# STEP Support Programme

# 2022 STEP 2 Worked Paper

## General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the Cambridge Assessment Admissions Testing website. These are the general comments for the STEP 2022 exam from the Examiner's report:

*Candidates appeared to be generally well prepared for most topics within the examination, but there were a few situations in questions where some did not appear to be as proficient in standard techniques as needed. In particular, the method for finding invariant lines required in question 8 and the manipulation of trigonometric functions that were needed in question 10 caused considerable difficulties for some candidates.*

*An additional issue that occurred at numerous points in the paper relates to the direction in which a deduction is required. It is important that candidates make sure that they know which statement is the one that they should start from as they deduce the other and that it is clear in their solution that the logic has gone in the correct direction. Clarity of solution is also an issue that candidates should be aware of, especially in the situations where the result to be reached has been given. It is important to check that there are no special cases that need to be considered separately, and when dividing by a function it is necessary to confirm that the function cannot be equal to 0 (and in the case of inequalities that the function always has the same sign).*

*When drawing diagrams and sketching graphs it is useful if significant points that need to be clear are not drawn over the lines on the page as these can be difficult to interpret during the marking process.*

Please send any corrections, comments or suggestions to step@maths.org.

## Question 1

**1** **(i)** By integrating one of the two terms in the integrand by parts, or otherwise, find

$$\int \left( 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}} \right) \, \mathrm{d}x \, .$$

**(ii)** Find

$$\int (x^2+2)\frac{\sin x}{x^3} \, \mathrm{d}x \, .$$

**(iii)** **(a)** Sketch the graph with equation $y = \dfrac{\mathrm{e}^x}{x}$, giving the coordinates of any stationary points.

**(b)** Find $a$ if

$$\int_a^{2a} \frac{\mathrm{e}^x}{x} \, \mathrm{d}x = \int_a^{2a} \frac{\mathrm{e}^x}{x^2} \, \mathrm{d}x \, .$$

**(c)** Show that it is not possible to find distinct integers $m$ and $n$ such that

$$\int_m^n \frac{\mathrm{e}^x}{x} \, \mathrm{d}x = \int_m^n \frac{\mathrm{e}^x}{x^2} \, \mathrm{d}x \, .$$

### Examiner's report

Most candidates attempted this question. While a few did not recognise that the process of integration by parts applied to one of the terms would then lead to the integral of the other term appearing in the answer in such a way that they could be combined, many candidates were able to show the result in the first part clearly. Many then realised that the second part would follow from a similar process but using two applications of integration by parts.

Part **(iii)(a)** proved relatively straightforward for many candidates and some who had struggled with the first two parts were able to successfully complete this section. Some candidates struggled to deduce the correct behaviour of the graph, in many cases assuming that the $x$-axis would be an asymptote on both sides of the graph.

Many realised that part **(iii)(b)** would follow from application of integration by parts and were able to follow through the process carefully to produce a clear deduction of the required value. In part **(iii)(c)**, while most candidates were able to identify the equation that needed to be solved, many did not notice the link with part **(iii)(a)**, which provided the easiest explanation of why two such integers could not exist. Attempts to justify through other arguments were not often sufficiently convincing to achieve the final mark.

**Solution**

**(i)** Consider integrating the second term of the integral (as this looks more like a product of two functions, try this one first!). We can also do some manipulation so that we can integrate the $(1+x^3)^{-\frac{1}{2}}$ term. Note that $\dfrac{\mathrm{d}}{\mathrm{d}x}\sqrt{1+x^3} = \frac{1}{2} \times \dfrac{3x^2}{\sqrt{1+x^3}}$.

$$\begin{aligned}
\int \frac{3x^3}{\sqrt{1+x^3}}\,\mathrm{d}x &= \int x \times \frac{3x^2}{\sqrt{1+x^3}}\,\mathrm{d}x \\
&= \left[x \times 2\sqrt{1+x^3}\right] - \int 2\sqrt{1+x^3}\,\mathrm{d}x
\end{aligned}$$

Therefore we have:

$$\begin{aligned}
\int \left(2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}}\right)\,\mathrm{d}x &= \int 2\sqrt{1+x^3}\,\mathrm{d}x + \int \frac{3x^3}{\sqrt{1+x^3}}\,\mathrm{d}x \\
&= \int 2\sqrt{1+x^3}\,\mathrm{d}x + \left[2x\sqrt{1+x^3}\right] - \int 2\sqrt{1+x^3}\,\mathrm{d}x \\
&= 2x\sqrt{1+x^3} \;+c
\end{aligned}$$

**(ii)** It's probably a good idea to try the same idea again. Consider the second term again:

$$\begin{aligned}
\int \frac{2\sin x}{x^3}\,\mathrm{d}x &= \left[-x^{-2}\sin x\right] - \int \left(-x^{-2}\cos x\right)\,\mathrm{d}x \\
&= -\frac{\sin x}{x^2} + \int x^{-2}\cos x\,\mathrm{d}x
\end{aligned}$$

This doesn't look promising yet, so lets go around again...

$$\begin{aligned}
\int \frac{2\sin x}{x^3}\,\mathrm{d}x &= -\frac{\sin x}{x^2} + \int x^{-2}\cos x\,\mathrm{d}x \\
&= -\frac{\sin x}{x^2} + \left[-x^{-1}\cos x - \int -x^{-1}(-\sin x)\,\mathrm{d}x\right] \\
&= -\frac{\sin x}{x^2} - \frac{\cos x}{x} - \int \frac{\sin x}{x}\,\mathrm{d}x
\end{aligned}$$

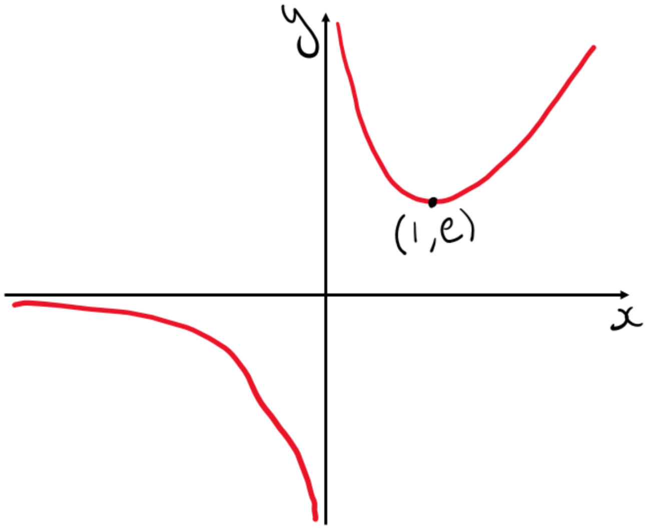
Therefore we have:

$$\begin{aligned}
\int (x^2+2)\frac{\sin x}{x^3}\,\mathrm{d}x &= \int \frac{\sin x}{x}\,\mathrm{d}x + \int \frac{2\sin x}{x^3}\,\mathrm{d}x \\
&= \int \frac{\sin x}{x}\,\mathrm{d}x + -\frac{\sin x}{x^2} - \frac{\cos x}{x} - \int \frac{\sin x}{x}\,\mathrm{d}x \\
&= -\frac{\sin x}{x^2} - \frac{\cos x}{x} \;+c
\end{aligned}$$

**(iii)** **(a)** Differentiating gives:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{e}^x}{x}\right) &= \frac{x\mathrm{e}^x - \mathrm{e}^x}{x^2}\\ &= \frac{(x-1)\mathrm{e}^x}{x^2}\end{aligned}$$

Therefore we have a stationary point at $(1, \mathrm{e})$. The graph looks like this:




Note that your sketch does not have to be perfect. The asymptotes should look vaguely correct, and should not meet the axes, but if yours do actually touch then you can add a note on the sketch saying that the curve is asymptotic to the axes, or that it shouldn't touch the axes, to make sure that your intention is clear to your examiner.

**(b)** Integrating the second integral gives:

$$\int_a^{2a} \frac{\mathrm{e}^x}{x^2}\,\mathrm{d}x = \Big[-x^{-1}\mathrm{e}^x\Big]_a^{2a} + \int_a^{2a} \frac{\mathrm{e}^x}{x}\,\mathrm{d}x$$

Therefore if the two integrals are going to be the same then we need:

$$\begin{aligned}\Big[-x^{-1}\mathrm{e}^x\Big]_a^{2a} &= 0\\ -\frac{\mathrm{e}^{2a}}{2a} + \frac{\mathrm{e}^a}{a} &= 0\\ \frac{\mathrm{e}^a}{a}\left(1 - \frac{\mathrm{e}^a}{2}\right) &= 0\end{aligned}$$

Therefore we have $\mathrm{e}^a = 2$, i.e. $a = \ln 2$.

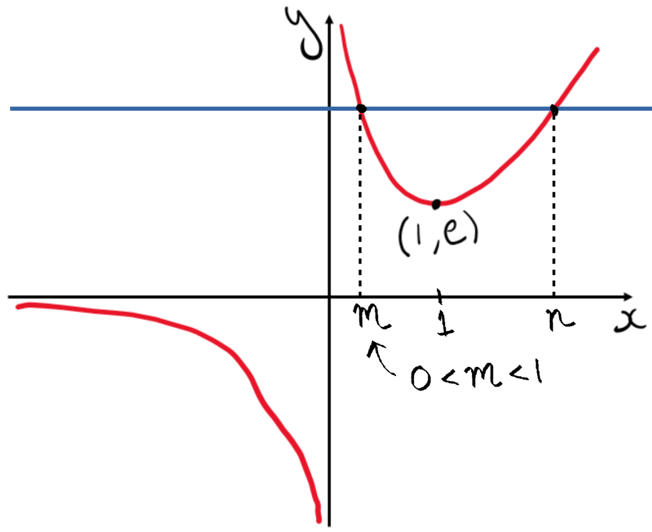
**(c)** In a similar way to before, if the two integrals are going to be the same then we need:

$$\left[-x^{-1}\mathrm{e}^x\right]_m^n = 0$$
$$-\frac{\mathrm{e}^n}{n} + \frac{\mathrm{e}^m}{m} = 0$$
$$\implies \frac{\mathrm{e}^n}{n} = \frac{\mathrm{e}^m}{m}$$

This means that we want to find two different values of $x$ which give the same value for $y = \dfrac{\mathrm{e}^x}{x}$. From the graph we can see that this means that the two values are positive, and one has to be in the range $0 < x < 1$, therefore at least one of the values cannot be an integer.

It is not necessary to show this on the graph, and just a line on the graph would not be enough without some annotations or explanations, but it might help you to put your thoughts into words, and can also help you to understand what is happening.

In the example below, a comment below $0 < m < 1$ to state that $m$ is not an integer would be enough.



## Question 2

**2** A sequence $u_n$, where $n = 1, 2, \ldots$, is said to have *degree* $d$ if $u_n$, as a function of $n$, is a polynomial of degree $d$.

**(i)** Show that, in any sequence $u_n$ $(n = 1, 2, \ldots)$ that satisfies $u_{n+1} = \frac{1}{2}(u_{n+2} + u_n)$ for all $n \geqslant 1$, there is a constant difference between successive terms.

Deduce that any sequence $u_n$ for which $u_{n+1} = \frac{1}{2}(u_{n+2} + u_n)$, for all $n \geqslant 1$, has degree at most 1.

**(ii)** The sequence $v_n$ $(n = 1, 2, \ldots)$ satisfies $v_{n+1} = \frac{1}{2}(v_{n+2} + v_n) - p$ for all $n \geqslant 1$, where $p$ is a non-zero constant. By writing $v_n = t_n + pn^2$, show that the sequence $v_n$ has degree 2.

Given that $v_1 = v_2 = 0$, find $v_n$ in terms of $n$ and $p$.

**(iii)** The sequence $w_n$ $(n = 1, 2, \ldots)$ satisfies $w_{n+1} = \frac{1}{2}(w_{n+2} + w_n) - an - b$ for all $n \geqslant 1$, where $a$ and $b$ are constants with $a \neq 0$. Show that the sequence $w_n$ has degree 3.

Given that $w_1 = w_2 = 0$, find $w_n$ in terms of $n$, $a$ and $b$.

**Examiner's report**

Many candidates who attempted this question struggled to achieve high marks as there were several points within this question where the reasoning required careful explanation.

Part **(i)** was often completed successfully by candidates who recognised that the relationship given could easily be rearranged to show the required result. Approaches which did not recognise this and tried to use the relationship expressed for different triples of terms were unable to make any significant progress. Candidates who attempted to argue that higher powers would not lead to a common difference were generally not successful in showing that the sequence has degree at most one.

The first section of part **(ii)** was generally well done by those who attempted it, and the relationship with the first part was often seen although in some cases candidates omitted to observe that $p$ was not equal to zero. Candidates were also often able to deduce the formula for the sequence, either by substituting a general form or by looking at the differences between terms. Many of the candidates who attempted part **(iii)** recognised that a similar approach to part **(ii)** would be likely to work. However, many assumed that the required coefficient of $n^3$ would be $a$, rather than using a variable so that the correct coefficient could be deduced at a later point. The algebra for this part was more complicated and some struggled to follow through the work accurately. Having reached the point where the correct value of $k$ could be deduced it was then necessary to consider when the new sequence would be of the form in part **(ii)** and when it would be of the form in part **(i)** and the analysis of these cases was not always completed fully.

**Solution**

**(i)** If there is going to be a constant difference between the terms then we need $u_{n+1} - u_n = u_n - u_{n-1}$. Taking the given relationship we have:

$$\begin{aligned} u_{n+1} &= \tfrac{1}{2}(u_{n+2} + u_n) \\ 2u_{n+1} &= u_{n+2} + u_n \\ u_{n+1} - u_n &= u_{n+2} - u_{n+1} \end{aligned}$$

Therefore the gaps between successive terms is equal.

Let $u_n - u_{n-1} = d$. Then we have $u_n = u_1 + (n-1)d$ and so $u_n$ has degree at most 1. Note that is $d = 0$ then the sequence has degree 0.

**(ii)** Substituting $v_n = t_n + pn^2$ into the given relationship gives:

$$\begin{aligned} v_{n+1} &= \frac{1}{2}(v_{n+2} + v_n) - p \\ t_{n+1} + p(n+1)^2 &= \frac{1}{2}\left[t_{n+2} + p(n+2)^2 + t_n + pn^2\right] - p \\ t_{n+1} + p(n^2 + 2n + 1) &= \frac{1}{2}\left[t_{n+2} + p(n^2 + 4n + 4) + t_n + pn^2\right] - p \\ t_{n+1} + p(n^2 + 2n + 1) &= \frac{1}{2}\left(t_{n+2} + t_n\right) + p(n^2 + 2n + 2) - p \\ t_{n+1} &= \frac{1}{2}\left(t_{n+2} + t_n\right) \end{aligned}$$

Like in part **(ii)**, we know that $t_n$ has degree at most 1. Then since $p \neq 0$ we know that $v_n$ has degree 2.

Since $v_n$ has degree 2 we can write $v_n = pn^2 + qn + r$, where $p \neq 0$. Using $v_1 = v_2 = 0$ gives:

$$\begin{aligned} p + q + r &= 0 \\ 4p + 2q + r &= 0 \\ \therefore\ 3p + q &= 0 \end{aligned}$$

This means we have $q = -3p$ and $r = -p - q = 2p$. Therefore we have $v_n = pn^2 - 3pn + 2p$.

**(iii)** Using a similar idea to before, let $w_n = t_n + kn^3$. Substituting this gives:

$$\begin{aligned} w_{n+1} &= \tfrac{1}{2}(w_{n+2} + w_n) - an - b \\ t_{n+1} + k(n+1)^3 &= \tfrac{1}{2}\left[t_{n+2} + k(n+2)^3 + tn + kn^3\right] - an - b \\ t_{n+1} + k(n^3 + 3n^2 + 3n + 1) &= \tfrac{1}{2}\left[t_{n+2} + t_n + k(2n^3 + 6n^2 + 12n + 8)\right] - an - b \\ t_{n+1} + k(\cancel{n^3} + \cancel{3n^2} + 3n + 1) &= \tfrac{1}{2}\left[t_{n+2} + t_n\right] + k(\cancel{n^3} + \cancel{3n^2} + 6n + 4) - an - b \\ t_{n+1} &= \tfrac{1}{2}\left[t_{n+2} + t_n\right] + n(3k - a) + (3k - b) \end{aligned}$$

Let $k = \frac{1}{3}a$. This means that we have $t_{n+1} = \frac{1}{2}\left[t_{n+2} + t_n\right] + (a - b)$, or perhaps more usefully $t_{n+1} = \frac{1}{2}\left[t_{n+2} + t_n\right] - (b - a)$. This is then the same situation as in part **(ii)**, with $p = b - a$. This means that $t_n$ has degree 2, and so as $a \neq 0$, $w_n$ has degree 3.

We have $w_n = \frac{1}{3}an^3 + (b-a)n^2 + qn + r$ (using the fact from part **(ii)** that the coefficient of $n^2$ is $p$). Substituting $w_1 = w_2 = 0$ gives:

$$\begin{aligned} \tfrac{1}{3}a + (b-a) + q + r &= 0 \\ \tfrac{8}{3}a + 4(b-a) + 2q + r &= 0 \\ \implies \tfrac{7}{3}a + 3(b-a) + q &= 0 \end{aligned}$$

Therefore we have $q = \frac{2}{3}a - 3b$ and $r = -q - \frac{1}{3}a - (b-a) = -\frac{2}{3}a + 3b - \frac{1}{2}a - b + a = 2b$. Hence we have:

$$w_n = \tfrac{1}{3}an^3 + (b-a)n^2 + \left(\tfrac{2}{3}a - 3b\right)n + 2b$$

## Question 3

**3** The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geqslant 0$, $F_{n+2} = F_{n+1} + F_n$.

**(i)** Prove that $F_r \leqslant 2^{r-n}F_n$ for all $n \geqslant 1$ and all $r \geqslant n$.

**(ii)** Let $S_n = \displaystyle\sum_{r=1}^{n} \frac{F_r}{10^r}$.

Show that

$$\sum_{r=1}^{n} \frac{F_{r+1}}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{10^{r-1}} = 89S_n - 10F_1 - F_0 + \frac{F_n}{10^n} + \frac{F_{n+1}}{10^{n-1}}.$$

**(iii)** Show that $\displaystyle\sum_{r=1}^{\infty} \frac{F_r}{10^r} = \frac{10}{89}$ and that $\displaystyle\sum_{r=7}^{\infty} \frac{F_r}{10^r} < 2 \times 10^{-6}$. Hence find, with justification, the first six digits after the decimal point in the decimal expansion of $\frac{1}{89}$.

**(iv)** Find, with justification, a number of the form $\dfrac{r}{s}$ with $r$ and $s$ both positive integers less than 10000 whose decimal expansion starts

$$0.00010102030508132134 55\ldots.$$

**Examiner's report**

There were several good attempts to this question, particularly in the middle section. However, the induction required in the first part of the question caused difficulties for many candidates. In particular, many did not realise which variable it was necessary to perform the induction on.

Attempts to the second part of the question were much better and candidates were generally able to demonstrate a good level of skill in manipulating summations, including changing the variable over which the sum ranges. Unfortunately, some slips in the algebra were seen in several cases, but many candidates were able to reach the given result convincingly.

Many candidates recognised that the left-hand side of the previous result must be equal to zero and were therefore able to show the values of the sums successfully. Some care was needed with the explanation of how the second sum led to certainty about the first six digits of the expansion.

Candidates who had successfully completed part **(iii)** of the question were often then able to identify the correct approach for the final part. However, the justifications often failed to calculate the value of the error when only considering some of the terms of the sequence.

**Solution**

**(i)** This feels like it should be a proof by induction, but there are two variables in the question so it is a little tricky to work out how to proceed with this.

Start by taking $r = n$. We have $F_n = 2^{n-n}F_n$, so $F_r \leqslant 2^{r-n}F_n$ is true when $r = n$.

Assume that statement is true when $r = k$, where $k \geqslant n$. This means that we have $F_k \leqslant 2^{k-n}F_n$ for all $n \geqslant 1$.

Considering the case $r = k + 1$ we have:

$$\begin{aligned}
F_{k+1} &= F_k + F_{k-1} \\
F_{k+1} &\leqslant 2F_k && \text{as } F_k \geqslant F_{k-1} \\
&\leqslant 2 \times 2^{k-n}F_n && \text{using the } r = k \text{ case} \\
\therefore F_{k+1} &\leqslant 2^{(k+1)-n}F_n
\end{aligned}$$

Therefore we have $F_r \leqslant 2^{r-n}F_n$ when $r = n$, and if it is true when $r = k$ (where $k \geqslant n$) then it is also true when $r = k + 1$. Therefore $F_r \leqslant 2^{r-n}F_n$ is true for all integers $r \geqslant n$.

**(ii)** For this part we want to try to force some $S_n$ terms out. Starting from the LHS we have:

$$\begin{aligned}
&\sum_{r=1}^{n} \frac{F_{r+1}}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{10^{r-1}} \\
&= 100\sum_{r=1}^{n} \frac{F_{r+1}}{10^{r+1}} - 10\sum_{r=1}^{n} \frac{F_r}{10^r} - \sum_{r=1}^{n} \frac{F_{r-1}}{10^{r-1}} \\
&= 100\left(\frac{F_{n+1}}{10^{n+1}} + \sum_{r=1}^{n} \frac{F_r}{10^r} - \frac{F_1}{10^1}\right) - 10\sum_{r=1}^{n} \frac{F_r}{10^r} - \left(\sum_{r=1}^{n} \frac{F_r}{10^r} - \frac{F_n}{10^n} + \frac{F_0}{10^0}\right) \\
&= \frac{100F_{n+1}}{10^{n+1}} + 100S_n - \frac{100F_1}{10} - 10S_n - S_n + \frac{F_n}{10^n} - \frac{F_0}{10^0} \\
&= 89S_n - 10F_1 - F_0 + \frac{F_n}{10^n} + \frac{F_{n+1}}{10^{n-1}}
\end{aligned}$$

as required.

**(iii)** We can rewrite the LHS of the relationship in part **(ii)** as:

$$\sum_{r=1}^{n} \frac{F_{r+1}}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{10^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{10^{r-1}} = \sum_{r=1}^{n} \frac{1}{10^{r-1}} \left(F_{r+1} - F_r - F_{r-1}\right)$$

which is equal to 0 as we have $F_{r+1} = F_r + F_{r-1}$ for $r \geqslant 1$.

Therefore we have:

$$89S_n - 10F_1 - F_0 + \frac{F_n}{10^n} + \frac{F_{n+1}}{10^{n-1}} = 0$$

From part **(i)** we have $F_r \leqslant 2^{r-1}F_1$, so we have $F_n \leqslant \frac{1}{2} \times 2^n$. This means that we have $\frac{F_n}{10^n} \leqslant \frac{1}{2} \times \frac{2^n}{10^n}$, and so as $n \to \infty$ we have $\frac{F_n}{10^n} \to 0$. Similarly $\frac{F_{n+1}}{10^{n-1}} \leqslant \frac{1}{2} \times \frac{2^{n+1}}{10^{n-1}}$, i.e. $\frac{F_{n+1}}{10^{n-1}} \leqslant 2 \times \frac{2^{n-1}}{10^{n-1}}$ and we have $\frac{F_{n+1}}{10^{n-1}} \to 0$ as $n \to \infty$.

Considering

$$89S_n - 10F_1 - F_0 + \frac{F_n}{10^n} + \frac{F_{n+1}}{10^{n-1}} = 0$$

in the limit as $n \to \infty$ gives:

$$89S_\infty = 10F_1 + F_0$$

and since $F_1 = 1$, $F_0 = 0$ this gives $S_\infty = \frac{10}{89}$ as required.

From part **(i)** we have $F_r \leqslant 2^{r-7}F_7$. Using this we have:

$$\sum_{r=7}^{\infty} \frac{F_r}{10^r} \leqslant F_7 \sum_{r=7}^{\infty} \frac{2^{r-7}}{10^r}$$

$$\sum_{r=7}^{\infty} \frac{F_r}{10^r} \leqslant \frac{F_7}{10^7} \sum_{r=7}^{\infty} \frac{2^{r-7}}{10^{r-7}}$$

$$\sum_{r=7}^{\infty} \frac{F_r}{10^r} \leqslant \frac{F_7}{10^7} \sum_{i=0}^{\infty} \frac{2^i}{10^i}$$

$$\sum_{r=7}^{\infty} \frac{F_r}{10^r} \leqslant \frac{F_7}{10^7} \sum_{i=0}^{\infty} \left(\frac{1}{5}\right)^i$$

Using the fact that $F_7 = 13$ and the formula for the infinite sum of a Geometric Progression gives:

$$\sum_{r=7}^{\infty} \frac{F_r}{10^r} \leqslant \frac{13}{10^7} \times \frac{5}{4}$$

$$\sum_{r=7}^{\infty} \frac{F_r}{10^r} \leqslant \frac{65}{40} \times 10^{-6}$$

$$\sum_{r=7}^{\infty} \frac{F_r}{10^r} < 2 \times 10^{-6}$$

We have:

$$\frac{10}{89} = \sum_{r=1}^{\infty} \frac{F_r}{10^r}$$

$$\frac{10}{89} = \frac{1}{10} + \frac{1}{10^2} + \frac{2}{10^3} + \frac{3}{10^4} + \frac{5}{10^5} + \frac{8}{10^6} + \sum_{r=7}^{\infty} \frac{F_r}{10^r}$$

$$\frac{10}{89} = 0.112358 + \sum_{r=7}^{\infty} \frac{F_r}{10^r}$$

$$\frac{1}{89} = 0.0112358 + \frac{1}{10}\sum_{r=7}^{\infty} \frac{F_r}{10^r}$$

We have $\frac{1}{10}\sum_{r=7}^{\infty} \frac{F_r}{10^r} < 2 \times 10^{-7}$, and so this will not be enough when combined with the $8 \times 10^{-7}$ term to roll over into the sixth place, hence the first 6 digits of the decimal expansion of $\frac{1}{89}$ are 0.011235.

**(iv)** Comparing this with the last answer, it looks as if the Fibonacci numbers are being divided by powers of 100. In a similar way to part **(ii)** let $T_n = \sum_{r=1}^{\infty} \frac{F_r}{100^r}$.

We have:

$$\begin{aligned}
0 &= \sum_{r=1}^{n} \frac{F_{r+1}}{100^{r-1}} - \sum_{r=1}^{n} \frac{F_r}{100^{r-1}} - \sum_{r=1}^{n} \frac{F_{r-1}}{100^{r-1}} \\
&= 100^2 \sum_{r=1}^{n} \frac{F_{r+1}}{100^{r+1}} - 100 \sum_{r=1}^{n} \frac{F_r}{100^r} - \sum_{r=1}^{n} \frac{F_{r-1}}{100^{r-1}} \\
&= 100^2 \left( T_n + \frac{F_{n+1}}{100^{n+1}} - \frac{1}{100} \right) - 100T_n - \left( T_n - \frac{F_n}{100^n} \right) \\
\implies 9899T_n &= 100 - \frac{F_n}{100^n} - \frac{F_n + 1}{100^{n-1}}
\end{aligned}$$

Therefore we have $T_\infty = \frac{100}{9899}$, i.e. we have $\frac{100}{9899} = 0.01010203050813....$ The required fraction is this divided by 100, so we have:

$$\frac{1}{9899} = 0.000101020305081321 3455...$$

## Question 4

**4** **(i)** Show that the function f, given by the single formula $\mathrm{f}(x) = |x| - |x-5| + 1$, can be written without using modulus signs as

$$\mathrm{f}(x) = \begin{cases} -4 & x \leqslant 0\,, \\ 2x-4 & 0 \leqslant x \leqslant 5\,, \\ 6 & 5 \leqslant x\,. \end{cases}$$

Sketch the graph with equation $y = \mathrm{f}(x)$.

**(ii)** The function g is given by:

$$\mathrm{g}(x) = \begin{cases} -x & x \leqslant 0\,, \\ 3x & 0 \leqslant x \leqslant 5\,, \\ x+10 & 5 \leqslant x\,. \end{cases}$$

Use modulus signs to write $\mathrm{g}(x)$ as a single formula.

**(iii)** Sketch the graph with equation $y = \mathrm{h}(x)$, where $\mathrm{h}(x) = x^2 - x - 4|x| + |x(x-5)|$.

**(iv)** The function k is given by:

$$\mathrm{k}(x) = \begin{cases} 10x & x \leqslant 0\,, \\ 2x^2 & 0 \leqslant x \leqslant 5\,, \\ 50 & 5 \leqslant x\,. \end{cases}$$

Use modulus signs to write $\mathrm{k}(x)$ as a single formula, explicitly verifying that your formula is correct.

**Examiner's report**

Almost all candidates attempted this question, and many of these did so very successfully. The majority of attempts managed to make some good progress, especially in the first three parts.

Overall, the way in which each of the given modulus forms changed sign was generally well grasped, although explanations were often rather lacking in detail. The graphs to be drawn in parts **(i)** and **(iii)** were managed well overall, although in some cases the fact that horizontal lines were drawn along the horizontal lines of the answer booklet meant that the intention was not always clear on the scanned script.

In part **(i)** some candidates failed to give sufficient explanation to show that the single formula could be written as the given set of three equations. Although a small number of candidates omitted to sketch the graph, most were able to draw the correct three straight line segments.

Part **(ii)** was well attempted, and many candidates were able to identify that the new function was very closely related to the one in the first part of the question meaning that they were able to write down the correct formula. Those who wrote a general form were almost always able to follow through the process to reach the correct final answer.

The introduction of quadratic terms to the function did not cause too much difficulty for candidates and many were able to deduce the correct equations for the sections of the function in part **(iii)**. While many correctly identified the shape of the two quadratic sections, in several cases the graphs presented were symmetric.

Part **(iv)** presented more of a challenge, but many candidates who wrote down a general form were able to work through to achieve the correct final answer. A few candidates were able to write down the answer, but in some cases their answer was not verified even though the question explicitly asked for verification that the formula is correct.

**Solution**

**(i)** When $x \leqslant 0$ we have:

$$\begin{aligned} \mathrm{f}(x) &= |x| - |x-5| + 1 \\ &= -x - (5-x) + 1 \\ &= -x - 5 + x + 1 = -4 \end{aligned}$$

When $0 \leqslant x \leqslant 5$ we have:

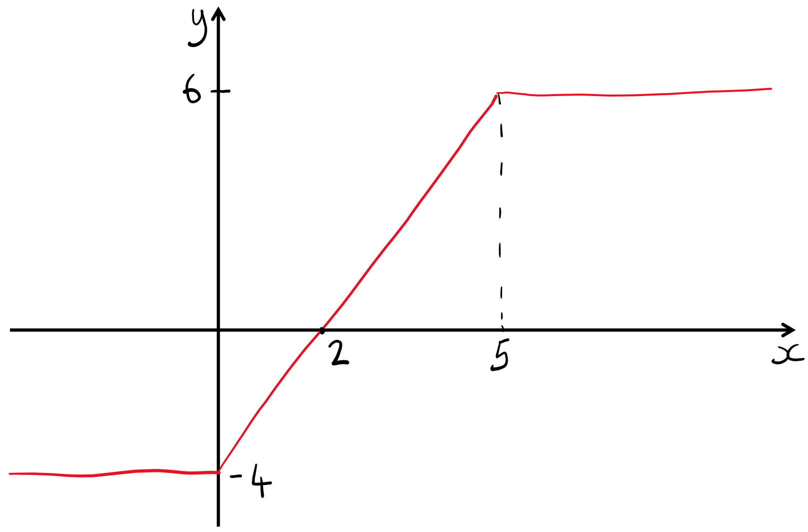
$$\begin{aligned} \mathrm{f}(x) &= |x| - |x-5| + 1 \\ &= x - (5-x) + 1 \\ &= x - 5 + x + 1 = 2x - 4 \end{aligned}$$

When $x \geqslant 5$ we have:

$$\begin{aligned} \mathrm{f}(x) &= |x| - |x-5| + 1 \\ &= x - (x-5) + 1 \\ &= x - x + 5 + 1 = 6 \end{aligned}$$

Hence we can write $\mathrm{f}(x)$ in the form required.

To sketch the graph we can sketch three straight lines. Note that the function is continuous at the points where $x = 0$ and $x = 5$.




**(ii)** Looking at the ranges of $x$, it looks like $\mathrm{g}(x)$ might have the form $\mathrm{g}(x) = a|x| + b|x-5| + c$. Looking at the different ranges of $x$ we have:

$$\begin{aligned}
x \leqslant 0 &\implies \mathrm{g}(x) = -ax + b(5-x) + c \equiv -x \\
&\implies a + b = 1 \text{ and } 5b + c = 0 \\
0 \leqslant x \leqslant 5 &\implies \mathrm{g}(x) = ax + b(5-x) + c \equiv 3x \\
&\implies a - b = 3 \text{ and } 5b + c = 0 \\
x \geqslant 5 &\implies \mathrm{g}(x) = ax + b(x-5) + c \equiv x + 10 \\
&\implies a + b = 1 \text{ and } c - 5b = 10
\end{aligned}$$

Solving $a + b = 1$ and $a - b = 3$ simultaneously gives $a = 2, b = -1$. Substituting $b = -1$ into $5b + c = 0$ gives $c = 5$, and we can also check that $c = 5, b = -1$ satisfies $c - 5b = 10$.

This gives:

$$\mathrm{g}(x) = 2|x| - |x-5| + 5$$

It is a good idea to double check that your solution gives the correct expressions for $\mathrm{g}(x)$ in the various ranges. For this part this can be done in your head, but note that in the last part of the question you need to do this explicitly (i.e. actually write the working to check the solution down!).

**(iii)** The "critical values" are $x = 0$ and $x = 5$, similar to the previous two parts.

In the range $x \leqslant 0$ we have:

$$\begin{aligned}
\mathrm{h}(x) &= x^2 - x - 4|x| + |x(x-5)| \\
&= x^2 - x + 4x + x(x-5) \\
&= 2x^2 - 2x \\
&= 2x(x-1)
\end{aligned}$$

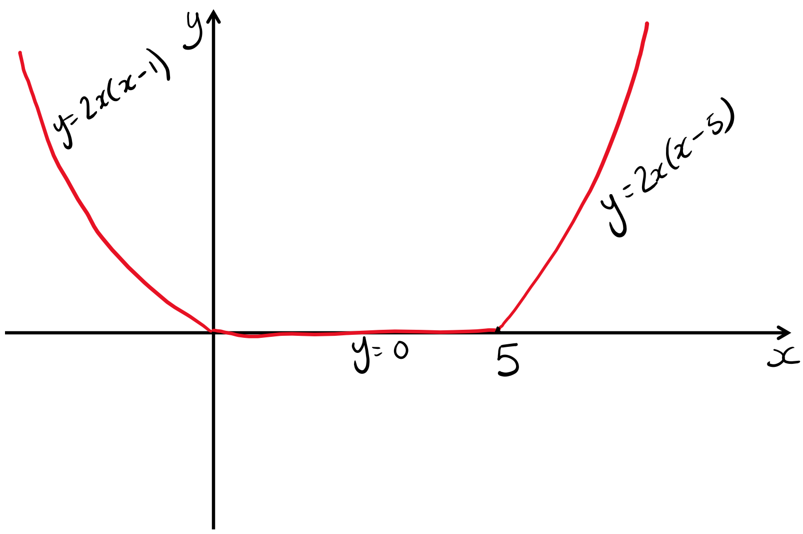
In the range $0 \leqslant x \leqslant 5$ we have:

$$\begin{aligned}
\mathrm{h}(x) &= x^2 - x - 4|x| + |x(x-5)| \\
&= x^2 - x - 4x - x(x-5) \\
&= x^2 - 5x - x^2 + 5x \\
&= 0
\end{aligned}$$

In the range $x \geqslant 5$ we have:

$$\begin{aligned}
\mathrm{h}(x) &= x^2 - x - 4|x| + |x(x-5)| \\
&= x^2 - x - 4x + x(x-5) \\
&= x^2 - 5x + x^2 - 5x \\
&= 2x(x-5)
\end{aligned}$$

This function is continuous at the points $x = 0$ and $x = 5$.




**(iv)** One way to attempt this part is to start with $\mathrm{k}(x) = ax^2 + b|x(x-5)| + c|x| + d|x-5| + e$, and then find some simultaneous equations as before.

Instead of this method, I am going to start by noticing that there is an $x^2$ term in the middle range of $x$ only, which happens to be the opposite way around to the function in part **(iii)**.

Consider $\mathrm{k}_1(x) = x^2 - |x(x-5)|$. This can be written as:

$$\mathrm{k}_1(x) = \begin{cases} 5x & x \leqslant 0\,, \\ 2x^2 - 5x & 0 \leqslant x \leqslant 5\,, \\ 5x & 5 \leqslant x\,. \end{cases}$$

We then need to find a function $\mathrm{k}_2$ which satisfies $\mathrm{k}(x) = \mathrm{k}_1(x) + \mathrm{k}_2(x)$. This means we want to find $\mathrm{k}_2(x)$ which satisfies:

$$\mathrm{k}_2(x) = \begin{cases} 5x & x \leqslant 0\,, \\ 5x & 0 \leqslant x \leqslant 5\,, \\ 50 - 5x & 5 \leqslant x\,. \end{cases}$$

Noting that this only changes across the $x = 5$ boundary means that we have $k_2(x) = a\left|x-5\right| + b$. We can immediately see that we must have $a = -5$, which means that we must have $b = 25$.

Using $\mathrm{k}(x) = \mathrm{k}_1(x) + \mathrm{k}_2(x)$ we have:

$$\mathrm{k}(x) = x^2 - \left|x(x-5)\right| - 5\left|x-5\right| + 25\,.$$

The question says that we have to explicitly verify that the formula is correct. For $x \leqslant 0$ we have:

$$\begin{aligned}\mathrm{k}(x) &= x^2 - \left|x(x-5)\right| - 5\left|x-5\right| + 25\\ &= x^2 - x(x-5) + 5(x-5) + 25\\ &= x^2 - x^2 + 5x + 5x - 25 + 25\\ &= 10x\end{aligned}$$

For $0 \leqslant x \leqslant 5$ we have:

$$\begin{aligned}\mathrm{k}(x) &= x^2 - \left|x(x-5)\right| - 5\left|x-5\right| + 25\\ &= x^2 + x(x-5) + 5(x-5) + 25\\ &= x^2 + x^2 - 5x + 5x - 25 + 25\\ &= 2x^2\end{aligned}$$

For $x \geqslant 5$ we have:

$$\begin{aligned}\mathrm{k}(x) &= x^2 - \left|x(x-5)\right| - 5\left|x-5\right| + 25\\ &= x^2 - x(x-5) - 5(x-5) + 25\\ &= x^2 - x^2 + 5x - 5x + 25 + 25\\ &= 50\end{aligned}$$

Therefore $\mathrm{k}(x) = x^2 - \left|x(x-5)\right| - 5\left|x-5\right| + 25$ gives the correct expressions in each of the three ranges for $x$.

## Question 5

**5** **(i)** Given that $a > b > c > 0$ are constants, and that $x$, $y$, $z$ are non-negative variables, show that

$$ax + by + cz \leqslant a(x + y + z).$$

In the acute-angled triangle $ABC$, $a$, $b$ and $c$ are the lengths of sides $BC$, $CA$ and $AB$, respectively, with $a > b > c$. $P$ is a point inside, or on the sides of, the triangle, and $x$, $y$ and $z$ are the perpendicular distances from $P$ to $BC$, $CA$ and $AB$, respectively. The area of the triangle is $\Delta$.

**(ii)** **(a)** Find $\Delta$ in terms of $a$, $b$, $c$, $x$, $y$ and $z$.

**(b)** Find both the minimum value of the sum of the perpendicular distances from $P$ to the three sides of the triangle and the values of $x$, $y$ and $z$ which give this minimum sum, expressing your answers in terms of some or all of $a$, $b$, $c$ and $\Delta$.

**(iii)** **(a)** Show that, for all real $a$, $b$, $c$, $x$, $y$ and $z$,

$$(a^2+b^2+c^2)(x^2+y^2+z^2) = (bx-ay)^2+(cy-bz)^2+(az-cx)^2+(ax+by+cz)^2.$$

**(b)** Find both the minimum value of the sum of the squares of the perpendicular distances from $P$ to the three sides of the triangle and the values of $x$, $y$ and $z$ which give this minimum sum, expressing your answers in terms of some or all of $a$, $b$, $c$ and $\Delta$.

**(iv)** Find both the maximum value of the sum of the squares of the perpendicular distances from $P$ to the three sides of the triangle and the values of $x$, $y$ and $z$ which give this maximum sum, expressing your answers in terms of some or all of $a$, $b$, $c$ and $\Delta$.

### Examiner's report

Many candidates appear to have spent a little time on this question before deciding to concentrate on others, meaning that the marks in general for this question were very low. In part **(i)** it was often unclear whether the candidate was using all of the conditions and in many cases the inequalities were written as strict when they should not have been.

In part **(ii)** most candidates were able to write down a correct formula for the area and part **(b)** was also generally answered well, although in some cases the candidates appeared to guess the best point, rather than deduce it from the previous part. The incentre was a common incorrect guess for at least one of the parts.

Candidates were generally most successful in part **(iii)(a)** with many convincing attempts seen. Fewer than half of the scripts progressed beyond this point, however. Those who did attempt part **(iii)(b)** managed to complete it quite well, although a common mistake was to simply square the answer from part **(ii)(b)** and claim that it all works out.

There were very few attempts at part **(iv)**, mostly trying to use previous parts or to guess a point. For both of the last two parts, some candidates correctly identified the maximum or minimum, but did not convincingly show where it was attained.

**Solution**

**(i)** Since $b < a$, and $y$ is non-negative, we have $by \leqslant ay$ (we could have equality if $y = 0$). Similarly we have $c < a$ and $z$ is non-negative so we have $cz \leqslant az$.

Therefore we have:

$$\begin{aligned} ax + by + cz &\leqslant ax + ay + az \\ ax + by + cz &\leqslant a(x + y + z) \end{aligned}$$

**(ii)** **(a)** The lines $AP$, $BP$ and $CP$ can be used to divide the triangle into three smaller triangles. We can find the area of each of the smaller triangles to give:

$$\Delta = \tfrac{1}{2}ax + \tfrac{1}{2}by + \tfrac{1}{2}cz$$

I haven't drawn a diagram for this part, but you might like to if you are finding the triangles hard to picture.

**(b)** We want to minimize $x + y + z$, subject to the condition $ax + by + cz = 2\Delta$. Using part **(i)** we have:

$$\begin{aligned} ax + by + cz &= 2\Delta \\ a(x + y + z) &\geqslant 2\Delta \\ x + y + z &\geqslant \frac{2\Delta}{a} \end{aligned}$$

Hence the minimum value of $x+y+z$ is $\dfrac{2\Delta}{a}$. This is achieved when $x = \dfrac{2\Delta}{a}, y = 0, z = 0$.

**(iii)** **(a)** Considering the right hand side we have:

$$\begin{aligned} \text{RHS} =&(bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 + (ax + by + cz)^2 \\ =&b^2x^2 - \cancel{2abxy} + a^2y^2 + c^2y^2 - \cancel{2bcyz} + b^2z^2 + a^2z^2 - \cancel{2acxz} + c^2x^2 \\ &+ a^2x^2 + b^2y^2 + c^2z^2 + \cancel{2abxy} + \cancel{2bcyz} + \cancel{2acxz} \\ =&(a^2 + b^2 + c^2)x^2 + (a^2 + b^2 + c^2)y^2 + (a^2 + b^2 + c^2)z^2 \\ =&(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \\ =&\text{LHS} \end{aligned}$$

**(b)** Using **(iii)(a)** and **(ii)(a)** we have:

$$\begin{aligned} &(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (bx - ay)^2 + (cy - bz)^2 + (az - cx)^2 + (2\Delta)^2 \\ \implies &(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \geqslant (2\Delta)^2 \end{aligned}$$

Therefore the minimum value of $x^2+y^2+z^2$ is $\dfrac{4\Delta^2}{a^2+b^2+c^2}$, and when it takes this value the other three squared brackets must be zero.

This means that at the minimum value we have $bx = ay$, $cy = bz$ and $az = cx$ or equivalently $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$. Letting $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = k$ gives $x = ak$, $y = bk$ and $z = ck$.

Substituting this into the area of the triangle condition gives:

$$\begin{aligned} \Delta &= \tfrac{1}{2}a^2k + \tfrac{1}{2}b^2k + \tfrac{1}{2}c^2k \\ 2\Delta &= k(a^2+b^2+c^2) \\ \implies k &= \frac{2\Delta}{a^2+b^2+c^2} \end{aligned}$$

Therefore the minimum value of $x^2+y^2+z^2$ occurs when $x = \frac{2a\Delta}{a^2+b^2+c^2}$, $y = \frac{2b\Delta}{a^2+b^2+c^2}$, $z = \frac{2c\Delta}{a^2+b^2+c^2}$.

**(iv)** Here we want to find the maximum value of $x^2+y^2+z^2$, so it would be useful to find an upper bound, i.e. something like $x^2+y^2+z^2 \leqslant \alpha$ (in the previous part we found a lower bound).

In a similar way to part **(i)** we have $ax+by+cz \geqslant c(x+y+z)$. As both sides are non-negative we can square both sides to give:

$$(ax+by+cz)^2 \geqslant c^2(x+y+z)^2$$

Since $x, y, z$ are non-negative we also have $(x+y+z)^2 \geqslant x^2+y^2+z^2$. Putting these two inequalities together we have:

$$\begin{aligned} c^2(x^2+y^2+z^2) &\leqslant (ax+by+cz)^2 \\ c^2(x^2+y^2+z^2) &\leqslant (2\Delta)^2 \\ x^2+y^2+z^2 &\leqslant \frac{4\Delta^2}{c^2} \end{aligned}$$

The maximum value of $x^2+y^2+z^2$ is $\dfrac{4\Delta^2}{c^2}$, and it is obtained when $x = 0, y = 0, z = \dfrac{2\Delta}{c}$.

Note that you cannot use a similar method to this to find the minimum value of $x^2+y^2+z^2$ as you cannot combine the inequalities $(ax+by+cz)^2 \leqslant a^2(x+y+z)^2$ and $(x+y+z)^2 \geqslant x^2+y^2+z^2$.

## Question 6

**6** In this question, you should consider only points lying in the first quadrant, that is with $x > 0$ and $y > 0$.

**(i)** The equation $x^2 + y^2 = 2ax$ defines a *family* of curves in the first quadrant, one curve for each positive value of $a$. A second family of curves in the first quadrant is defined by the equation $x^2 + y^2 = 2by$, where $b > 0$.

**(a)** Differentiate the equation $x^2 + y^2 = 2ax$ implicitly with respect to $x$, and hence show that every curve in the first family satisfies the differential equation

$$2xy\frac{\mathrm{d}y}{\mathrm{d}x} = y^2 - x^2.$$

Find similarly a differential equation, independent of $b$, for the second family of curves.

**(b)** Hence, or otherwise, show that, at every point with $y \neq x$ where a curve in the first family meets a curve in the second family, the tangents to the two curves are perpendicular.

A curve in the first family meets a curve in the second family at $(c\,,\ c)$, where $c > 0$. Find the equations of the tangents to the two curves at this point. Is it true that where a curve in the first family meets a curve in the second family on the line $y = x$, the tangents to the two curves are perpendicular?

**(ii)** Given the family of curves in the first quadrant $y = c \ln x$, where $c$ takes any non-zero value, find, by solving an appropriate differential equation, a second family of curves with the property that at every point where a curve in the first family meets a curve in the second family, the tangents to the two curves are perpendicular.

**(iii)** A family of curves in the first quadrant is defined by the equation $y^2 = 4k(x+k)$, where $k$ takes any non-zero value.

Show that, at every point where one curve in this family meets a second curve in the family, the tangents to the two curves are perpendicular.

**Examiner's report**

This was one of the better answered questions on the paper and most candidates produced substantial responses to the question. Part **(i)(a)** was completed well in general, with most candidates achieving good marks for this section. When considering the second family of curves, some candidates forgot to include the $\frac{dy}{dx}$ on the right-hand side following their differentiation with respect to $x$. In part **(i)(b)** many candidates failed to justify their division by $(x^2 - y^2)$ when considering the case where $x \neq y$. Most candidates were however able to show that the tangents do remain perpendicular even in the case where $x = y$.

In part **(ii)** many candidates failed to recognise that $c$ needed to be eliminated from the differential equation in order to find the second family of curves. This mistake often led to the candidate only being able to achieve one of the marks available in this section.

Many candidates were able to make good progress on part **(iii)** of the question and many good solutions to this part of the question were seen.

**Solution**

At first sight this looks like a very long question, but there are a lot of conditions and clarifications throughout, and not as much work as it first appears to be.

**(i)** **(a)** Differentiating with respect to $x$ gives:

$$2x + 2y\frac{dy}{dx} = 2a$$
$$x + y\frac{dy}{dx} = a$$

Substituting this expression for $a$ into the equation for the family of curves gives:

$$x^2 + y^2 = 2x\left(x + y\frac{dy}{dx}\right)$$
$$y^2 - x^2 = 2xy\frac{dy}{dx}$$

as required.

For the second family we have:

$$2x + 2y\frac{dy}{dx} = 2b\frac{dy}{dx}$$

Multiplying the original equation for this family by $\frac{dy}{dx}$ and substituting gives:

$$(x^2 + y^2)\frac{dy}{dx} = 2by\frac{dy}{dx}$$
$$(x^2 + y^2)\frac{dy}{dx} = y\left(2x + 2y\frac{dy}{dx}\right)$$
$$(x^2 - y^2)\frac{dy}{dx} = 2xy$$

**(b)** Providing that $y \neq x$ (note that we are only considering points where $x > 0$ and $y > 0$ as mentioned in the stem of the question) we can write the gradients of the curves as:

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} = \frac{y^2 - x^2}{2xy}$$
$$\frac{\mathrm{d}y_2}{\mathrm{d}x} = \frac{2xy}{x^2 - y^2}$$

and multiplying these together gives:

$$\frac{y^2 - x^2}{2xy} \times \frac{2xy}{x^2 - y^2} = -1\,.$$

Therefore as long as $y \neq x$ the tangents to the curves are perpendicular where they meet.

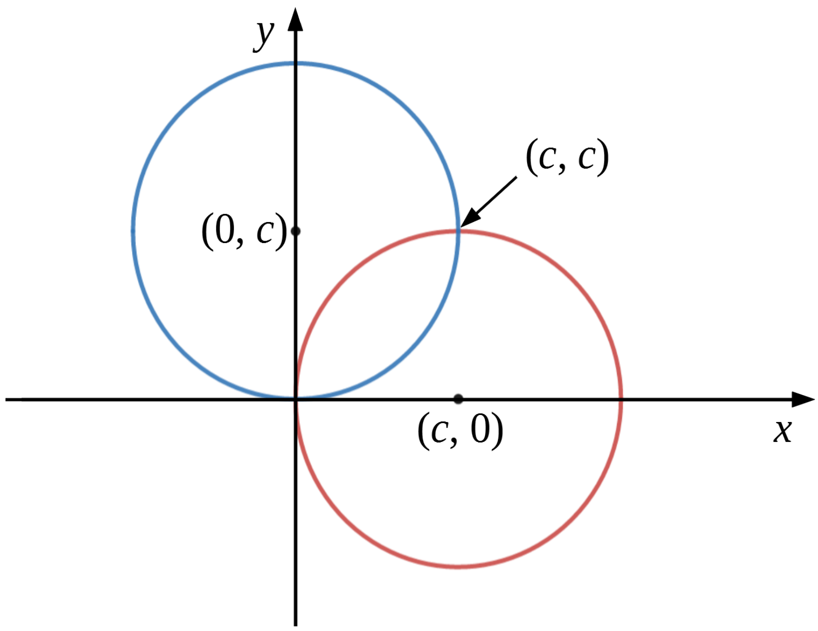
To consider the tangents at the points where $x = y$, it might be helpful to consider the equations of the curves more generally. The equation for the first family can be rewritten as:

$$x^2 + y^2 = 2ax$$
$$x^2 - 2ax + y^2 = 0$$
$$(x - a)^2 + y^2 = a^2$$

and hence the first family of curves are circles, centre $(a, 0)$ with radius $a$. Similarly the second family of curves are circles, centre $(0, b)$ with radius $b$.

If $x = y = c$ then the first equation (in it's original form) becomes $2c^2 = 2ca$. Since $c > 0$, this means that we must have $a = c$, and similarly we must have $b = c$.

A quick sketch of the situation might help us see what is happening now.




From this sketch we can see that the tangent to the first curve has equation $y = c$, and the tangent to the second one has equation $x = c$, so they are perpendicular.

In the stem of the question it told us to only consider points with $x > 0, y > 0$, so we can ignore the intersection at the origin.

**(ii)** In this part, the first family of curves satisfies $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{c}{x}$. Since $y = c\ln x$, we can write this equation as $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{y}{x\ln x}$.

Therefore we want the second curve to satisfy $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-x\ln x}{y}$.

Note that we cannot use the first differential equation in terms of $c$, as we want the two general families of curves to be perpendicular, not just one of each family at any one time.

Separating variables, and integrating by parts gives:

$$\begin{aligned} y\frac{\mathrm{d}y}{\mathrm{d}x} &= -x\ln x \\ \int y\,\mathrm{d}y &= -\int x\ln x\,\mathrm{d}x \\ \tfrac{1}{2}y^2 &= -\left[\tfrac{1}{2}x^2\ln x\right] + \tfrac{1}{2}\int x^2 \times \frac{1}{x}\,\mathrm{d}x \\ \tfrac{1}{2}y^2 &= -\tfrac{1}{2}x^2\ln x + \tfrac{1}{2}\int x\,\mathrm{d}x \\ \tfrac{1}{2}y^2 &= -\tfrac{1}{2}x^2\ln x + \tfrac{1}{4}x^2 + k \end{aligned}$$

**(iii)** Let the $k$ values for the two curves be $k_1$ and $k_2$, where $k_1 \neq k_2$. Where the two curves meet the $y$ coordinates are the same, so we have:

$$\begin{aligned} 4k_1(x+k_1) &= 4k_2(x+k_2) \\ (k_1-k_2)x &= (k_2^2-k_1^2) \\ x &= -\frac{(k_2+k_1)(k_2-k_1)}{k_2-k_1} \qquad k_1 \neq k_2 \\ x &= -(k_1+k_2) \end{aligned}$$

The $y$ coordinates satisfy:

$$y^2 = 4k\Big[-(k_1+k_2)+k\Big]$$

and for both $k = k_1$ and $k = k_2$ we have $y^2 = -4k_1k_2$.

Differentiating $y^2 = 4k(x+k)$ gives:

$$2y\frac{\mathrm{d}y}{\mathrm{d}x} = 4k \implies \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2k}{y}$$

Multiplying the gradients of the two curves gives:

$$\frac{\mathrm{d}y_1}{\mathrm{d}x} \times \frac{\mathrm{d}y_2}{\mathrm{d}x} = \frac{2k_1}{y} \times \frac{2k_2}{y} = \frac{4k_1k_2}{y^2} = -1$$

and so the tangents are perpendicular.

# Question 7

**7** Let $\mathrm{h}(z) = nz^6 + z^5 + z + n$, where $z$ is a complex number and $n \geqslant 2$ is an integer.

**(i)** Let $w$ be a root of the equation $\mathrm{h}(z) = 0$.

**(a)** Show that $|w^5| = \sqrt{\dfrac{\mathrm{f}(w)}{\mathrm{g}(w)}}$, where

$$\mathrm{f}(z) = n^2 + 2n\mathrm{Re}(z) + |z|^2 \text{ and } \mathrm{g}(z) = n^2|z|^2 + 2n\mathrm{Re}(z) + 1.$$

**(b)** By considering $\mathrm{f}(w) - \mathrm{g}(w)$, prove by contradiction that $|w| \geqslant 1$.

**(c)** Show that $|w| = 1$.

**(ii)** It is given that the equation $\mathrm{h}(z) = 0$ has six distinct roots, none of which is purely real.

**(a)** Show that $\mathrm{h}(z)$ can be written in the form

$$\mathrm{h}(z) = n(z^2 - a_1z + 1)(z^2 - a_2z + 1)(z^2 - a_3z + 1),$$

where $a_1$, $a_2$ and $a_3$ are real constants.

**(b)** Find $a_1 + a_2 + a_3$ in terms of $n$.

**(c)** By considering the coefficient of $z^3$ in $\mathrm{h}(z)$, find $a_1a_2a_3$ in terms of $n$.

**(d)** How many of the six roots of the equation $\mathrm{h}(z) = 0$ have a negative real part? Justify your answer.

### Examiner's report

Only a small number of candidates attempted this question, and many of those struggled to achieve good marks.

In part **(i)(a)** many candidates failed to spot the useful way of writing $w^5$ and were unable to secure any marks. Many of those who achieved few marks overall were able to obtain a mark by writing down an expression for $\mathrm{f}(w) - \mathrm{g}(w)$. Of those who did score well, many lost a mark for not stating that $\mathrm{g}(w) > 0$ when dividing through by it in their inequalities. While there were some successful alternative solutions to part **(i)(c)**, many of those who did not mimic part **(i)(b)** and tried to deduce the result directly were unsuccessful.

In part **(ii)(a)** some candidates failed to observe that the coefficients of the polynomial were real when stating that the roots occur in complex conjugate pairs. The remainder of part **(ii)** was dealt with easily by most candidates, including many of those who had otherwise obtained few marks.

**Solution**

**(i)** **(a)** We have $nw^6 + w^5 + w + n = 0$. We want $|w^5|$, so lets try some manipulation.

$$\begin{aligned} nw^6 + w^5 &= -w - n \\ w^5(nw+1) &= -w - n \\ w^5 &= -\frac{w+n}{nw+1} \end{aligned}$$

Using the fact that $|z|^2 = zz^*$ we have:

$$\begin{aligned} |w^5|^2 &= \frac{w+n}{nw+1} \times \left(\frac{w+n}{nw+1}\right)^* \\ |w^5| &= \sqrt{\frac{w+n}{nw+1} \times \frac{w^*+n}{nw^*+1}} \\ &= \sqrt{\frac{ww^* + n(w+w^*) + n^2}{n^2ww^* + n(w+w^*) + 1}} \\ &= \sqrt{\frac{|w|^2 + 2n\mathrm{Re}(w) + n^2}{n^2|w|^2 + 2n\mathrm{Re}(w) + 1}} \end{aligned}$$

as required.

**(b)** Considering $\mathrm{f}(w) - \mathrm{g}(w)$ gives:

$$\begin{aligned} \mathrm{f}(w) - \mathrm{g}(w) &= |w|^2 + n^2 - n^2|w|^2 - 1 \\ &= \left(n^2 - 1\right)\left(1 - |w|^2\right) \end{aligned}$$

We have $n \geqslant 2$, so $n^2 - 1 \geqslant 0$. If we have $|w| < 1$ then this gives $\mathrm{f}(w) - \mathrm{g}(w) > 0$, and so $\mathrm{f}(w) > \mathrm{g}(w)$. Since $\mathrm{f}(w)$ and $\mathrm{g}(w)$ are both equal to the square of a modulus of a non-zero complex number they must both be positive, and so we have $\mathrm{f}(w) > \mathrm{g}(w) > 0$.

However we also have $|w^5| = |w|^5 = \sqrt{\dfrac{\mathrm{f}(w)}{\mathrm{g}(x)}}$, and so since $\mathrm{f}(w) > \mathrm{g}(w) > 0$ this implies that $|w|^5 > 1$, and so $|w| > 1$. This is a contradiction of the initial statement $|w| < 1$, so we must have $|w| \geqslant 1$.

**(c)** In a similar way to part **(b)**, start by assuming that $|w| > 1$. This then gives $\mathrm{f}(w) - \mathrm{g}(w) < 0$, and so we have $\mathrm{g}(w) > \mathrm{f}(w) > 0$. This gives $|w| < 1$, contradicting the first statement, so we must have $|w| \leqslant 1$.

We must have $|w| \geqslant 1$ **and** $|w| \leqslant 1$ therefore we must have $|w| = 1$.

**(ii)** **(a)** The equation $\mathrm{h}(z) = 0$ is a polynomial of degree 6 and it has real coefficients. Since we are told that none of the roots are real this means that they occur in complex conjugate pairs.

Let two of the roots be $a \pm \mathrm{i}b$. By the factor theorem, that means that the product of two of the factors of $\mathrm{h}(z)$ is given by:

$$\begin{aligned}\big(z-(a+\mathrm{i}b)\big)\big(z-(a-\mathrm{i}b)\big) &= \big((z-a)-\mathrm{i}b\big)\big((z-a)+\mathrm{i}b\big)\\ &= (z-a)^2+b^2\\ &= z^2-2az+a^2+b^2\end{aligned}$$

Note that the constant term is $a^2+b^2=|z|^2$, and we know that if $w$ is a root we have $|w|=1$. Therefore we have $a^2+b^2=1$ and this factor is of the form $z^2-a_1z+1$. Repeating this with the other roots gives:

$$\begin{aligned}\mathrm{h}(z) &= nz^6+z^5+z+n\\ &= n(z^2-a_1z+1)(z^2-a_2z+1)(z^2-a_3z+1)\end{aligned}$$

where the $n$ outside the brackets is to ensure that the coefficient of $z^6$ is $n$.

**(b)** By equating coefficients of $z^5$ we have $1=n(-a_1-a_2-a_3)$, so we have $a_1+a_2+a_3=-\frac{1}{n}$.

**(c)** The coefficients of $z^3$ are formed by considering products of the form $z^2\times z\times 1$ or $z\times z\times z$, so equating coefficients of $z^3$ gives:

$$\begin{aligned}0 &= n\Big[-a_1a_2a_3-2a_1-2a_2-2a_3\Big]\\ 0 &= -na_1a_2a_3-2n(a_1+a_2+a_3)\\ 0 &= -na_1a_2a_3+2\\ \implies a_1a_2a_3 &= \frac{2}{n}\end{aligned}$$

**(d)** The real parts of the roots are equal to $\dfrac{a_i}{2}$, with each different real part being repeated twice in the six roots.

Using $a_1a_2a_3=\dfrac{2}{n}>0$ we know that either none or two of the $a_i$ are negative. However we also have $a_1+a_2+a_3=-\dfrac{1}{n}<0$, so at least one must be negative. Hence exactly two of $a_1,a_2,a_3$ are negative and so four of the six roots have a negative real part.

In many ways part **(i)(a)** is the hardest part of this question. However the rest of the question, and the rest of the marks (16 in total), are still available even if you cannot do the first part. Part **(ii)** of the question can also be done without having completed any of part **(i)** as part **(i)** has lots of "show that's". Doing part **(ii)** only as a sixth question could have provided a useful 8 marks. It is always a good idea to read through entire questions to see which parts can be done independently of the others, which might be useful as fifth or sixth questions towards the end of the exam time.

# Question 8

**8** Let $\mathbf{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a real matrix with $a \neq d$. The transformation represented by $\mathbf{M}$ has exactly two distinct invariant lines through the origin.

**(i)** Show that, if neither invariant line is the $y$-axis, then the gradients of the invariant lines are the roots of the equation

$$bm^2 + (a-d)m - c = 0.$$

If one invariant line is the $y$-axis, what is the gradient of the other?

**(ii)** Show that, if the angle between the two invariant lines is 45°, then

$$(a-d)^2 = (b-c)^2 - 4bc.$$

**(iii)** Find a necessary and sufficient condition, on some or all of $a$, $b$, $c$ and $d$, for the two invariant lines to make equal angles with the line $y = x$.

**(iv)** Give an example of a matrix which satisfies both the conditions in parts **(ii)** and **(iii)**.

**Examiner's report**

There were a large number of attempts at this question, but high scores were rare. One of the major reasons for solutions losing marks was a lack of understanding that the case where one invariant line is the $y$-axis needs to be addressed separately (as is highlighted in part **(i)** of the question). Many candidates were, however, able to demonstrate excellent understanding of the geometry involved.

In part **(i)** many candidates set up a matrix equation to find invariant points rather than invariant lines. Additionally, a significant number of candidates ignored the fact that the invariant lines must go through the origin (Note that this is the case for this question, in general invariant lines do not have to pass through the origin), meaning that the algebra was more complicated. The special case of one invariant line being the $y$-axis was often dealt with incorrectly, with many candidates believing that this meant that the other invariant line must be the $x$-axis.

There were many possible approaches to parts **(ii)** and **(iii)**, the simplest of which involved the tangent addition formulae for part **(ii)** and the realisation that the two lines were reflections of one another in the line $y = x$ for part **(iii)**. However, many candidates did not appreciate that the angle between the invariant lines is not necessarily the same as the angle between two vectors representing their directions. Additionally, many solutions did not check carefully that the quantities being used were well defined.

Those candidates who had scored well on parts **(ii)** and **(iii)** were often able to link them together and find a suitable matrix. A very small number of candidates produced the relevant calculations but failed to write down a matrix.

**Solution**

There can be some confusion between *invariant lines* and *lines of invariant points*. For an invariant line, individual points can move to other points on the same line under the transformation represented by the matrix, but for a line of invariant points the points must stay in the same place. In this question we are told that the invariant lines pass through the origin, but this is not always the case. There is a more detailed discussion about invariant lines and invariant points in the STEP 2 Matrices topic notes.

The fact that the lines are distinct is important! It might be an idea to highlight key words in the stem, so that when you are in the thick of working through the question you can glance back to see what facts you have been given. Do be careful not to highlight the entire question though, as this won't be as useful. In this case I would highlight "$a \neq d$" and "distinct", and maybe "origin" as well.

**(i)** We are told that there are exactly two invariant lines though the origin, and if they are not the $y$ axis then they have the form $y = mx$ (note that $m = 0$ gives the $x$ axis).

Lets assume that a point $(x, mx)$ maps onto another point $(x', mx')$ (i.e. they are both on the same line, but $x$ and $x'$ can be different. This means we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x' \\ mx' \end{pmatrix}$$
$$\begin{pmatrix} ax + bmx \\ cx + dmx \end{pmatrix} = \begin{pmatrix} x' \\ mx' \end{pmatrix}$$

This gives us two equations in $x$ and $x'$. Eliminating $x'$ gives:

$$cx + dmx = m(ax + bmx)$$
$$cx + dmx = max + bm^2x$$

If the $y$ axis is **not** one of the lines, then $x = 0$ is not a solution, and so we can divide throughout by $x$. Doing this and rearranging gives $bm^2 + (a-d)m - c = 0$, as required.

It the $y$ axis is invariant, then we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y' \end{pmatrix}$$

and so this means that we must have $b = 0$. The other line then satisfies:

$$\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx \end{pmatrix} = \begin{pmatrix} x' \\ mx' \end{pmatrix}$$

and eliminating $x'$ as before gives $cx + dmx = max$. Since the two lines are distinct then this one cannot be the $y$ axis, so $x = 0$ is not a solution and therefore we have $c + dm = ma \implies m = \dfrac{a-d}{c}$.

**(ii)** If one of the lines is the $y$ axis, then if the other line is at $45°$ it has gradient $\pm 1$. This means that we have:

$$\begin{aligned} \frac{a-d}{c} &= \pm 1 \\ \implies (a-d)^2 &= c^2 \end{aligned}$$

We also have $b = 0$ (from before), and so the equation $(a-d)^2 = (b-c)^2 - 4bc$ is satisfied.

If neither of the lines are the $y$ axis, then the directions of the lines can be written in vector form as $\begin{pmatrix} 1 \\ m_1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ m_2 \end{pmatrix}$. Using $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, and noting that if the angle between the lines is $45°$ so $\cos\theta = \pm\frac{1}{\sqrt{2}}$, we have:

$$\begin{aligned} 1 + m_1 m_2 &= \sqrt{1+m_1^2}\sqrt{1+m_2^2} \times \pm\frac{1}{\sqrt{2}} \\ (1+m_1m_2)^2 &= \frac{1}{2}\left(1+m_1^2\right)\left(1+m_2^2\right) \\ 2(1+m_1m_2)^2 &= 1 + m_1^2 + m_2^2 + m_1^2m_2^2 \\ 2(1+m_1m_2)^2 &= (m_1+m_2)^2 - 2m_1m_2 + 1 + (m_1m_2)^2 \qquad (*) \end{aligned}$$

If the angle between the two lines is $45°$, then the angles between the two vectors is either $45°$ or $135°$, since the vectors have a direction. This is why we have the two possible values of $\theta$.

The reason that we are rearranging the equation in this way is so that we can use the sum and product of roots equations (also know as Vieta's formulae).

We know that the values of $m$ satisfy $bm^2 + (a-d)m - c = 0$, so we have

$$\begin{aligned} m_1m_2 &= -\frac{c}{b} \\ m_1 + m_2 &= \frac{d-a}{b} \end{aligned}$$

Substituting these into (*) gives:

$$\begin{aligned} 2(1+m_1m_2)^2 &= (m_1+m_2)^2 - 2m_1m_2 + 1 + (m_1m_2)^2 \\ 2\left(1 - \frac{c}{b}\right)^2 &= \left(\frac{d-a}{b}\right)^2 + \frac{2c}{b} + 1 + \frac{c^2}{b^2} \\ 2(b-c)^2 &= (d-a)^2 + 2cb + b^2 + c^2 \\ 2(b-c)^2 &= (d-a)^2 + 2cb + (b-c)^2 + 2cb \\ \implies (a-d)^2 &= (b-c)^2 - 4bc \end{aligned}$$

**(iii)** If one of the lines is the $y$ axis then we have $b = 0$, and if they are to make equal angles with $y = x$ then the other line must be the $x$ axis. This means that the matrix must satisfy

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x' \\ 0 \end{pmatrix}$$

and so we must also have $c = 0$. So in this case the condition is $b = c = 0$.

In the other case, the lines with directions $\begin{pmatrix}1\\m_1\end{pmatrix}$ and $\begin{pmatrix}1\\m_2\end{pmatrix}$ must make equal angles with the line with direction $\begin{pmatrix}1\\1\end{pmatrix}$. Using the dot product and squaring to deal with the direction along the line the vectors point in gives:

$$\begin{aligned}\frac{(1+m_1)^2}{2(1+m_1^2)} &= \frac{(1+m_2)^2}{2(1+m_2^2)}\\(1+m_1)^2(1+m_2^2) &= (1+m_2)^2(1+m_1^2)\\1+2m_1+m_1^2+m_2^2+2m_1m_2^2+m_1^2m_2^2 &= 1+2m_2+m_2^2+m_1^2+2m_1^2m_2+m_1^2m_2^2\\2m_1+2m_1m_2^2 &= 2m_2+2m_1^2m_2\end{aligned}$$

There was a lot of cancelling to be done there! Dividing by 2 and moving everything over to one side gives:

$$\begin{aligned}m_1m_2^2-m_1^2m_2+m_1-m_2 &= 0\\m_1m_2(m_2-m_1)-(m_2-m_1) &= 0\\(m_2-m_1)(m_1m_2-1) &= 0\end{aligned}$$

We cannot have $m_1 = m_2$ as the lines are distinct, therefore we have $m_1m_2 = 1$, and so the condition is $-\frac{c}{b} = 1 \implies b + c = 0$. If $b = 0$ this condition gives $c = 0$, and so the condition $b + c = 0$ holds for both cases.

### An alternative approach:

If the two invariant lines make equal angles with the line $y = x$, then they are reflections of each other in the line $y = x$. This means that the product of their gradients is equal to 1, i.e. $m_1m_2 = 1$.[1]

Using the equation found in part **(i)**, we know that the product of the roots of the equation (and so the product of the gradients) is equal to $\frac{-c}{b}$. Putting this together we have:

$$\begin{aligned}m_1m_2 &= 1\\\frac{-c}{b} &= 1\\b+c &= 0\end{aligned}$$

As in the first method, the special case when the invariant lines are the $x$ and $y$ axes needs to be considered separately.

[1]To see why this is true, firstly note that both lines pass through the origin. Then if the point $(a, b)$ is on one invariant line, then the reflections of this point in the line $y = x$ lies on the other invariant line, i.e. $(b, a)$ lies on the other line. The gradients of the two lines are then $\frac{b}{a}$ and $\frac{a}{b}$ and so the product of these is equal to 1. As is often the case, a diagram makes this clearer but unfortunately this footnote is too small for one.

**(iv)** If we are to satisfy both conditions then we need:

$$b + c = 0$$
$$(a-d)^2 = (b-c)^2 - 4bc$$

Substituting $c = -b$ gives:

$$(a-d)^2 = (2b)^2 + 4b^2$$
$$(a-d)^2 = 8b^2$$

At this point it is slightly tempting to set $b = 0$, but if we do that then the matrix has the form $\mathbf{M} = k\mathbf{I}$, which has infinitely many invariant lines (it should be fairly obvious that it has three, the $x$ and $y$ axes and $y = x$, and with a little more thought it can be seen that every straight line through the origin is an invariant line).

So that we get a perfect square, let $b = \sqrt{2}$, so that we have $c = -\sqrt{2}$ and $(a-d)^2 = 16$. This last equation can be satisfied by $a = 5, d = 1$ so one possible matrix is:

$$\mathbf{M} = \begin{pmatrix} 5 & \sqrt{2} \\ -\sqrt{2} & 1 \end{pmatrix}$$

# Question 9

**9** A rectangular prism is fixed on a horizontal surface. A vertical wall, parallel to a vertical face of the prism, stands at a distance $d$ from it. A light plank, making an acute angle $\theta$ with the horizontal, rests on an upper edge of the prism and is in contact with the wall below the level of that edge of the prism and above the level of the horizontal plane. You may assume that the plank is long enough and the prism high enough to make this possible.

The contact between the plank and the prism is smooth, and the coefficient of friction at the contact between the plank and the wall is $\mu$. When a heavy point mass is fixed to the plank at a distance $x$, along the plank, from its point of contact with the wall, the system is in equilibrium.

**(i)** Show that, if $x = d\sec^3\theta$, then there is no frictional force acting between the plank and the wall.

**(ii)** Show that, if $x > d\sec^3\theta$, it is necessary that

$$\mu \geqslant \frac{x - d\sec^3\theta}{x\tan\theta}$$

and give the corresponding inequality if $x < d\sec^3\theta$.

**(iii)** Show that

$$\frac{x}{d} \geqslant \frac{\sec^3\theta}{1 + \mu\tan\theta}.$$

Show also that, if $\mu < \cot\theta$, then

$$\frac{x}{d} \leqslant \frac{\sec^3\theta}{1 - \mu\tan\theta}.$$

**(iv)** Show that if $x$ is such that the point mass is fixed to the plank somewhere between the edge of the prism and the wall, then $\tan\theta < \mu$.

## Examiner's report

Many of the attempts at this question failed to achieve good marks. Some candidates struggled to convert the text into a suitable diagram, while others were confused about whether to label the forces that other objects exerted on the plank or the forces exerted by the plank itself. These diagrams then often led to confusion about the directions of the forces in play.

In part **(i)** there were many attempts that used backwards logic – assuming that there was no frictional force and finding the value of $x$ required for this to be true. There were also a number of candidates who assumed that the frictional force would always be equal to $\mu N$, rather than taking this as the maximum.

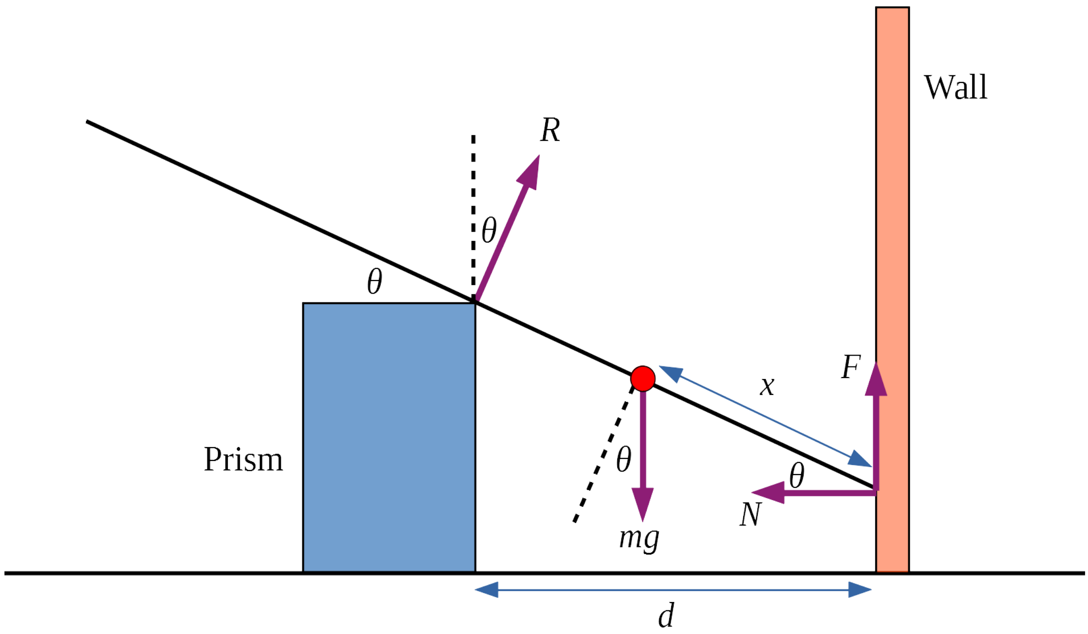
Part **(ii)** was often done poorly because it was often not clear that the solution was using the magnitudes of the forces when using $F = \mu N$.

In part **(iii)** candidates often failed to consider all possible cases for the value of $x$ and it was common to see the first result shown in one of the cases and the second result shown in the other.

Solutions to part **(iv)** were more successful from the candidates who reached this point.

**Solution**

As is often the case with mechanics questions, the first thing to do is to draw a large, clear diagram.




In the diagram above I have indicated several other angles which are equal to $\theta$ as these will be helpful later. It might also be helpful to note that the length of plank between the wall and the prism is equal to $\dfrac{d}{\cos\theta} = d\sec\theta$.

**(i)** Resolving forces gives:

$$F + R\cos\theta = mg \tag{1}$$
$$R\sin\theta = N \tag{2}$$

Using moments about the point where the plank meets the wall gives:

$$x \times mg\cos\theta = \frac{d}{\cos\theta} \times R \tag{3}$$

Rearranging (3) gives $R = \dfrac{mgx\cos^2\theta}{d}$, and substituting this into (2) gives $N = \dfrac{mgx\cos^2\theta\sin\theta}{d}$.

We also have:

$$\begin{aligned} F &= mg - R\cos\theta \\ &= mg - \frac{mgx\cos^3\theta}{d} \\ &= mg\left(1 - \frac{x\cos^3\theta}{d}\right) \end{aligned}$$

Therefore if $x = d\sec^3\theta$ we have $F = 0$. Note the direction of the implication!

**(ii)** If $x > d\sec^3\theta$ then the frictional force is negative (i.e. it acts downwards). Using $F \leqslant \mu N$ with the downwards frictional force gives:

$$\begin{aligned} mg\left(\frac{x\cos^3\theta}{d} - 1\right) &\leqslant \mu \times \frac{mgx\cos^2\theta\sin\theta}{d} \\ x\cos^3\theta - d &\leqslant \mu x\cos^2\theta\sin\theta \\ \implies \mu &\geqslant \frac{x\cos^3\theta - d}{x\cos^2\theta\sin\theta} \\ \mu &\geqslant \frac{x - d\sec^3\theta}{x\tan\theta} \end{aligned}$$

If $x < d\sec^3\theta$ then the frictional force acts upwards and using $F \leqslant \mu N$ gives:

$$\begin{aligned} mg\left(1 - \frac{x\cos^3\theta}{d}\right) &\leqslant \mu \times \frac{mgx\cos^2\theta\sin\theta}{d} \\ d - x\cos^3\theta &\leqslant \mu x\cos^2\theta\sin\theta \\ \implies \mu &\geqslant \frac{d - x\cos^3\theta}{x\cos^2\theta\sin\theta} \\ \mu &\geqslant \frac{d\sec^3\theta - x}{x\tan\theta} \end{aligned}$$

**(iii)** If $x \geqslant d\sec^3\theta$ then $\dfrac{x}{d} \geqslant \sec^3\theta$. Since $1 + \mu\tan\theta \geqslant 1$ we have $\dfrac{x}{d} \geqslant \dfrac{\sec^3\theta}{1 + \mu\tan\theta}$.

If $x < d\sec^3\theta$ then we have $\mu \geqslant \dfrac{d\sec^3\theta - x}{x\tan\theta}$. This rearranges to give $\dfrac{x}{d} \geqslant \dfrac{\sec^3\theta}{1 + \mu\tan\theta}$, as required.

For the second statement, if $x \leqslant d\sec^3\theta$ then we have $\dfrac{x}{d} \leqslant \sec^3\theta$. Since $0 \leqslant \mu < \cot\theta$ we have $0 \leqslant \mu\tan\theta < 1$, and so $0 < 1 - \mu\tan\theta \leqslant 1$ and so we have $\dfrac{x}{d} \leqslant \dfrac{\sec^3\theta}{1 - \mu\tan\theta}$.

If we have $x > d\sec^3\theta$ then we have $\mu \geqslant \dfrac{x - d\sec^3\theta}{x\tan\theta}$ which rearranges to give $\dfrac{x}{d} \leqslant \dfrac{\sec^3\theta}{1 - \mu\tan\theta}$.

**(iv)** If the point mass is between the edge of the prism and the wall (as shown in the diagram at the start of this solution) then we have $x < d\sec\theta$.

From part **(iii)** we have:

$$\begin{aligned}
\frac{x}{d} &\geqslant \frac{\sec^3\theta}{1+\mu\tan\theta} \\
x &\geqslant \frac{d\sec\theta \times \sec^2\theta}{1+\mu\tan\theta} \\
x &> \frac{x \times \sec^2\theta}{1+\mu\tan\theta} \\
1 &> \frac{\sec^2\theta}{1+\mu\tan\theta} \\
1+\mu\tan\theta &> \sec^2\theta \\
1+\mu\tan\theta &> 1+\tan^2\theta \\
\mu\tan\theta &> \tan^2\theta \\
\mu &> \tan\theta
\end{aligned}$$

as required.

Note that $d > 0$, $\tan\theta > 0$ and $1+\mu\tan\theta > 0$, so all of the manipulations above are correct, we don't need to worry about the sign changing at any point.

# Question 10

**10** **(i)** Show that, if a particle is projected at an angle $\alpha$ above the horizontal with speed $u$, it will reach height $h$ at a horizontal distance $s$ from the point of projection where

$$h = s\tan\alpha - \frac{gs^2}{2u^2\cos^2\alpha}.$$

The remainder of this question uses axes with the $x$- and $y$-axes horizontal and the $z$-axis vertically upwards. The ground is a sloping plane with equation $z = y\tan\theta$ and a road runs along the $x$-axis. A cannon, which may have any angle of inclination and be pointed in any direction, fires projectiles from ground level with speed $u$. Initially, the cannon is placed at the origin.

**(ii)** Let a point $P$ on the plane have coordinates $(x\,,\ y\,,\ y\tan\theta)$. Show that the condition for it to be possible for a projectile from the cannon to land at point $P$ is

$$x^2 + \left(y + \frac{u^2\tan\theta}{g}\right)^2 \leqslant \frac{u^4\sec^2\theta}{g^2}.$$

**(iii)** Show that the furthest point directly up the plane that can be reached by a projectile from the cannon is a distance

$$\frac{u^2}{g(1+\sin\theta)}$$

from the cannon.

How far from the cannon is the furthest point directly down the plane that can be reached by a projectile from it?

**(iv)** Find the length of road which can be reached by projectiles from the cannon.

The cannon is now moved to a point on the plane vertically above the $y$-axis, and a distance $r$ from the road. Find the value of $r$ which maximises the length of road which can be reached by projectiles from the cannon. What is this maximum length?

### Examiner's report

While the first part of the question was answered well in general, candidates did not perform well on the question as a whole. Many candidates only answered the first part of the question, although some misunderstood and attempted to consider the maximum height reached. Others assigned the sign to acceleration incorrectly in the vertical motion equation.

Many candidates also struggled with the trigonometric functions and failed to identify that trigonometric identities could be applied; this often meant that part **(ii)** was not possible. The lack of familiarity with trigonometric manipulations caused further marks to be lost later in the question also.

**Solution**

**(i)** The horizontal ($s$) and vertical ($h$) distances are given by:

$$\begin{aligned} s &= ut\cos\alpha \\ h &= ut\sin\alpha - \tfrac{1}{2}gt^2 \end{aligned}$$

Using the first equation to eliminate $t$ gives:

$$\begin{aligned} h &= \frac{s}{\cos\alpha} \times \sin\alpha - \tfrac{1}{2}g\left(\frac{s}{u\cos\alpha}\right)^2 \\ h &= s\tan\alpha - \frac{gs^2}{2u^2\cos^2\alpha} \qquad (*) \end{aligned}$$

**(ii)** It might have seemed rather odd that the first part used $s$ and $h$ (rather than $x$ and $y$), but in this part this choice makes a bit more sense. The cannon can pivot around, and the horizontal distance travelled by the projective is given by $s = \sqrt{x^2+y^2}$. We want the height to be $h = y\tan\theta$. Substituting these into (*) gives:

$$\begin{aligned} y\tan\theta &= \tan\alpha\sqrt{x^2+y^2} - \frac{g(x^2+y^2)}{2u^2\cos^2\alpha} \\ y\tan\theta &= \tan\alpha\sqrt{x^2+y^2} - \frac{g(x^2+y^2)}{2u^2}(1+\tan^2\alpha) \\ \implies 0 &= \frac{g(x^2+y^2)}{2u^2}\tan^2\alpha - \tan\alpha\sqrt{x^2+y^2} + y\tan\theta + \frac{g(x^2+y^2)}{2u^2} \end{aligned}$$

This is a quadratic equation in $\tan\alpha$ which will have (real) solutions if:

$$\begin{aligned} (x^2+y^2) - 4\frac{g(x^2+y^2)}{2u^2}\left(y\tan\theta + \frac{g(x^2+y^2)}{2u^2}\right) &\geqslant 0 \\ 1 - \frac{2g}{u^2}\left(y\tan\theta + \frac{g(x^2+y^2)}{2u^2}\right) &\geqslant 0 \\ 2g\left(y\tan\theta + \frac{g(x^2+y^2)}{2u^2}\right) &\leqslant u^2 \\ 2gu^2y\tan\theta + g^2(x^2+y^2) &\leqslant u^4 \\ x^2+y^2+\frac{2u^2\tan\theta}{g}y &\leqslant \frac{u^4}{g^2} \end{aligned}$$

Completing the square with the $y$ terms gives:

$$\begin{aligned} x^2 + \left(y+\frac{u^2\tan\theta}{g}\right)^2 - \left(\frac{u^2\tan\theta}{g}\right)^2 &\leqslant \frac{u^4}{g^2} \\ x^2 + \left(y+\frac{u^2\tan\theta}{g}\right)^2 &\leqslant \frac{u^4}{g^2} + \left(\frac{u^2\tan\theta}{g}\right)^2 \\ x^2 + \left(y+\frac{u^2\tan\theta}{g}\right)^2 &\leqslant \frac{u^4}{g^2}\sec^2\theta \end{aligned}$$

**(iii)** The maximum distance up the plane will be when $x = 0$, when we have:

$$\left(y + \frac{u^2 \tan\theta}{g}\right)^2 \leqslant \frac{u^4}{g^2}\sec^2\theta$$

Therefore:

$$-\frac{u^2 \sec\theta}{g} - \frac{u^2 \tan\theta}{g} \leqslant y \leqslant \frac{u^2 \sec\theta}{g} - \frac{u^2 \tan\theta}{g}$$

The distance up the plane is given by $d = y\sec\theta$, and so we have:

$$d \leqslant \frac{u^2 \sec^2\theta}{g} - \frac{u^2 \tan\theta\sec\theta}{g}$$

The greatest distance is:

$$\begin{aligned}
d_{\text{max}} &= \frac{u^2}{g}\left(\sec^2\theta - \tan\theta\sec\theta\right)\\
&= \frac{u^2}{g}\left(\frac{1-\sin\theta}{\cos^2\theta}\right)\\
&= \frac{u^2}{g}\left(\frac{1-\sin\theta}{1-\sin^2\theta}\right)\\
&= \frac{u^2}{g}\left(\frac{1}{1+\sin\theta}\right)
\end{aligned}$$

The minimum value of $d$ is given by:

$$\begin{aligned}
d_{\text{min}} &= -\frac{u^2 \sec^2\theta}{g} - \frac{u^2 \tan\theta\sec\theta}{g}\\
&= -\frac{u^2}{g}\left(\sec^2\theta + \tan\theta\sec\theta\right)\\
&= -\frac{u^2}{g}\left(\frac{1+\sin\theta}{\cos^2\theta}\right)\\
&= -\frac{u^2}{g}\left(\frac{1}{1-\sin\theta}\right)
\end{aligned}$$

The greatest down the slope is therefore $\dfrac{u^2}{g}\left(\dfrac{1}{1-\sin\theta}\right)$.

**(iv)** If we have $y = 0$, then we have $x^2 \leqslant \dfrac{u^4}{g^2}$, so we have $-\dfrac{u^2}{g} \leqslant x \leqslant \dfrac{u^2}{g}$ and the total length of road that can be reached is $\dfrac{2u^2}{g}$.

If the cannon is moved a distance $r$ up the slope, then the coordinates of the cannon will now be $(0, r\cos\theta, r\sin\theta)$. We can find the positions that the projectile can land by replacing $y$ with $y - r\cos\theta$, in the result from part **(ii)** to get:

$$x^2 + \left(y - r\cos\theta + \frac{u^2 \tan\theta}{g}\right)^2 \leqslant \frac{u^4}{g^2}\sec^2\theta$$

When $y = 0$ this gives:

$$x^2 \leqslant \frac{u^4}{g^2}\sec^2\theta - \left(\frac{u^2\tan\theta}{g} - r\cos\theta\right)^2$$

This will be maximised when the second square bracket is equal to 0, i.e. when $r = \dfrac{u^2\tan\theta\sec\theta}{g}$. The length of road which can then by reached is given by $\dfrac{2u^2\sec\theta}{g}$.

A quick sanity check reassures us that when the cannon in placed up the slope, it can reach more of the road (as $\sec\theta \geqslant 1$).

# Question 11

**11** A batch of $N$ USB sticks is to be used on a network. Each stick has the same unknown probability $p$ of being infected with a virus. Each stick is infected, or not, independently of the others.

The network manager decides on an integer value of $T$ with $0 \leqslant T < N$. If $T = 0$ no testing takes place and the $N$ sticks are used on the network, but if $T > 0$, the batch is subject to the following procedure.

- Each of $T$ sticks, chosen at random from the batch, undergoes a test during which it is destroyed.
- If any of these $T$ sticks is infected, all the remaining $N - T$ sticks are destroyed.
- If none of the $T$ sticks is infected, the remaining $N - T$ sticks are used on the network.

If any stick used on the network is infected, the network has to be disinfected at a cost of £$D$, where $D > 0$. If no stick used on the network is infected, there is a gain of £1 for each of the $N - T$ sticks. There is no cost to testing or destroying a stick.

**(i)** Find an expression in terms of $N$, $T$, $D$ and $q$, where $q = 1 - p$, for the expected net loss.

**(ii)** Let $\alpha = \dfrac{DT}{N(N - T + D)}$. Show that $0 \leqslant \alpha < 1$.

Show that, for fixed values of $N$, $D$ and $T$, the greatest value of the expected net loss occurs when $q$ satisfies the equation $q^{N-T} = \alpha$.

Show further that this greatest value is £$\dfrac{D(N - T)\,\alpha^k}{N}$, where $k = \dfrac{T}{N - T}$.

**(iii)** For fixed values of $N$ and $D$, show that there is some $\beta > 0$ so that for all $p < \beta$, the expression for the expected loss found in part **(i)** is an increasing function of $T$. Deduce that, for small enough values of $p$, testing no sticks minimises the expected net loss.

**Examiner's report**

Some candidates made good progress with this question. However, a large proportion of attempts did not make much progress and many were unable to score more than one or two marks.

Many candidates were able to correctly identify the way in which the required expression in part **(i)** should be constructed and they were often successful in putting these together correctly.

In part **(ii)** most candidates were able to consider the definition of $\alpha$ and demonstrate the inequality required, although some candidates only showed one side of the inequality. Candidates who recognised that differentiation of the expected net loss function was a useful approach were often able to make good progress towards finding the required results for the remainder of this question.

A small number of candidates attempted part **(iii)** of the question and many of these were able to show the first result. The explanations of the final deduction were often not convincing however.

**Solution**

**(i)** If any of the first $T$ sticks is infected then there is no gain or loss. If none of the sticks at all are infected then there is a gain of $N-T$ pounds. If none of the first $T$ sticks is infected, but then at least one of the $N-T$ are infected then there is a loss of $D$ pounds. We are given $q = 1-p$, which means that $q$ is the probability that a stick is not infected. The probability that none of the first $T$ sticks are infected is $q^T$, and the probability that at least one of the other $N-T$ sticks are infected is $1-q^{N-T}$.

The expected net loss is therefore:

$$\begin{aligned} L &= q^T(1-q^{N-T})D - q^N(N-T) \\ &= Dq^T + q^N(T-D-N) \end{aligned}$$

**(ii)** We know that $D, N, T > 0$ and also that $N > T$. Therefore if $\alpha = \dfrac{DT}{N(N-T+D)}$ both the numerator and denominator are positive so we have $\alpha \geqslant 0$ (in fact we have $\alpha > 0$).

We are also asked to show that $\alpha < 1$, i.e. $\dfrac{DT}{N(N-T+D)} < 1$. When trying to show an inequality is true it is often easiest to rearrange so that instead we are showing that something is positive or negative. Consider $DT - N(N-T+D)$:

$$\begin{aligned} DT - N(N-T+D) &= DT - N^2 + NT - ND \\ &= DT - ND + NT - N^2 \\ &= D(T-N) + N(T-N) \\ &< 0 \end{aligned}$$

Where the last line is because we have $D, N > 0$ and $T < N$. Hence we have $DT < N(N-T+D)$ and so $\alpha < 1$.

Differentiating our expression for $L$ with respect to $q$ gives:

$$\begin{aligned} \frac{\mathrm{d}L}{\mathrm{d}q} &= TDq^{T-1} + N(T-D-N)q^{N-1} \\ 0 &= q^{T-1}\left[TD + N(T-D-N)q^{N-T}\right] \end{aligned}$$

Assuming $0 < q < 1$, then the stationary point will be when the square bracket is equal to 0, as $q^{T-1} > 0$. The $q = 0$ case is when all the sticks are infected and so they will all be destroyed with net loss 0. The $q = 1$ case is when none of the sticks are infected, so there will be a gain of $N-T$.

The only possible stationary point is where $q^{N-T} = \dfrac{TD}{N(D+N-T)} = \alpha$. Since $0 < \alpha < 1$ there is a possible value of $q$ which satisfies this (remembering that $0 < q < 1$).

Rewriting the derivative gives:

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}q} &= q^{T-1}\left[TD + N(T-D-N)q^{N-T}\right] \\ &= N(D+N-T)q^{T-1}\left[\frac{TD}{N(D+N-T)} - q^{N-T}\right] \\ &= N(D+N-T)q^{T-1}\left[\alpha - q^{N-T}\right]\end{aligned}$$

The expression outside the square brackets is positive, so when $q^{N-T} < \alpha$ the gradient is positive and when $q^{N-T} > \alpha$ the gradient is negative. This means that to the left of the stationary point $L$ in increasing and to the right it is decreasing, hence the stationary point is a maximum.

The maximum value of $L$ is therefore:

$$\begin{aligned}L_{\max} &= Dq^T + q^N(T-D-N) \\ &= q^T\left[D + q^{N-T}(T-D-N)\right] \\ &= q^T\left[D + \alpha(T-D-N)\right] \\ &= q^T\left[D - \frac{DT}{N}\right] \\ &= \frac{Dq^T}{N}\left[N-T\right]\end{aligned}$$

where the penultimate line above follows from the definition of $\alpha$.

We have $q^T = \left[\sqrt[N-T]{\alpha}\right]^T = \alpha^{\frac{T}{N-T}} = \alpha^k$, and so the maximum net loss is:

$$L_{\max} = \frac{D\alpha^k(N-T)}{N}$$

as required.

**(iii)** In this part, we are considering the expected loss as a function of $T$ (in part **(i)** $T$ was fixed and $q$ was varying). Since $T$ is discrete (it is an integer!), we know that $L$ is a increasing function in $T$ if $L(T+1) > L(T)$ for all values if $T$. In context, this means every time we test one more stick, the expected loss is increased.

Consider $L(T+1) - L(T)$ (which we want to show is positive).

$$\begin{aligned}&L(T+1) - L(T) \\ =&\Big(Dq^{T+1} + q^N[(T+1)-D-N]\Big) - \Big(Dq^T + q^N[T-D-N]\Big) \\ =&Dq^{T+1} + q^N - Dq^T \\ =&Dq^T(q-1) + q^N \\ =&q^T(q^{N-T} - Dp)\end{aligned}$$

This is then positive as long as $q^{N-T} > Dp$. As $p \to 0$, $Dp \to 0$ and $q^{N-T} \to 1$. This means that as long as $p$ is small enough we will have $q^{N-T} > Dp$ for all $T$. Therefore if $p < \beta$, (for some appropriate $\beta$), $L$ will be an increasing function and so the optimal strategy will be to test no sticks (as each time we test one more stick, the expected loss increases).

Note that in part **(ii)** we were optimising over a continuous quantity ($q$), and so differentiation was a valid method. In part **(iii)** we were optimising over $T$ which takes discrete values, and so considering what happens to $L$ as $T$ increases by 1 each time is the valid method here.

# Question 12

**12** The random variable $X$ has probability density function

$$f(x) = \begin{cases} kx^n(1-x) & 0 \leqslant x \leqslant 1\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $n$ is an integer greater than 1.

**(i)** Show that $k = (n+1)(n+2)$ and find $\mu$, where $\mu = \mathrm{E}(X)$.

**(ii)** Show that $\mu$ is less than the median of $X$ if

$$6 - \frac{8}{n+3} < \left(1 + \frac{2}{n+1}\right)^{n+1}.$$

By considering the first four terms of the expansion of the right-hand side of this inequality, or otherwise, show that the median of $X$ is greater than $\mu$.

**(iii)** You are given that, for positive $x$, $\left(1 + \dfrac{1}{x}\right)^{x+1}$ is a decreasing function of $x$.

Show that the mode of $X$ is greater than its median.

**Examiner's report**

Part **(i)** of the question was generally well attempted, although some candidates opted to integrate by parts rather than noticing that the integral was simple once the brackets were expanded. A small number of candidates failed to simplify the mean fully.

Part **(ii)** was found to be difficult. Most candidates attempted to compute an expression for the median rather than comparing the cumulative density function of the mean to $\frac{1}{2}$. Those who did follow the intended approach were generally able to work through the algebra well. Many candidates were also confused about the direction of the logic in this question and instead showed the converse of the required result. In many cases the argument provided was reversible so many of the marks could still be awarded.

Expansion of the binomial expression was generally done well, but the algebra of the part that followed proved difficult, with most candidates either giving up early on or making mistakes that either rendered the conclusion trivial or impossible to obtain. Only a handful of candidates successfully reached the correct condition and convincingly showed it to be true.

Many candidates did not attempt part **(iii)**. Those who had been successful in part **(ii)** almost always realised that they needed to consider the cumulative density function of the mode and compare it to $\frac{1}{2}$ and almost all these candidates managed to deduce the argument completely and gain full credit.

**Solution**

**(i)** Equating the total probability to 1 gives:

$$\begin{aligned}
\int_0^1 kx^n(1-x)\,\mathrm{d}x &= 1 \\
\int_0^1 kx^n - kx^{n+1}\,\mathrm{d}x &= 1 \\
\left[\frac{k}{n+1}x^{n+1} - \frac{k}{n+2}x^{n+2}\right]_0^1 &= 1 \\
\frac{k}{n+1} - \frac{k}{n+2} &= 1 \\
\frac{k[(n+2)-(n+1)]}{(n+1)(n+2)} &= 1 \\
\frac{k}{(n+1)(n+2)} &= 1
\end{aligned}$$

Therefore we have $k = (n+1)(n+2)$, as required.

Using $\mu = \int xf(x)\,\mathrm{d}x$ gives:

$$\begin{aligned}
\int_0^1 x \times kx^n(1-x)\,\mathrm{d}x &= k\int_0^1 x^{n+1} - x^{n+2}\,\mathrm{d}x \\
&= k\left[\frac{1}{n+2}x^{n+2} - \frac{1}{n+3}x^{n+3}\right]_0^1 \\
&= k\left[\frac{1}{n+2} - \frac{1}{n+3}\right] \\
&= \frac{k}{(n+2)(n+3)} \\
&= \frac{(n+1)(n+2)}{(n+2)(n+3)} \\
&= \frac{n+1}{n+3}
\end{aligned}$$

**(ii)** If $\mu$ is less than the median then we have:

$$\begin{aligned}
\int_0^{\mu} \mathrm{f}(x)\,\mathrm{d}x &< 0.5 \\
k\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^{\mu} &< 0.5 \\
k\mu^{n+1}\left[\frac{1}{n+1} - \frac{\mu}{n+2}\right] &< 0.5 \\
(n+1)(n+2)\left(\frac{n+1}{n+3}\right)^{n+1}\left[\frac{1}{n+1} - \frac{n+1}{(n+2)(n+3)}\right] &< 0.5 \\
\left(\frac{n+1}{n+3}\right)^{n+1}\left[(n+2) - \frac{(n+1)^2}{n+3}\right] &< 0.5 \\
2\left[(n+2) - \frac{(n+1)^2}{n+3}\right] &< \left(\frac{n+3}{n+1}\right)^{n+1} \\
2\left[\frac{(n+2)(n+3)-(n+1)^2}{n+3}\right] &< \left(\frac{n+3}{n+1}\right)^{n+1} \\
2\left[\frac{n^2+5n+6-n^2-2n-1}{n+3}\right] &< \left(\frac{n+1+2}{n+1}\right)^{n+1} \\
2\left[\frac{3n+5}{n+3}\right] &< \left(1+\frac{2}{n+1}\right)^{n+1} \\
2\left[\frac{3(n+3)-4}{n+3}\right] &< \left(1+\frac{2}{n+1}\right)^{n+1} \\
6 - \frac{8}{n+3} &< \left(1+\frac{2}{n+1}\right)^{n+1}
\end{aligned}$$

Since all of the terms in the binomial expansion of the right hand side are positive, this inequality will be true if it is true for just the first 4 terms, i.e. if:

$$\begin{aligned}
6 - \frac{8}{n+3} &< 1 + (n+1)\left(\frac{2}{n+1}\right) + \tfrac{1}{2}(n+1)n\left(\frac{2}{n+1}\right)^2 + \tfrac{1}{6}(n+1)n(n-1)\left(\frac{2}{n+1}\right)^3 \\
6 - \frac{8}{n+3} &< 1 + 2 + \frac{2n}{n+1} + \frac{4n(n-1)}{3(n+1)^2} \\
3 - \frac{8}{n+3} &< \frac{2n}{n+1} + \frac{4n(n-1)}{3(n+1)^2}
\end{aligned}$$

Multiplying throughout by $3(n+1)^2(n+3)$ (which is positive) gives:

$$\begin{aligned}
9(n+3)(n+1)^2 - 24(n+1)^2 &< 6n(n+1)(n+3) + 4n(n-1)(n+3) \\
9(n^3+5n^2+7n+3) - 24(n^2+2n+1) &< 6(n^3+4n^2+3n) + 4(n^3+2n^2-3n) \\
9n^3+21n^2+15n+3 &< 10n^3+32n^2+6n \\
0 &< n^3+11n^2-9n-3 \\
0 &< (n-1)(n^2+12n+3)
\end{aligned}$$

Since we know $n > 1$ (which is given in the stem of the question!) this inequality is true and therefore we have shown that $\mu$ is less than the median of $X$.

**(iii)** The mode of $X$ is the maximum value of the pdf[2], i.e. where $f'(x) = 0$, so satisfies $k\Big(nx^{n-1} - (n+1)x^n\Big) = 0$. Since $n > 1$ this has solutions $x = 0$ and $x = \dfrac{n}{n+1}$. By considering what the graph of $f(x) = kx^n(1-x)$ looks like for $0 \leqslant x \leqslant 1$ we can see that the mode is given by $m = \dfrac{n}{n+1}$.

If the mode of $X$ is greater than the median then we have $\displaystyle\int_0^m f(x)\,\mathrm{d}x > \frac{1}{2}$.

$$\begin{aligned}
\int_0^m kx^n(1-x)\,\mathrm{d}x &= k\left[\frac{x^{n+1}}{n+1} - \frac{x^{n+2}}{n+2}\right]_0^m \\
&= k\left(\frac{m^{n+1}}{n+1} - \frac{m^{n+2}}{n+2}\right) \\
&= km^{n+1}\left(\frac{1}{n+1} - \frac{m}{n+2}\right) \\
&= k\left(\frac{n}{n+1}\right)^{n+1}\left(\frac{1}{n+1} - \frac{n}{(n+2)(n+1)}\right) \\
&= (n+1)(n+2)\left(\frac{n}{n+1}\right)^{n+1}\left(\frac{\cancel{n}+2-\cancel{n}}{(n+2)(n+1)}\right) \\
&= 2\left(\frac{n}{n+1}\right)^{n+1}
\end{aligned}$$

We are told that $\left(1+\dfrac{1}{x}\right)^{x+1}$ is a decreasing function of $x$. This means that for $x > 1$ we have:

$$\begin{aligned}
\left(1+\frac{1}{x}\right)^{x+1} &< \left(1+\frac{1}{1}\right)^{1+1} \\
\left(\frac{x+1}{x}\right)^{x+1} &< 4 \\
\implies \left(\frac{x}{x+1}\right)^{x+1} &> \frac{1}{4}
\end{aligned}$$

Using this fact we have:

$$\begin{aligned}
\int_0^m kx^n(1-x)\,\mathrm{d}x &= 2\left(\frac{n}{n+1}\right)^{n+1} \\
&> 2 \times \frac{1}{4} \\
&> \frac{1}{2}
\end{aligned}$$

and therefore we have shown that the mode of $X$ is greater than the median.

[2]Probability Generating Function