{1}{n}\end{aligned}$$

And so we have $0 < \mathrm{f}(n) < \dfrac{1}{n}$ as required.

(ii) We have:

$$\begin{aligned}&\frac{1}{n+1} - \frac{1}{(n+1)(n+2)} > 0 \\ &\frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)} > 0\end{aligned}$$

etc., and so we have $\mathrm{g}(n) > 0$.

Also we have:

$$\begin{aligned}&\frac{1}{(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)} > 0 \\ &\frac{1}{(n+1)(n+2)(n+3)(n+4)} - \frac{1}{(n+1)(n+2)(n+3)(n+4)(n+5)} > 0\end{aligned}$$

and so $\mathrm{g}(n) = \dfrac{1}{n+1} - \sum(\text{positive terms}) \implies \mathrm{g}(n) < \dfrac{1}{n+1}$.

Therefore we have $0 < \mathrm{g}(n) < \dfrac{1}{n+1}$ as required.

(iii) Considering the first expression we have:

$$\begin{aligned}(2n)!\mathrm{e} - \mathrm{f}(2n) &= (2n)!\left(1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right) \\ &\quad - \left(\frac{1}{2n+1} + \frac{1}{(2n+1)(2n+2)} + \frac{1}{(2n+1)(2n+2)(2n+3)} + \cdots\right) \\ &= (2n)! + (2n)! + \frac{(2n)!}{2!} + \frac{(2n)!}{3!} + \frac{(2n)!}{4!} + \cdots + \frac{(2n)!}{(2n)!} + \frac{(2n)!}{(2n+1)!} + \frac{(2n)!}{(2n+2)!} + \cdots \\ &\quad - \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} - \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots \\ &= (2n)! + (2n)! + \frac{(2n)!}{2!} + \frac{(2n)!}{3!} + \frac{(2n)!}{4!} + \cdots + \frac{(2n)!}{(2n)!}\end{aligned}$$

which is an integer.

The second expression is:

$$\begin{aligned}\frac{(2n)!}{\mathrm{e}} + \mathrm{g}(2n) &= (2n)!\mathrm{e}^{-1} + \mathrm{g}(2n)\\ &= (2n)!\left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots\right)\\ &\qquad + \left(\frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} + \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots\right)\\ &= (2n)! - (2n)! + \frac{(2n)!}{2!} - \frac{(2n)!}{3!} + \frac{(2n)!}{4!} - \cdots + \frac{(2n)!}{(2n)!} - \frac{(2n)!}{(2n+1)!} + \frac{(2n)!}{(2n+2)!} - \cdots\\ &\qquad + \frac{1}{2n+1} - \frac{1}{(2n+1)(2n+2)} + \frac{1}{(2n+1)(2n+2)(2n+3)} - \cdots\\ &= (2n)! - (2n)! + \frac{(2n)!}{2!} - \frac{(2n)!}{3!} + \frac{(2n)!}{4!} - \cdots + \frac{(2n)!}{(2n)!}\end{aligned}$$

which is also an integer.

(iv) Using the results shown in the previous part, and since p and q are integers we know that both $q\big[(2n)!\mathrm{e} - \mathrm{f}(2n)\big]$ and $p\big[(2n)!\mathrm{e}^{-1} + \mathrm{g}(2n)\big]$ are integers. Hence

$$p\big[(2n)!\mathrm{e}^{-1} + \mathrm{g}(2n)\big] - q\big[(2n)!\mathrm{e} - \mathrm{f}(2n)\big] \quad \text{is an integer}$$

I got this expression by looking at my results from part **(iii)** and also considering what I wanted to show in this part. The fact that I was looking to find something of the form $p\mathrm{g}(2n) + q\mathrm{f}(2n)$ was what prompted how I combined the previous results.

Rearranging gives:

$$p\mathrm{g}(2n) + q\mathrm{f}(2n) + (2n)!\left[p\mathrm{e}^{-1} - q\mathrm{e}\right] = p\mathrm{g}(2n) + q\mathrm{f}(2n)$$

using $q\mathrm{e} = \dfrac{p}{\mathrm{e}}$ to cancel the square bracket. Hence we have $p\mathrm{g}(2n) + q\mathrm{f}(2n)$ is an integer as required.

(v) To answer this last part we probably need to look back at the previous parts. In parts **(i)** and **(ii)** we justified some limits on $\mathrm{f}(n)$ and $\mathrm{g}(n)$.

In part **(iv)** we showed that if $q\mathrm{e} = \dfrac{p}{\mathrm{e}}$ for some positive integers p and q then $p\mathrm{g}(2n) + q\mathrm{f}(2n)$ is a positive integer for all positive integers n. Let n be the maximum of p and q. We then have:

$$p\mathrm{g}(2n) < \frac{p}{2n+1} \leqslant \frac{n}{2n+1} < \frac{1}{2}$$

and

$$q\mathrm{f}(2n) < \frac{q}{2n} \leqslant \frac{n}{2n} \leqslant \frac{1}{2}$$

Therefore we have $p\mathrm{g}(2n) + q\mathrm{f}(2n) < 1$, and we also have $p\mathrm{g}(2n) + q\mathrm{f}(2n) > 0$. This means that $p\mathrm{g}(2n) + q\mathrm{f}(2n)$ is not an integer for sufficiently large values of n. This contradicts part **(iv)**, and so there are no integers such that $\dfrac{p}{\mathrm{e}} = q\mathrm{e}$, i.e. there are no integers such that $\mathrm{e}^2 = \dfrac{p}{q}$. Hence e^2 is irrational.

Question 8

8 **(i)** Explain why the equation $(y-x+3)(y+x-5)=0$ represents a pair of straight lines with gradients 1 and -1. Show further that the equation

$$y^2 - x^2 + py + qx + r = 0$$

represents a pair of straight lines with gradients 1 and -1 if and only if $p^2 - q^2 = 4r$.

In the remainder of this question, C_1 is the curve with equation $x = y^2 + 2sy + s(s+1)$ and C_2 is the curve with equation $y = x^2$.

(ii) Explain why the coordinates of any point which lies on both of the curves C_1 and C_2 also satisfy the equation

$$y^2 + 2sy + s(s+1) - x + k(y - x^2) = 0$$

for any real number k.

Given that s is such that C_1 and C_2 intersect at four distinct points, show that choosing $k = 1$ gives an equation representing a pair of straight lines, with gradients 1 and -1, on which all four points of intersection lie.

(iii) Show that if C_1 and C_2 intersect at four distinct points, then $s < -\frac{3}{4}$.

(iv) Show that if $s < -\frac{3}{4}$, then C_1 and C_2 intersect at four distinct points.

Examiner's report

One of the least popular questions in the Pure Mathematics section, candidates did slightly less well here than on question 7. There were some excellent answers to this question, but also some answers that were lacking in clear explanation. There were sometimes issues with candidates not understanding the direction of implication required by the various question parts. The best solutions used the structure of the question to help find appropriate and efficient methods to solve the problem but there were also some inventive solutions using other techniques.

Part **(i)** was generally done well, though some candidates did not show sufficient working to justify the given answer fully.

Part **(ii)** was also generally done well, but some candidates did not take advantage of the work done in the previous part to show that the given equation represented a pair of straight lines. A small minority of candidates instead tried to show that if the equation represented a pair of straight lines then $k = 1$.

Parts **(iii)** and **(iv)** were found to be more difficult.

In Part **(iii)** the most successful candidates tended to follow the lead of the previous parts and factorised the equation in part **(ii)** to find the equations of two straight lines. A considerable number of candidates made a sign error while doing this: expanding to check a factorisation is correct is always a good idea. Those that factorised usually could see how to set up two quadratic equations in x and so find a condition of s. Some candidates set up a quartic equation in x but only a small number of these could complete an argument to show that $s < -0.75$, and these candidates often were confused on the direction of implication needed in this part.

The direction of implication required in part **(iv)** confused a lot of candidates, with some stating that they had already answered this in the previous part and others repeating a proof that four distinct points implies $s < -0.75$. Some other candidates recognised that there must be two distinct points of intersection of the curves and each line but did not realise that one of these points of intersection could be where both curves and both lines meet. A sketch was often a good idea to help clarify the geometry of the situation. A handful of candidates managed to consider the "if and only if" situation by considering where the two straight lines were tangential to $y = x^2$ answering both of the last two parts in one go.

Solution

(i) If we have $(y - x + 3)(y + x - 5) = 0$ then we have either:

$$\begin{aligned} y - x + 3 = 0 &\implies y = x - 3 \\ y + x - 5 = 0 &\implies y = -x + 5 \end{aligned}$$

and hence we have a pair of lines with gradients 1 and -1.

The equation $y^2 - x^2 + py + qx + r = 0$ represents a pair of straight lines with gradients ± 1 if and only if we can rewrite $y^2 - x^2 + py + qx + r$ in the form $(y - x + a)(y + x + b)$

It is important that you don't reuse letters which have already been used in the question!

Equating the different expressions gives:

$$\begin{aligned} y^2 - x^2 + py + qx + r &\equiv (y - x + a)(y + x + b) \\ y^2 - x^2 + py + qx + r &\equiv y^2 + \cancel{xy} + by - \cancel{xy} - x^2 - bx + ay + ax + ab \end{aligned}$$

Equating coefficients gives:

$$\begin{aligned} p &= a + b \\ q &= a - b \\ r &= ab \end{aligned}$$

We then have:

$$\begin{aligned} p^2 - q^2 &= (p + q)(p - q) \\ &= 2a \times 2b \\ &= 4ab \\ &= 4r \end{aligned}$$

Hence $y^2 - x^2 + py + qx + r = 0$ represents a pair of straight lines if and only if we have $p^2 - q^2 = 4r$.

(ii) A point on C_1 satisfies $y^2 + 2sy + 2(s+1) - x = 0$ and a point on C_2 satisfies $y - x^2 = 0$. Taking a linear combination of these equations gives:

$$y^2 + 2sy + 2(s+1) - x + k(y - x^2) = 0$$

as required.

If we take $k = 1$ we have:

$$y^2 - x^2 + (2s+1)y - x + s(s+1) = 0$$

This looks like the form in part **(i)** with $p = 2s+1$, $q = -1$ and $r = s(s+1)$. Considering $p^2 - q^2$ we have:

$$\begin{aligned} p^2 - q^2 &= (2s+1)^2 - (-1)^2 \\ &= 4s^2 + 4s + 1 - 1 \\ &= 4s(s+1) \\ &= 4r \end{aligned}$$

Therefore this equation satisfies $p^2 - q^2 = 4r$ and so it represents a pair of straight lines with gradients ± 1, and the points of intersection of C_1 and C_2 must lie on one (or both) of these lines.

(iii) For the last two parts of this question you are being asked to show that an implication works in two different directions. It can be quite tricky to put both directions into one argument so the request has been split into two parts to help you! It is very important that you are very clear about which condition you are assuming is true, and which one you are trying to prove, in each case.

If the curves intersect at four distinct points then C_2 (i.e. $y = x^2$) must meet the two straight lines at four distinct points. This means that the curve must meet each of the straight lines at two distinct points (as a quadratic has at most two points of intersection with a straight line).

Consider the equation for the points of intersection with $k = 1$:

$$\begin{aligned} y^2 - x^2 + (2s+1)y - x + s(s+1) &= 0 \\ (y + x + s + 1)(y - x + s) &= 0 \end{aligned}$$

It is possible to factorise the expression by inspection **BUT** make sure you expand it to double check that you have not made a sign error!

Therefore the straight lines have equations $y = x - s$ and $y = -x - s - 1$.

If C_1 and C_2 intersect at 4 distinct points then we must have two distinct points of intersection between C_2 and $y = x - s$ and two distinct points of intersection of C_2 and $y = -x - s - 1$. This means that the following quadratic equations must each have a positive discriminant:

$$\begin{aligned} x^2 &= x - s \\ x^2 &= -x - s - 1 \end{aligned}$$

Considering the discriminant of $x^2 - x + s = 0$ gives $1 - 4s > 0 \implies s < \frac{1}{4}$.

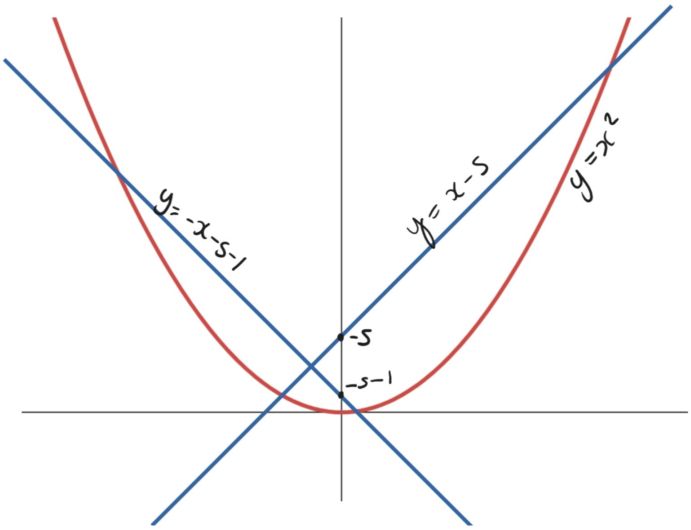
Considering the discriminant of $x^2 + x + s + 1 > 0$ gives $1 - 4(s+1) > 0 \implies -4s > 3 \implies s < -\frac{3}{4}$.

Therefore if C_1 and C_2 intersect in 4 distinct points then we must have $s < -\frac{3}{4}$.

(iv) In this part we are working in the opposite direction, this time we are starting with the assumption that $s < -\frac{3}{4}$.

If $s < -\frac{3}{4}$ then, from the work done in the previous part, we know that there are two distinct points of intersection between C_2 and $y = -x - s - 1$. Since $s < -\frac{3}{4}$ then we also have $s < \frac{1}{4}$ then we know that there are two distinct points of intersection between C_2 and $y - x + s = 0$.

A sketch of the situation will help us see what is going on here (**when stuck a sketch is often a good idea!**). Remember that we have $s < -\frac{3}{4}$, so we know that s is negative.



In the sketch above you can see that there are 4 distinct points of intersection of the two straight lines and C_2, but as s varies the situation changes. We know that there are two distinct points of intersection of each straight line with C_2, but if the two straight lines themselves intersect on C_2 then there will only be at most three distinct points of intersection of the two straight lines of C_2.

The point of intersection of the two straight lines is where $x - s = -x - s - 1 \implies x = -\frac{1}{2}$ and so the point of intersection of the two lines is at $\left(-\frac{1}{2}, -s - \frac{1}{2}\right)$. This will lie on C_2 if and only if

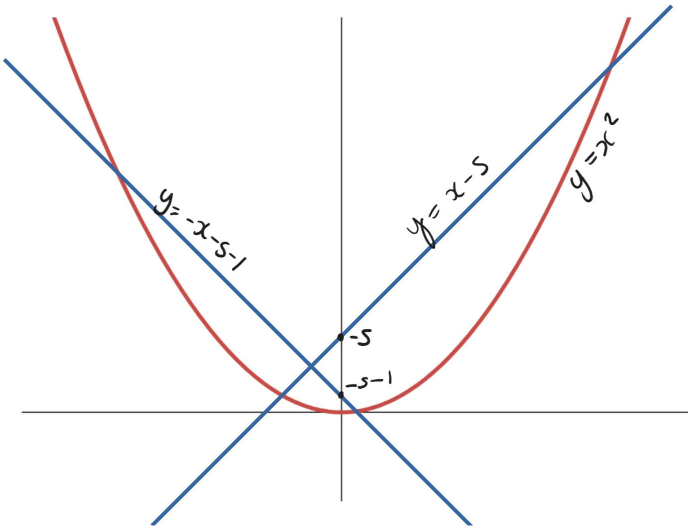
$$\begin{aligned} y &= x^2 \\ -s - \frac{1}{2} &= \left(-\frac{1}{2}\right)^2 \\ \implies s &= -\frac{3}{4} \end{aligned}$$

Which is not allowed as we have $s < -\frac{3}{4}$.

The following method could be use to provide an "if and only if" proof for parts **(iii)** & **(iv)**.

C_1 and C_2 intersect at four distinct points if and only if C_2 meets the two lines $y = x - s$ and $y = -x - s - 1$ at four distinct points.

If s is negative then the graph of C_2 and the straight lines looks like this:



As s increases (i.e. as it gets "less negative" and then positive) the straight lines will move "downwards" as the y intercepts decrease. As s increases the straight lines will go from meeting the curve $y = x^2$ at two distinct points, to meeting at just one (when the straight line is a tangent to C_2) to not meeting at all.

From the sketch we can see that the first line to become a tangent to C_2 will be $y = -x - s - 1$.

When the line is tangential to the curve the gradients are the same, so we want the point on C_2 when the gradient is equal to -1. This happens when $2x = -1$, and so the point on C_2 with gradient -1 is $\left(-\frac{1}{2}, \frac{1}{4}\right)$. We need to find the value of s which means that the straight line intersects C_2 at this point, so we need:

$$\begin{aligned} y &= -x - s - 1 \\ \frac{1}{4} &= \frac{1}{2} - s - 1 \\ s &= -\frac{1}{2} - \frac{1}{4} \\ s &= -\frac{3}{4} \end{aligned}$$

So when $s = -\frac{3}{4}$ the line $y = -x - s - 1$ is tangential to C_2 and so only has one point of intersection.

Therefore C_2 has four distinct points of intersection with the straight lines if and only if $s < -\frac{3}{4}$ and so C_1 and C_2 have four distinct points of intersection if and only if $s < -\frac{3}{4}$.

Note that this method combined with finding the point of intersection of the straight lines can be used to show that C_1 and C_2 can never intersect at exactly three distinct points.

Question 9

9 The origin O of coordinates lies on a smooth horizontal table and the x- and y-axes lie in the plane of the table. A smooth sphere A of mass m and radius r is at rest on the table with its lowest point at the origin.

A second smooth sphere B has the same mass and radius and also lies on the table. Its lowest point has y-coordinate $2r\sin\alpha$, where α is an acute angle, and large positive x-coordinate.

Sphere B is now projected parallel to the x-axis, with speed u, so that it strikes sphere A. The coefficient of restitution in this collision is $\frac{1}{3}$.

(i) Show that, after the collision, sphere B moves with velocity

$$\begin{pmatrix} -\frac{1}{3}u\left(1+2\sin^2\alpha\right) \\ \frac{2}{3}u\sin\alpha\cos\alpha \end{pmatrix}.$$

(ii) Show further that the lowest point of sphere B crosses the y-axis at the point $(0,\,Y)$, where $Y = 2r\,(\cos\alpha\tan\beta + \sin\alpha)$ and

$$\tan\beta = \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}.$$

A third sphere C of radius r is at rest with its lowest point at $(0,h)$ on the table, where $h > 0$.

(iii) Show that, if $h > Y + 2r\sec\beta$, sphere B will not strike sphere C in its motion after the collision with sphere A.

(iv) Show that $Y < 2r\sec\beta$.

Hence show that sphere B will not strike sphere C for any value of α, if $h > \dfrac{8r}{\sqrt{3}}$.

Examiner's report

This was an unpopular question, only being attempted by about a seventh of the candidates. It was also the second least successful with a mean score of only 4/20. There were mixed responses, and it mostly depended on how the diagram was set up, that is in which directions candidates chose to label the velocities. Many candidates struggled to understand how to apply the restitution law when the particles collide obliquely rather than directly along the line of centres. Some tried to use total speeds of the particles rather than the speeds along the line of contact, and some tried to use the horizontal speeds. Many also did not use vectors correctly, drawing vectors in certain directions then not introducing necessary negative signs.

Other than that, part **(i)** was done well and most understood how to rotate the solution back into usual $x - y$ directions.

Those who got to part **(ii)** generally did it easily.

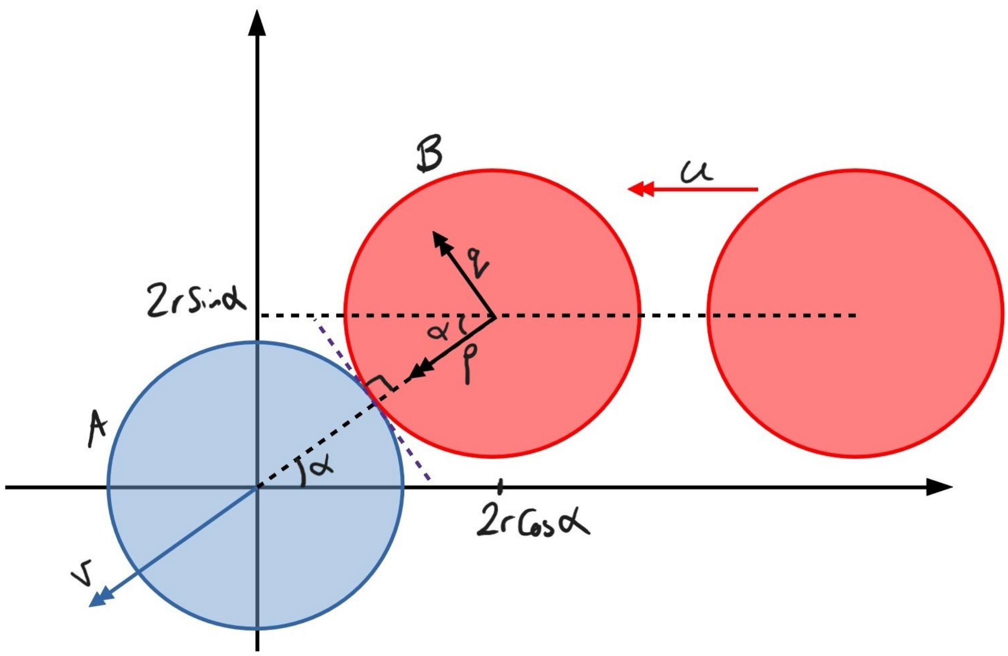
Most found part **(iii)** trickier, and it tended to be either done well or not really started. Once the diagram was set up, it was found to be straightforward, and most who got that far saw how to proceed.

There were very few significant attempts at part **(iv)**.

Solution

In this question there was a bit of a trade off in how to describe the situation. "The lowest point of the sphere" feels a little inelegant, but means the point where the sphere meets the table. It will be vertically below the centre of the sphere. Alternatively the question could have talked about "pucks" which are disc shaped, i.e. short cylinders, but "pucks" are not necessarily a globally known word. "Discs" feel like they would be 2D objects, and the idea of cylinders being pushed around feels odd - we are told that everything is smooth so they would just glide about, but that doesn't tally with our world view, and perhaps using cylinders would make some candidates worry about toppling.

It's a mechanics question, so the first thing to do is to draw a diagram! At first reading the initial coordinates of the lowest point of B seem a little odd, but when you add in the position of B when it hits A the coordinates make more sense.



After the collision sphere A will move in the direction of the line of centres, and let the speed in this direction be v. I have chosen to resolve the velocity of B after the collision in the directions parallel and perpendicular to the line of centres, but you could instead resolve horizontally and vertically. I initially used a and b for the components of B's velocity, but decided that a and α were a bit too similar for my handwriting to cope with.

(i) Conservation of momentum along the line of centres gives:

$$mu\cos\alpha = mv + mp \implies u\cos\alpha = v + p$$

Newton's law of restitution gives:

$$v - p = \frac{1}{3}u\cos\alpha$$

Subtracting the second equation from the first gives $2p = \frac{2}{3}u\cos\alpha \implies p = \frac{1}{3}u\cos\alpha$.

Conservation of momentum perpendicular to the lines of centres gives:

$$mu\sin\alpha = mq \implies q = u\sin\alpha$$

The velocity of B with respect to the axes is given by:

$$\begin{aligned}\begin{pmatrix}-p\cos\alpha - q\sin\alpha \\ q\cos\alpha - p\sin\alpha\end{pmatrix} &= \begin{pmatrix}-\frac{1}{3}u\cos^2\alpha - u\sin^2\alpha \\ u\sin\alpha\cos\alpha - \frac{1}{3}u\cos\alpha\sin\alpha\end{pmatrix} \\ &= \begin{pmatrix}-\frac{1}{3}u\left(\cos^2\alpha + 3\sin^2\alpha\right) \\ \frac{2}{3}u\sin\alpha\cos\alpha\end{pmatrix} \\ &= \begin{pmatrix}-\frac{1}{3}u\left(1 + 2\sin^2\alpha\right) \\ \frac{2}{3}u\sin\alpha\cos\alpha\end{pmatrix}\end{aligned}$$

as required.

(ii) If t is the time after the collision then the position of B is given by:

$$\begin{pmatrix}2r\cos\alpha - \frac{1}{3}ut\left(1 + 2\sin^2\alpha\right) \\ 2r\sin\alpha + \frac{2}{3}ut\sin\alpha\cos\alpha\end{pmatrix}$$

At the point where B crosses the y-axis (i.e where $x = 0$) we have:

$$\begin{aligned}0 &= 2r\cos\alpha - \frac{1}{3}ut\left(1 + 2\sin^2\alpha\right) \\ ut\left(1 + 2\sin^2\alpha\right) &= 6r\cos\alpha \\ t &= \frac{6r\cos\alpha}{u\left(1 + 2\sin^2\alpha\right)}\end{aligned}$$

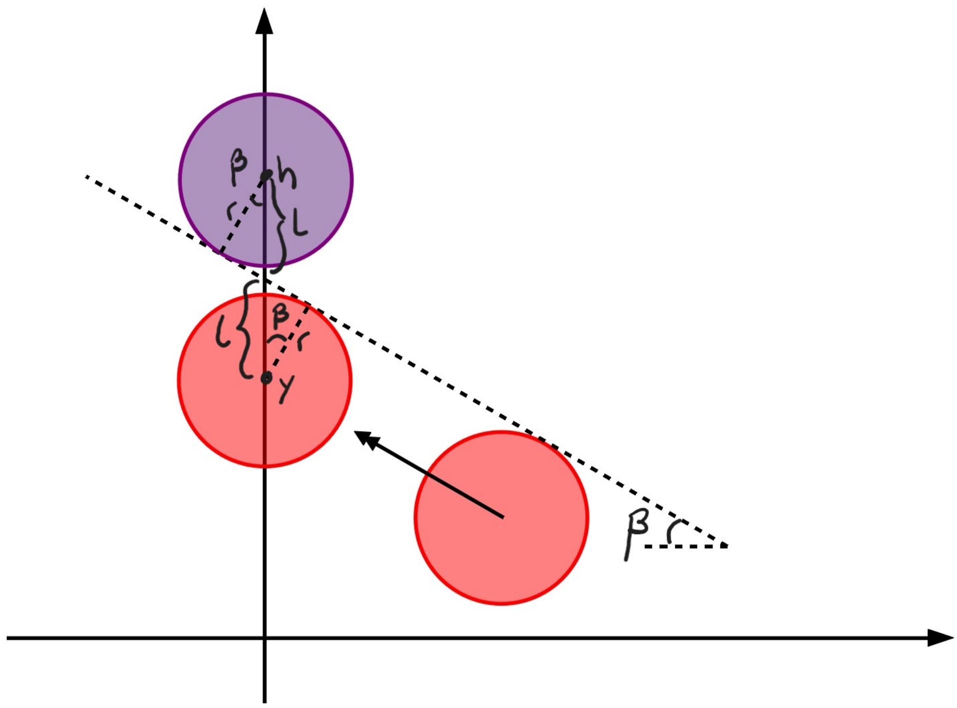
Substituting this value of t into the y coordinate gives:

$$\begin{aligned}y &= 2r\sin\alpha + \frac{2}{3}ut\sin\alpha\cos\alpha \\ Y &= 2r\sin\alpha + \frac{2}{3}u\sin\alpha\cos\alpha \times \frac{6r\cos\alpha}{u\left(1 + 2\sin^2\alpha\right)} \\ &= 2r\sin\alpha + \frac{4r\cos^2\alpha\sin\alpha}{\left(1 + 2\sin^2\alpha\right)} \\ &= 2r\sin\alpha + 2r\cos\alpha \times \frac{2\sin\alpha\cos\alpha}{1 + 2\sin^2\alpha} \\ &= 2r\sin\alpha + 2r\cos\alpha\tan\beta\end{aligned}$$

where $\tan\beta = \dfrac{2\sin\alpha\cos\alpha}{1 + 2\sin^2\alpha}$.

The final form required for Y might seem a little odd, but it is likely that being asked to write it in this form will be useful later!

(iii) After the collision, sphere B travels in a straight line in a direction parallel to the velocity found in part **(i)**. If sphere B just brushes against sphere C then the picture will look like this:



Notice that I have labelled some angles as being equal to β. If we consider the gradient of the direction of velocity, we find that this is equal to:

$$\frac{\frac{2}{3}u\sin\alpha\cos\alpha}{-\frac{1}{3}u\left(1+2\sin^2\alpha\right)} = -\frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha} = -\tan\beta$$

Hence the angle the direction of B makes with the horizontal is β, and chasing round the right-angled triangles shows why the other marked angles are also equal to β.

By symmetry,the two lengths marked l are equal and we have $l = \dfrac{r}{\cos\beta} = r\sec\beta$. Hence if we have $h > Y + 2r\sec\beta$ then B will miss C.

(iv) We have:

$$\begin{aligned} Y &= 2r\left(\cos\alpha\tan\beta + \sin\alpha\right) \\ &= 2r\sec\beta\left(\cos\alpha\sin\beta + \sin\alpha\cos\beta\right) \\ &= 2r\sec\beta\sin\left(\alpha+\beta\right) \\ \implies Y &\leqslant 2r\sec\beta \end{aligned}$$

However we are asked to find a strict inequality so we need to eliminate the case where $\sin(\alpha+\beta) = 1$. If we have $\sin(\alpha+\beta) = 1$ then we have $\alpha + \beta = \frac{\pi}{2}$ (we are told that α is acute, and from the result in part **(i)** we have that β is acute).

Substituting $\beta = \frac{\pi}{2} - \alpha$ into the expression for $\tan\beta$ gives:

$$\begin{aligned} \tan\left(\frac{\pi}{2} - \alpha\right) &= \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha} \\ \cot\alpha &= \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha} \\ \cos\alpha\left(1+2\sin^2\alpha\right) &= 2\sin^2\alpha\cos\alpha \\ \implies \cos\alpha &= 0 \end{aligned}$$

Which implies that $\alpha = \frac{\pi}{2}$, which is not possible as we are told that α is acute.

Therefore we have $Y < 2r\sec\beta$.

For the last part, we know that there is no collision if $h > Y + 2r\sec\beta$ so if we can find the maximum value of $\sec\beta$ as α varies we can find a value of h where spheres B and C can never collide.

We know that β is acute, and we have $\sec^2\beta = \tan^2\beta + 1$, so the maximal value of $\sec\beta$ will occur when $\tan\beta$ is at a maximum.

We have:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\alpha}\tan\beta &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha}\right) \\ &= \frac{\left(1+2\sin^2\alpha\right)\times 2\left(\cos^2\alpha - \sin^2\alpha\right) - 2\sin\alpha\cos\alpha\times 4\sin\alpha\cos\alpha}{\left(1+2\sin^2\alpha\right)^2} \\ &= \frac{2\left(1+2\sin^2\alpha\right)\left(1-2\sin^2\alpha\right) - 8\sin^2\alpha(1-\sin^2\alpha)}{\left(1+2\sin^2\alpha\right)^2} \\ &= \frac{2 - 8\sin^4\alpha - 8\sin^2\alpha + 8\sin^4\alpha}{\left(1+2\sin^2\alpha\right)^2} \\ &= \frac{2-8\sin^2\alpha}{\left(1+2\sin^2\alpha\right)^2} \end{aligned}$$

The derivative is equal to zero when $2 - 8\sin^2\alpha = 0$, i.e. when $\sin\alpha = \frac{1}{2}$ (and so $\cos\alpha = \frac{\sqrt{3}}{2}$). This gives:

$$\begin{aligned} \tan\beta &= \frac{2\sin\alpha\cos\alpha}{1+2\sin^2\alpha} \\ &= \frac{\frac{\sqrt{3}}{2}}{1+\frac{1}{2}} \\ &= \frac{\sqrt{3}}{3} \end{aligned}$$

We know that this must be a maximum as when $\alpha = 0$ or $\alpha = \frac{\pi}{2}$ we have $\tan\beta = 0$. We then have $\sec^2\beta = 1 + \tan^2\beta \implies \sec^2\beta = \frac{12}{9} \implies \sec\beta = \frac{2\sqrt{3}}{3}$.

This means that there will be no collision for any value of α as long as:

$$\begin{aligned}
h &> Y + 2r\sec\beta \\
h &> 4r\sec\beta \\
h &> 4r \times \frac{2\sqrt{3}}{3} \\
h &> \frac{8r}{\sqrt{3}} \qquad \text{as required.}
\end{aligned}$$

Question 10

10 A cube of uniform density ρ is placed on a horizontal plane and a second cube, also of uniform density ρ, is placed on top of it. The lower cube has side length 1 and the upper cube has side length a, with $a \leqslant 1$. The centre of mass of the upper cube is vertically above the centre of mass of the lower cube and all the edges of the upper cube are parallel to the corresponding edges of the lower cube. The contacts between the two cubes, and between the lower cube and the plane, are rough, with the same coefficient of friction $\mu < 1$ in each case. The midpoint of the base of the upper cube is X and the midpoint of the base of the lower cube is Y.

A horizontal force P is exerted, perpendicular to one of the vertical faces of the upper cube, at a point halfway between the two vertical edges of this face, and a distance h, with $h < a$, above the lower edge of this face.

(i) Show that, if the two cubes remain in equilibrium, the normal reaction of the plane on the lower cube acts at a point which is a distance

$$\frac{P(1+h)}{(1+a^3)\rho g}$$

from Y, and find a similar expression for the distance from X of the point at which the normal reaction of the lower cube on the upper cube acts.

The force P is now gradually increased from zero.

(ii) Show that, if neither cube topples, equilibrium will be broken by the slipping of the upper cube on the lower cube, and not by the slipping of the lower cube on the ground.

(iii) Show that, if $a = 1$, then equilibrium will be broken by the slipping of the upper cube on the lower cube if $\mu(1+h) < 1$ and by the toppling of the lower and upper cube together if $\mu(1+h) > 1$.

(iv) Show that, in a situation where $a < 1$ and $h\big(1+a^3(1-a)\big) > a^4$, and no slipping occurs, equilibrium will be broken by the toppling of the upper cube.

(v) Show, by considering $a = \frac{1}{2}$ and choosing suitable values of h and μ, that the situation described in (**iv**) can in fact occur.

Examiner's report

This was the least popular question on the paper by some way, being attempted by fewer than 6% of the candidates. It was attempted only a little more successfully than question 9 scoring a mean of about 5.5/20. Some of the few attempts were little more than a poor diagram and nothing further. If it was setup correctly, the candidates did fairly well, despite losing marks for not drawing everything required on the diagram, though there was some leniency about drawing equal and opposite forces (e.g. the reaction force from the top cube down onto the bottom cube). It should be stressed that very few did this so it could be a point of focus when preparing candidates for STEP mechanics. The only common error found once the first part was complete was mostly to do with reading carefully.

In part **(iii)**, most did not check that the upper cube could not topple without the lower toppling first, they just compared toppling of bottom cube and slipping.

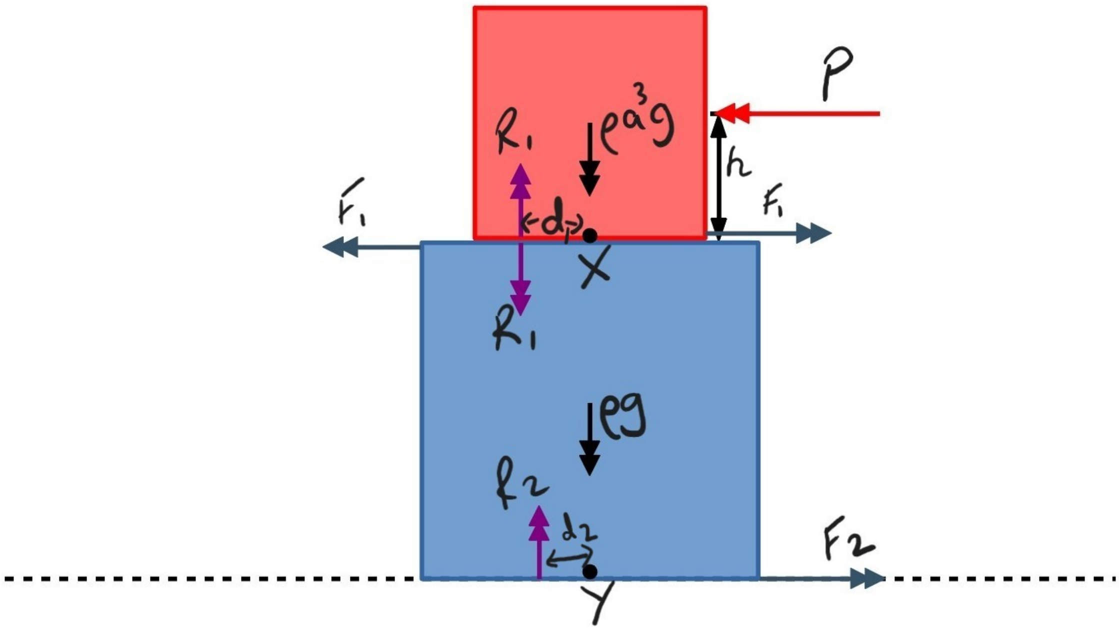
The main challenge in this sort of question is in the initial setup, after which the techniques required are not particularly difficult. Candidates who were able to interpret the context and setup the situation usually did very well.

Solution

No prizes for what I am going to say - it's a mechanics question so the first thing is to draw a good, large, clear diagram! There is a lot of information in the stem of this question, and quite a lot of it is saying that the top cube is placed as you might expect onto the bottom cube, i.e. in the middle with everything lined up nicely. However the wording in the stem is more precise and accurate (if a bit lengthy!).

A bit of thinking about how the cubes will react when the top one is pushed is needed to get the frictional forces acting in the right direction, and to get the reaction forces on the correct side of X and Y.

The cubes and forces look like this:



For some reason I have decided to push from right to left - this might be because I have just finished question 9 where everything was going in that direction!

(i) We have:

$R_1 = \rho a^3 g$	Vertical forces on upper cube	(1)
$P = F_1$	Horizontal forces on upper cube	(2)
$Ph = R_1 d_1$	Taking moments for the upper cube about X	(3)
$R_2 = \rho g + R_1$	Vertical forces on lower cube	(4)
$F_1 = F_2$	Horizontal forces on lower cube	(5)
$R_2 d_2 = F_1 + R_1 d_1$	Taking moments for the lower cube about X	(6)

There are lots of equations here so it's a good idea to number them!

In this part we are being asked to find d_1 and d_2 (the second of which is a "show that").

Substituting equation (1) into equation (3) gives:

$$\begin{aligned} R_1 d_1 &= Ph \\ \rho a^3 g d_1 &= Ph \\ d_1 &= \frac{Ph}{\rho a^3 g} \end{aligned}$$

Also:

$$
\begin{aligned}
R_2d_2 &= F_1 + R_1d_1 && \text{equation (6)}\\
R_2d_2 &= F_1 + Ph && \text{using equation (3)}\\
(\rho g + R_1)d_2 &= P + Ph && \text{using equations (2) and (4)}\\
(\rho g + \rho a^3 g)d_2 &= P + Ph && \text{using equation (1)}\\
\rho g(1 + a^3)d_2 &= P(1 + h) && \text{factorising}\\
d_2 &= \frac{P(1+h)}{\rho g(1+a^3)} && \text{as required.}
\end{aligned}
$$

(ii) In this part we are told that neither cube topples. If we are in equilibrium then we have $F_1 = F_2 = P$, so as P increases one of the frictions will reach it's limiting value first.

The limiting frictions are given by:

$$
\begin{aligned}
F_1(\max) &= \mu R_1\\
F_2(\max) &= \mu R_2 = \mu(\rho g + R_1) > \mu R_1
\end{aligned}
$$

Hence $F_1(\max)$ is reached first and equilibrium is broken by the upper cube slipping on the lower cube.

(iii) When $a = 1$ the two cubes are the same size, and we have $R_1 = \rho g$.

The upper cube will slip on the lower cube when $P > \mu\rho g$.

From part **(ii)** we know that if nether cube topples the upper cube will slip before the lower cube, so we don't need to consider the lower cube slipping.

The upper cube will topple when $d_1 > \frac{1}{2}$ (i.e the reaction force has moved beyond the edge of the cube). Using the result $d_1 = \dfrac{Ph}{\rho a^3 g}$ (from part **(i)**) with $a = 1$ the upper cube will topple when $P > \dfrac{\rho g}{2h}$.

Both cubes will topple when $d_2 > \frac{1}{2}$ which, using the result $d_2 = \dfrac{P(1+h)}{\rho(1+a^3)g}$ with $a = 1$, tells us that both cubes will topple when $P > \dfrac{2\rho g}{2(1+h)}$ i.e. $P > \dfrac{\rho g}{1+h}$. Since we know that $h < 1$ we have $2h < 1 + h$ and so $\dfrac{\rho g}{2h} > \dfrac{\rho g}{1+h}$. This means that if equilibrium is broken by toppling it will be both cubes toppling together rather than just the upper one.

This means that equilibrium will be broken **either** by the upper cube slipping, when $P > \mu\rho g$, **or** by both cubes toppling, when $P > \dfrac{\rho g}{1+h}$, depending on which limit is reached first.

Equilibrium will be broken by the upper cube slipping if:

$$
\begin{aligned}
&\mu\rho g < \frac{\rho g}{1+h}\\
\implies &\mu(1+h) < 1
\end{aligned}
$$

and if $\mu(1 + h) > 1$ equilibrium will be broken by both cubes toppling.

(iv) This time $a < 1$. Using $d_1 = \dfrac{Ph}{\rho a^3 g}$ and $d_2 = \dfrac{P(1+h)}{\rho(1+a^3)g}$ The upper cube topples when $d_1 > \frac{a}{2}$, so when $P > \dfrac{\rho g a^4}{2h}$ and both cubes will topple when $d_2 > \frac{1}{2}$, so when $P > \dfrac{\rho g(1+a^3)}{2(1+h)}$.

So the upper cube will topple first when:

$$\begin{aligned}\frac{\rho g a^4}{2h} &< \frac{\rho g(1+a^3)}{2(1+h)}\\(1+h)a^4 &< h(1+a^3)\\a^4 &< h(1+a^3) - ha^4\\a^4 &< h\left[1+a^3-a^4\right]\\a^4 &< h\left[1+a^3(1-a)\right]\end{aligned}$$

(v) If $a = \frac{1}{2}$ then the condition in part **(iv)** becomes:

$$\begin{aligned}\frac{1}{16} &< h\left[1+\frac{1}{16}\right]\\\implies h &> \frac{1}{17}\end{aligned}$$

We also need $h < a$, and we need there to be no slipping. From part **(ii)** we know that the upper cube will slip first. If the upper cube topples before the upper cubes slips then we have:

$$\begin{aligned}\frac{\rho g a^4}{2h} &< \mu\rho g a^3\\\implies h\mu &> \frac{a}{2}\\h\mu &> \frac{1}{4}\end{aligned}$$

So altogether we need $h < \frac{1}{2}$ (from $h < a$), $h > \frac{1}{17}$ and $h\mu > \frac{1}{4}$. Since $\mu < 1$ we must have $h > \frac{1}{4}$. Taking $h = \frac{3}{8}$ and $\mu = \frac{3}{4}$ satisfies all the requirements.

Question 11

11 *In this question, you may use without proof the results*

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n \text{ and } \sum_{r=0}^{n} r\binom{n}{r} = n2^{n-1}.$$

(i) Show that

$$r\binom{2n}{r} = (2n+1-r)\binom{2n}{2n+1-r}$$

for $1 \leqslant r \leqslant 2n$. Hence show that

$$\sum_{r=0}^{2n} r\binom{2n}{r} = 2\sum_{r=n+1}^{2n} r\binom{2n}{r}.$$

(ii) A fair coin is tossed $2n$ times. The value of the random variable X is whichever is the larger of the number of heads and the number of tails shown. If n heads and n tails are shown, then $X = n$.

Show that

$$\mathrm{E}(X) = n\left(1 + \frac{1}{2^{2n}}\binom{2n}{n}\right).$$

(iii) Show that $\dfrac{1}{2^{2n}}\dbinom{2n}{n}$ decreases as n increases.

(iv) In a game, you choose a value of n and pay £n; then a fair coin is tossed $2n$ times. You win an amount in pounds equal to the larger of the number of heads and the number of tails shown. If n heads and n tails are shown, then you win £n. How should you choose n to maximise your expected winnings per pound paid?

Examiner's report

Very nearly 30% of the candidates attempted this, making it the most popular non-Pure question, and they did so relatively successfully with a mean score of nearly 11/20, better than all but question 1. A significant number of candidates gained full or close to full credit.

Part **(i)** was generally well executed, although using $r\binom{2n}{r} = (2n+1-r)\binom{2n}{2n+1-r}$ for $r = 0$ without justification was a common error.

In part **(ii)**, a common error was using an incorrect probability distribution for the random variable X, common examples included asserting that X itself was binomially distributed as $B(2n, \frac{1}{2})$, or asserting that either $\mathrm{P}(X = k) = \frac{1}{2^{2n}}\binom{2n}{r}$ or $\mathrm{P}(X = k) = \frac{2}{2^{2n}}\binom{2n}{r}$ for all $n \leqslant k \leqslant 2n$.

Showing that $\frac{1}{2^{2n}}\binom{2n}{n}$ is a decreasing function of n in part **(iii)** was generally well executed; a few students considered the difference between $\frac{1}{2^{2n}}\binom{2n}{n}$ and $\frac{1}{2^{2n+2}}\binom{2n+2}{n+1}$, rather than the ratio, which led to a largely similar, but slightly more involved, computation.

Part **(iv)** commonly saw candidates trying to maximise total expected winnings, rather than expected winnings per pound. However generally the standard of responses to this question was quite high.

Solution

(i) We have:

$$\begin{aligned} r\binom{2n}{r} &= r \times \frac{(2n)!}{(2n-r)!r!} \\ &= \frac{(2n)!}{(2n-r)!(r-1)!} \\ &= (2n-r+1) \times \frac{(2n)!}{(2n-r+1)!(r-1)!} \\ &= (2n-r+1)\binom{2n}{2n+1-r} \end{aligned}$$

Considering the sum we have:

$$\begin{aligned} \sum_{r=0}^{2n} r\binom{2n}{r} &= 0 + \sum_{r=1}^{n} r\binom{2n}{r} + \sum_{r=n+1}^{2n} r\binom{2n}{r} \\ &= \sum_{r=1}^{n}(2n-r+1)\binom{2n}{2n+1-r} + \sum_{r=n+1}^{2n} r\binom{2n}{r} \end{aligned}$$

Using the substitution $i = 2n - r + 1$ in the first sum gives:

$$\begin{aligned} \sum_{r=0}^{2n} r\binom{2n}{r} &= \sum_{i=n+1}^{2n} i\binom{2n}{i} + \sum_{r=n+1}^{2n} r\binom{2n}{r} \\ &= 2\sum_{r=n+1}^{2n} r\binom{2n}{r} \end{aligned}$$

(ii) Using $\mathrm{E}(X) = \sum x\mathrm{P}(X = x)$ we have:

$$\begin{aligned} \mathrm{E}(X) &= 2\sum_{r=n+1}^{2n} r\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + n\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \\ &= \sum_{r=0}^{2n} r\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} + n\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \end{aligned}$$

The first term is the same as the expectation of a Binomial distribution, $B\left(2n, \frac{1}{2}\right)$ and so is equal to $2n \times \frac{1}{2} = n$. Therefore we have:

$$\begin{aligned} \mathrm{E}(X) &= n + n\binom{2n}{n}\left(\frac{1}{2}\right)^{2n} \\ &= n\left(1 + \left(\frac{1}{2}\right)^{2n}\binom{2n}{n}\right) \end{aligned}$$

(iii) If we consider consecutive terms, $n = m+1$ and $n = m$, and look at the ratio between these we have:

$$\begin{aligned} \frac{\frac{1}{2^{2(m+1)}}\binom{2(m+1)}{m+1}}{\frac{1}{2^{2m}}\binom{2m}{m}} &= \frac{2^{2m}\binom{2m+2}{m+1}}{2^{2m+2}\binom{2m}{m}} \\ &= \frac{1}{4} \times \frac{(2m+2)!}{(m+1)!(m+1)!} \times \frac{m!m!}{(2m)!} \\ &= \frac{(2m+2)(2m+1)}{4(m+1)(m+1)} \\ &= \frac{2m+1}{2(m+1)} < 1 \end{aligned}$$

Since the ratio is less than one we know that $\frac{1}{2^{2n}}\binom{2n}{n}$ decreases as n increases.

(iv) The expected profit is equal to $\mathrm{E}(X) - n = \frac{1}{2^{2n}}\binom{2n}{n}$, so the expected profit per pound is $\frac{1}{2^{2n}}\binom{2n}{n}$. Since we know that this decreases as n increases we should choose $n = 1$.

Question 12

12 **(i)** A point is chosen at random in the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, so that the probability that a point lies in any region is equal to the area of that region. R is the random variable giving the distance of the point from the origin.

Show that the cumulative distribution function of R is given by

$$\mathrm{P}(R \leqslant r) = \sqrt{r^2 - 1} + \tfrac{1}{4}\pi r^2 - r^2 \cos^{-1}(r^{-1}),$$

when $1 \leqslant r \leqslant \sqrt{2}$. What is the cumulative distribution function when $0 \leqslant r \leqslant 1$?

(ii) Show that $\mathrm{E}(R) = \frac{2}{3}\displaystyle\int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r$.

(iii) Show further that $\mathrm{E}(R) = \frac{1}{3}\left(\sqrt{2} + \ln\left(\sqrt{2}+1\right)\right)$.

Examiner's report

A little over one fifth of the candidates attempted this, marginally less successfully than question 11 with a mean score of 10 marks. As with question 11, a significant number of candidates gained full or close to full credit. In the main, there was a dichotomy in student responses: for each of the parts, students were generally either unable to make any real progress with that part question or were able to produce a relatively full solution.

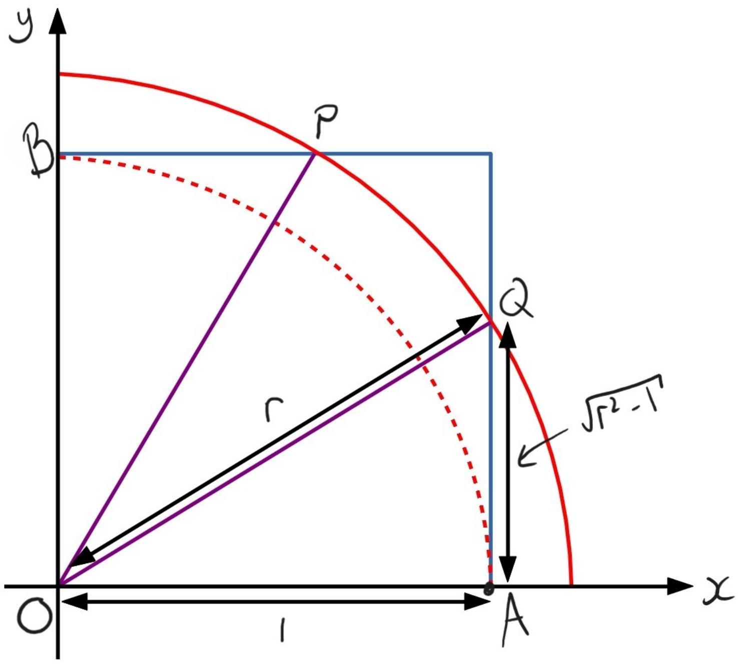
Parts **(i)** and **(ii)** were generally well done, although quite a common error was to incorrectly differentiate the cumulative distribution function from **(i)** to find the probability distribution required for **(ii)**. Another quite common error was attempting to use integration by change of variable rather than by parts to evaluate $\int r^2 \cos^{-1}(r^{-1})\mathrm{d}r$ in **(ii)**.

A number of students only attempted part **(iii)** of the question, in many of these cases, gaining full or close to full marks. For this part, by far the most common approach was to use the substitution $r = \cosh u$ to evaluate the integral. However, other solutions were also seen. Various different substitutions were used either successfully, or at least in some way productively, to evaluate the integral, including $r = \sec u$, $r = \operatorname{cosec} u$, $r = \coth u$, the double substitution $u = \sqrt{r^2-1}$ followed by $u = \sinh x$, and the double substitution $r = \sec u$ followed by $x = \sin u$. However, it was not uncommon to see an unproductive substitution such as $u = r^2 - 1$.

Solution

(i) The maximum possible value of r is $\sqrt{2}$.

The diagram below shows the case where $1 \leqslant r \leqslant \sqrt{2}$.



The area of the whole square is equal to 1, so the probability that $R \leqslant r$ is equal to the area of the shape $OAQPB$ (which, now I come to think about it, is a strange ordering of letters!).

Since $OA = 1$ and $OQ = r$ we have $AQ = \sqrt{r^2 - 1}$ are the area of the two triangles OAQ and OBP together is equal to $1 \times \sqrt{r^2 - 1}$.

The area of sector OPQ can be found by using the area of a sector formula $\frac{1}{2}r^2\theta$. We have $\angle POQ = \frac{1}{2}\pi - 2\cos^{-1}\left(\frac{1}{r}\right)$ and so the area of the sector is given by $\frac{1}{2}r^2\left(\frac{1}{2}\pi - 2\cos^{-1}\left(\frac{1}{r}\right)\right)$.

Putting these together we have:

$$\begin{aligned} \mathrm{P}(R \leqslant r) &= \text{ Area } OAQPB \\ &= \sqrt{r^2 - 1} + \tfrac{1}{2}r^2\left(\tfrac{1}{2}\pi - 2\cos^{-1}\left(\tfrac{1}{r}\right)\right) \\ &= \sqrt{r^2 - 1} + \tfrac{1}{4}\pi r^2 - r^2\cos^{-1}\left(r^{-1}\right) \end{aligned}$$

a required.

When $0 \leqslant r \leqslant 1$ then the area is just a quarter circle and we have:

$$\mathrm{P}(R \leqslant r) = \tfrac{1}{4}\pi r^2$$

(ii) To find $\mathrm{E}(R)$ we need to find the probability density function, so we need to differentiate the cumulative distribution function. We then have:

$$\mathrm{f}(r) = \frac{1}{2}\pi r \qquad \text{for } 0 \leqslant r \leqslant 1$$

and

$$\mathrm{f}(r) = \frac{\mathrm{d}}{\mathrm{d}r}\left[\sqrt{r^2-1} + \tfrac{1}{4}\pi r^2 - r^2\cos^{-1}\left(r^{-1}\right)\right] \qquad \text{for } 1 \leqslant r \leqslant \sqrt{2}$$

If you can remember the result when differentiating $\cos^{-1} x$ then you can just quote it, but it doesn't take took long to derive:

$$\begin{aligned} y &= \cos^{-1} x \\ \cos y &= x \\ -\sin y \times \frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-1}{\sin y} \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-1}{\sqrt{1-x^2}} \end{aligned}$$

Using this, along with the chain rule and product rule gives:

$$\begin{aligned} \mathrm{f}(r) &= \frac{\mathrm{d}}{\mathrm{d}r}\left[\sqrt{r^2-1} + \tfrac{1}{4}\pi r^2 - r^2\cos^{-1}\left(r^{-1}\right)\right] \\ &= r(r^2-1)^{-\frac{1}{2}} + \tfrac{1}{2}\pi r - 2r\cos^{-1}\left(r^{-1}\right) - r^2 \times \left(-r^{-2}\right) \times \frac{-1}{\sqrt{1-\left(r^{-1}\right)^2}} \\ &= \frac{r}{\sqrt{r^2-1}} + \tfrac{1}{2}\pi r - 2r\cos^{-1}\left(r^{-1}\right) - \frac{1}{\sqrt{1-\left(r^{-1}\right)^2}} \\ &= \tfrac{1}{2}\pi r - 2r\cos^{-1}\left(r^{-1}\right) \end{aligned}$$

So the expectation is given by:

$$\begin{aligned} \mathrm{E}(R) &= \int_0^1 r \times \tfrac{1}{2}\pi r \,\mathrm{d}r + \int_1^{\sqrt{2}} r \times \left[\tfrac{1}{2}\pi r - 2r\cos^{-1}\left(r^{-1}\right)\right]\,\mathrm{d}r \\ &= \int_0^{\sqrt{2}} \tfrac{1}{2}\pi r^2 \,\mathrm{d}r - \int_1^{\sqrt{2}} 2r^2\cos^{-1}\left(r^{-1}\right)\,\mathrm{d}r \\ &= \left[\frac{1}{6}\pi r^3\right]_0^{\sqrt{2}} - \left(\left[\frac{2}{3}r^3\cos^{-1}\left(r^{-1}\right)\right]_1^{\sqrt{2}} - \int_1^{\sqrt{2}} \frac{2}{3}r^3 \frac{r^{-2}}{\sqrt{1-\left(r^{-1}\right)^2}}\,\mathrm{d}r\right) \\ &= \frac{2\sqrt{2}\pi}{6} - \left(\frac{2}{3}(\sqrt{2})^2\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) - 0\right) + \int_1^{\sqrt{2}} \frac{2}{3}r^2\frac{1}{\sqrt{r^2-1}}\,\mathrm{d}r \\ &= \frac{\sqrt{2}\pi}{3} - \frac{4\sqrt{2}}{3} \times \frac{\pi}{4} + \int_1^{\sqrt{2}} \frac{2}{3}\frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r \\ &= \frac{2}{3}\int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r \end{aligned}$$

as required.

(iii) For the last part we need to evaluate the integral found in part **(ii)**. Lets start by ignoring the $\frac{2}{3}$, and using a substitution of $r = \sec\theta$. We have:

$$\begin{aligned}\frac{\mathrm{d}r}{\mathrm{d}\theta} &= \frac{\mathrm{d}(\cos\theta)^{-1}}{\mathrm{d}\theta} \\ &= -(\cos\theta)^{-2} \times (-\sin\theta) \\ &= \sec\theta\tan\theta\end{aligned}$$

and so:

$$\begin{aligned}\int \frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r &= \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\,\mathrm{d}\theta \\ &= \int \sec^3\theta\,\mathrm{d}\theta\end{aligned}$$

Letting I be the integral of $\sec^3\theta$:

$$\begin{aligned}I &= \int \sec^3\theta\,\mathrm{d}\theta \\ &= \int \sec^2\theta\sec\theta\,\mathrm{d}\theta \\ &= [\tan\theta\sec\theta] - \int \tan\theta \times \sec\theta\tan\theta\,\mathrm{d}\theta \quad \text{Using integration by parts} \\ &= \tan\theta\sec\theta - \int (\sec^2\theta - 1)\sec\theta\,\mathrm{d}\theta \\ &= \tan\theta\sec\theta - I + \int \sec\theta\,\mathrm{d}\theta \\ 2I &= \tan\theta\sec\theta + \int \sec\theta\,\mathrm{d}\theta \\ I &= \tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\int \sec\theta\,\mathrm{d}\theta\end{aligned}$$

The last thing we need to find is the integral of $\sec\theta$, and we would quite like to have a ln appearing! We know that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\sec\theta &= \sec\theta\tan\theta \quad \text{from previous work} \\ \frac{\mathrm{d}}{\mathrm{d}\theta}\tan\theta &= \sec^2\theta \\ \implies \frac{\mathrm{d}}{\mathrm{d}\theta}\big(\sec\theta + \tan\theta\big) &= \sec\theta\tan\theta + \sec^2\theta\end{aligned}$$

Using this we have:

$$\begin{aligned}\int \sec\theta\,\mathrm{d}\theta &= \int \frac{\sec\theta(\sec\theta + \tan\theta)}{\sec\theta + \tan\theta}\,\mathrm{d}\theta \\ &= \int \frac{\sec\theta\tan\theta + \sec^2\theta}{\sec\theta + \tan\theta}\,\mathrm{d}\theta \\ &= \ln|\sec\theta + \tan\theta)|\end{aligned}$$

Putting this altogether, along with the limits $r = 1 \implies \theta = 0$ and $r = \sqrt{2} \implies \theta = \frac{\pi}{4}$, we have:

$$\begin{aligned}
\mathrm{E}(R) &= \frac{2}{3}\int_1^{\sqrt{2}} \frac{r^2}{\sqrt{r^2-1}}\,\mathrm{d}r \\
&= \frac{2}{3}\int_0^{\frac{\pi}{4}} \sec^3\theta\,\mathrm{d}\theta \\
&= \frac{2}{3}\Big[\tfrac{1}{2}\tan\theta\sec\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta)|\Big]_0^{\frac{\pi}{4}} \\
&= \frac{1}{3}\,\Big[\Big(\tan\tfrac{\pi}{4}\sec\tfrac{\pi}{4} + \ln|\sec\tfrac{\pi}{4} + \tan\tfrac{\pi}{4}|\Big) - \ln|\sec 0|\Big] \\
&= \frac{1}{3}\,\Big[\Big(\sqrt{2} + \ln|\sqrt{2}+1|\Big)\Big] \\
&= \frac{1}{3}\Big(\sqrt{2} + \ln(\sqrt{2}+1)\Big)
\end{aligned}$$

There might be quicker ways to do this, maybe using a hyperbolic substitution rather than a trig one.