STEP Support Programme

2022 STEP 3 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the Cambridge Assessment Admissions Testing website. These are the general comments for the STEP 3 2022 exam from the Examiner's report:

One question was attempted by well over 90% of the candidates, two others by about 90%, and a fourth by over 80%. Two questions were attempted by about half the candidates and a further three questions by about a third of the candidates. Even the other three received attempts from a sixth of the candidates or more, meaning that even the least popular questions were markedly more popular than their counterparts in previous years.

Nearly 90% of candidates attempted no more than 7 questions.

Please send any corrections, comments or suggestions to step@maths.org.

Question 1

1 Let C_1 be the curve given by the parametric equations

$$x = ct\,, \quad y = \frac{c}{t}\,,$$

where $c > 0$ and $t \neq 0$, and let C_2 be the circle

$$(x-a)^2 + (y-b)^2 = r^2\,.$$

C_1 and C_2 intersect at the four points P_i $(i = 1, 2, 3, 4)$, and the corresponding values of the parameter t at these points are t_i.

(i) Show that t_i are the roots of the equation

$$c^2t^4 - 2act^3 + (a^2 + b^2 - r^2)t^2 - 2bct + c^2 = 0\,. \qquad (*)$$

(ii) Show that

$$\sum_{i=1}^{4} t_i^2 = \frac{2}{c^2}(a^2 - b^2 + r^2)$$

and find a similar expression for $\displaystyle\sum_{i=1}^{4} \frac{1}{t_i^2}$.

(iii) Hence show that $\displaystyle\sum_{i=1}^{4} OP_i^2 = 4r^2$, where OP_i denotes the distance of the point P_i from the origin.

(iv) Suppose that the curves C_1 and C_2 touch at two distinct points.

By considering the product of the roots of $(*)$, or otherwise, show that the centre of circle C_2 must lie on either the line $y = x$ or $y = -x$.

Examiner's report

This was the most popular question with 94% attempting it, and it was also the most successful with a mean mark of nearly 14/20. Apart from very occasional inaccuracies, part **(i)** was always successfully done.The first summation result in part **(ii)** was usually successfully done, though there was some poor summation notation which let some candidates down. The second summation was completed successfully by virtue of some heavy algebra or, more efficiently, by seeing the connection to the first result dividing the quartic by t^4 and comparing the quartic for the reciprocals with that in part **(i)**. Some candidates were penalised for not justifying their result, having clearly worked backwards from part **(iii)**.

Part **(iii)** was well done except when candidates disregarded their result from part **(ii)**. A lot of candidates managed to correctly interpret the implication of the curves touching at two distinct points in terms of the roots for t and the consequent result for the product of the four roots, but then struggled to reach the required result by algebra or poorly justified geometric arguments.

Solution

(i) Substituting the parametric equations into the equation for the circle gives:

$$\begin{aligned}(ct-a)^2 + \left(\frac{c}{t}-b\right)^2 &= r^2\\ \left(ct^2-at\right)^2 + (c-bt)^2 &= r^2t^2\\ c^2t^4 - 2act^3 + a^2t^2 + c^2 - 2bct + b^2t^2 &= r^2t^2\\ c^2t^4 - 2act^3 + \left(a^2+b^2-r^2\right)t^2 - 2bct + c^2 &= 0\end{aligned}$$

(ii) We have $(t_1+t_2+t_3+t_4)^2 = t_1^2+t_2^2+t_3^2+t_4^2+2\sum t_it_j$ and so:

$$\begin{aligned}\sum t_i^2 &= \left(\sum t_i\right)^2 - 2\left(t_it_j\right)\\ &= \left(\frac{2ac^2}{c}\right)^2 - 2\frac{a^2+b^2-r^2}{c^2}\\ &= \frac{4a^2-2(a^2+b^2-r^2)}{c^2}\\ &= \frac{2}{c^2}\left(a^2-b^2+r^2\right)\end{aligned}$$

Equation $(*)$ has roots t_i. If we divide throughout by t^4 (which is fine as $t \neq 0$) then we will get a quartic with roots $\frac{1}{t_i}$.

Dividing $(*)$ gives:

$$c^2 - \frac{2ac}{t} + \frac{a^2+b^2-r^2}{t^2} - \frac{2bc}{t^3} + \frac{c^2}{t^4} = 0$$

Letting $x = \frac{1}{t}$ this becomes:

$$c^2x^4 - 2bcx^3 + \left(a^2+b^2-r^2\right)x^2 - 2acx + c^2 = 0$$

This equation is the same as $(*)$, just with a and b swapped. This means that:

$$\sum \frac{1}{t_i}^2 = \frac{2}{c^2}(b^2-a^2+r^2)$$

You can also use the sum and product of roots formulae (sometimes known as Vieta's formulae) but the algebra gets slightly messy.

(iii) We have:

$$\begin{aligned}\sum OP_i^2 &= \sum (x_i^2 + y_i^2) \\ &= \sum \left(c^2 t_i^2 + \frac{c^2}{t_i^2} \right) \\ &= 2(a^2 - b^2 + r^2) + 2(b^2 - a^2 + r^2) \\ &= 4r^2\end{aligned}$$

(iv) If the curves touch at two distinct points then there must be two repeated roots of $(*)$, so let the roots be t_1, t_1, t_2, t_2.

When curves "touch" it means that where they meet they also share a tangent. This corresponds to a repeated root of the equation for where the curves meet.

From $(*)$ we have the product of the roots is equal to 1. This means we have:

$$(t_1 t_2)^2 = 1 \implies t_1^2 = \frac{1}{t_2^2}$$

From part **(ii)** we have:

$$\begin{aligned}\sum t_i^2 &= \frac{2}{c^2}(a^2 - b^2 + r^2) \\ \implies t_1^2 + t_2^2 &= \frac{1}{c^2}(a^2 - b^2 + r^2) \\ \sum \frac{1}{t_i^2} &= \frac{2}{c^2}(b^2 - a^2 + r^2) \\ \implies \frac{1}{t_1^2} + \frac{1}{t_2^2} &= \frac{1}{c^2}(b^2 - a^2 + r^2)\end{aligned}$$

Since we have $t_1^2 = \frac{1}{t_2^2}$, these two expressions are equal so we have:

$$\begin{aligned}\frac{1}{c^2}(a^2 - b^2 + r^2) &= \frac{1}{c^2}(b^2 - a^2 + r^2) \\ \implies a^2 &= b^2\end{aligned}$$

The centre of the circle C_2 is at (a, b), and if we have $a^2 = b^2$ then we have $a = \pm b$ and so the centre of C_2 lies on either $y = x$ or $= -x$.

Question 2

2 **(i)** Suppose that there are three non-zero integers a, b and c for which $a^3 + 2b^3 + 4c^3 = 0$. Explain why there must exist an integer p, with $|p| < |a|$, such that $4p^3 + b^3 + 2c^3 = 0$, and show further that there must exist integers p, q and r, with $|p| < |a|$, $|q| < |b|$ and $|r| < |c|$, such that $p^3 + 2q^3 + 4r^3 = 0$. Deduce that no such integers a, b and c can exist.

(ii) Prove that there are no non-zero integers a, b and c for which $9a^3+10b^3+6c^3 = 0$.

(iii) By considering the expression $(3n\pm1)^2$, prove that, unless an integer is a multiple of three, its square is one more than a multiple of three. Deduce that the sum of the squares of two integers can only be a multiple of three if each of the integers is a multiple of three.

Hence prove that there are no non-zero integers a, b and c for which $a^2+b^2 = 3c^2$.

(iv) Prove that there are no non-zero integers a, b and c for which $a^2+b^2+c^2 = 4abc$.

Examiner's report

The fifth most popular question, being attempted by just a little over half the candidates, it was the fourth most successful with a mean score of 9/20.

Whilst the algebra associated with this question was not difficult, the logic and communication required was certainly too much for many students.

In part **(i)** it required some justification that a had to be even. Contradiction or infinite descent could be used but either way the argument had to be made clear. Claiming "this can be continued forever" or moduli were always decreasing would eventually get to zero was not good enough. Successful candidates were able to explain why the integer nature of the solutions was vital to reach a contradiction.

In part **(ii)** many candidates were able to see that this was a similar problem to the first one, and most observed that divisibility by three was now the key idea.

In part **(iii)**, many candidates were able to consider the remainders when divided by 3, but again many struggled to communicate clearly an argument leading to the final contradiction.

By part **(iv)** most candidates were expecting to recreate the original equation again and the fact that this did not happen meant some came to a dead halt. Other were either oblivious to the issue or were bluffing their way through as a slightly more subtle argument was now required.

Solution

(i) Note that a, b and c are non-zero.

We can rearrange the equation to get $a^3 = -2(b^3 + 2c^3)$, and so we know that a^3 is even and hence a must be even. This means that we can write $a = 2p$ where $|p| < |a|$ and $p \neq 0$ (we are told that a is non zero). Substituting $a = 2p$ gives:

$$\begin{aligned}(2p)^3 + 2b^3 + 4c^3 &= 0\\ 8p^3 + 2b^3 + 4c^3 &= 0\\ 4p^3 + b^3 + 2c^3 &= 0\end{aligned}$$

Repeating the argument, b must be even so let $b = 2q$ where $|q| < |b|$ $(b \neq 0)$. Substituting and dividing by 2 gives:

$$2p^3 + 4q^3 + c^3 = 0$$

Repeating again, we know that c must be even and so let $c = 2r$ where $|r| < |c|$ $(c \neq 0)$. Substituting and dividing by 2 gives:

$$p^3 + 2q^3 + 4r^3 = 0$$

Hence if there exist non-zero integers a, b and c for which $a^3 + 2b^3 + 4c^3 = 0$, then there exists another three non-zero, "smaller" integers p, q and r which satisfy the same equation.

This argument can be repeated indefinitely, but the integers cannot keep getting smaller (as eventually the modulus will become less than 1). This is a contradiction, so there is no set of non-zero integers a, b and c which satisfy the original equation.

(ii) Again we have a, b and c are non-zero.

We can rearrange the equation to get $10b^3 = -3(2c^3 + 3a^3)$. 10 has no factors of 3, and so b^3 must have a factor of 3, and so b has a factor of 3.

Let $b = 3q$, where $|q| < |b|$ (and $q \neq 0$), and substitute this to give:

$$\begin{aligned}9a^3 + 270q^3 + 6c^3 &= 0\\ 3a^3 + 90q^3 + 2c^3 &= 0\end{aligned}$$

This means that $2c^3 = -3(a^3 + 30q^3)$, and so we can write $c = 3r$ (with $|r| < |c|$ and $r \neq 0$). Substitution gives:

$$\begin{aligned}3a^3 + 90q^3 + 54r^3 &= 0\\ a^3 + 30q^3 + 18r^3 &= 0\end{aligned}$$

Repeating again with $a = 3p$, where $|p| < |a|$ and $p \neq 0$ gives:

$$\begin{aligned}27p^3 + 30q^3 + 18r^3 &= 0\\ 9p^3 + 10q^3 + 6r^3 &= 0\end{aligned}$$

Which is the same as the original equation. Hence if non-zero integers a, b and c satisfy the equation then so do integers p, q and r where $|p| < |a|$, $|q| < |b|$ and $|r| < |c|$. This argument can be repeated indefinitely, but the integers cannot keep getting smaller indefinitely. Hence this is a contradiction and so there are no non-zero integers a, b and c which satisfy
$9a^3 + 10b^3 + 6c^3 = 0$.

(iii) If an integer is not a multiple of three, then it is either one more than, or one less than, a multiple of three, i.e. it has the form $3n+1$ or $3n-1$.

Squaring gives:

$$\begin{aligned}(3n \pm 1)^2 &= 9n^2 \pm 6n + 1 \\ &= 3(3n^2 \pm 2n) + 1\end{aligned}$$

This has the form $3k+1$, so if you square a number which is not a multiple of three, you get a number which is one more than a multiple of three.

If an integer is a multiple of three it can be written as $3n$ and squaring gives $9n^2$, which is a multiple of three (it's also a multiple of 9).

If we consider the equation $a^2+b^2=3c^2$, then the right hand side is a multiple of three.

If neither a nor b is a multiple of three, then a^2+b^2 has the form $(3p+1)+(3q+1)=3(p+q)+2$, which is **not** a multiple of three, so this case is not possible.

If one of a, b is a multiple of three and the other isn't then a^2+b^2 has the form $3p+(3q+1)=3(p+q)+1$ which is not a multiple of three.

The only case left is when both a and b are multiples of three. Let $a=3p$ and $b=3q$ (where a, b, p, q are non-zero):

$$\begin{aligned}a^2+b^2 &= 3c^2 \\ 9p^2+9q^2 &= 3c^2 \\ 3p^2+3q^2 &= c^2 \\ 3(p^2+q^2) &= c^2\end{aligned}$$

Hence we know that c must be a multiple of 3, so let $c=3r$ (where c, r are non-zero).

$$\begin{aligned}3(p^2+q^2) &= 9r^2 \\ p^2+q^2 &= 3r^2\end{aligned}$$

So if there exist non-zero integers a, b and c that satisfy $a^2+b^2=c^2$ then there also exist non-zero integers p, q and r which satisfy the equation, but where $|p|<|a|$, $|q|<|b|$ and $|r|<|c|$. Hence we have a contradiction as before and so there are no non-zero integers which satisfy the equation.

(iv) In this case the right hand side is a multiple of 4.

If a number is even, then when we square it we get $(2n)^2=4n^2$, i.e. it is a multiple of four.

If a number is odd, then when we square it we get $(2n+1)^2=4n^2+4n+1=4(n^2+n)+1$, i.e. one more than a multiple of four.

The left hand side consists of the sum of three square numbers. The cases are:

- a,b,c are all odd, then $a^2+b^2+c^2$ has the form $4k+3$
- two of a,b,c are odd, then $a^2+b^2+c^2$ has the form $4k+2$
- one of a,b,c is odd, then $a^2+b^2+c^2$ has the form $4k+1$
- a,b,c are all even, then $a^2+b^2+c^2$ has the form $4k$

Therefore only the last case is possible. Let $a = 2p$, $b = 2q$ and $c = 2r$. We then have:

$$a^2 + b^2 + c^2 = 4abc$$
$$4p^2 + 4q^2 + 4r^2 = 32pqr$$
$$p^2 + q^2 + r^2 = 8pqr$$

This argument can then be repeated again. Hence if there is a set of non-zero integers a, b, c such that $a^2 + b^2 + c^2 = 2^n abc$, then there exists a smaller set of integers that satisfy $a^2 + b^2 + c^2 = 2^{n+1} abc$. This cannot be repeated indefinitely by smaller and smaller integers, and so there cannot exist non-zero integer solutions of the original equation.

Question 3

3 **(i)** The curve C_1 has equation

$$ax^2 + bxy + cy^2 = 1$$

where $abc \neq 0$ and $a > 0$.

Show that, if the curve has two stationary points, then $b^2 < 4ac$.

(ii) The curve C_2 has equation

$$ay^3 + bx^2y + cx = 1$$

where $abc \neq 0$ and $b > 0$.

Show that the x-coordinates of stationary points on this curve satisfy

$$4cb^3x^4 - 8b^3x^3 - ac^3 = 0\,.$$

Show that, if the curve has two stationary points, then $4ac^6 + 27b^3 > 0$.

(iii) Consider the simultaneous equations

$$ay^3 + bx^2y + cx = 1$$
$$2bxy + c = 0$$
$$3ay^2 + bx^2 = 0$$

where $abc \neq 0$ and $b > 0$.

Show that, if these simultaneous equations have a solution, then $4ac^6 + 27b^3 = 0$.

Examiner's report

This was very popular, being attempted by over 90%, but not very successfully, with a mean score of about 5.5/20. In part **(i)**, candidates generally obtained a correct equation for x or y, but then failed to properly justify the manipulation of the inequality. Whilst the quartic was frequently correctly obtained in part **(ii)**, there were a number of different incorrect assumptions or assertions made regarding the two stationary points being repeated roots or the value of the quartic having different signs at the two stationary points. It was also common that the case when c is negative was not considered.

Whilst it was not uncommon for candidates to argue incorrectly for part **(iii)** that the three equations were equivalent to the curve C_2 in part **(ii)** having one stationary point, (often using $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$), in contrast, a pleasing number of candidates who made little progress in **(ii)** past obtaining the quartic, approached part **(iii)** by simply attempting to solve the equations by elimination, earning full or close to full marks.

Solution

(i) Note that the condition $abc \neq 0$ means that a, b and c are all non-zero.

Differentiation gives:

$$\begin{aligned} 2ax + by + bx\frac{\mathrm{d}y}{\mathrm{d}x} + 2cy\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ \implies \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2ax+by}{bx+2cy} \end{aligned}$$

Therefore the stationary points satisfy $2ax+by = 0$. Substituting $y = \frac{-2ax}{b}$ into the equation for C_1 gives:

$$\begin{aligned} ax^2 - 2ax^2 + c\left(\frac{-2ax}{b}\right)^2 &= 1 \\ ab^2x^2 - 2ab^2x^2 + 4a^2cx^2 &= b^2 \\ \left(4a^2c - ab^2\right)x^2 &= b^2 \end{aligned}$$

So if there are to be two stationary points then we need:

$$\begin{aligned} 4a^2c - ab^2 &> 0 \\ 4a^2c &> ab^2 \\ 4ac &> b^2 \end{aligned}$$

Where the last line is true since we have $a > 0$.

(ii) Differentiation gives:

$$\begin{aligned} 3ay^2\frac{\mathrm{d}y}{\mathrm{d}x} + bx^2\frac{\mathrm{d}y}{\mathrm{d}x} + 2bxy + c &= 0 \\ \implies \frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{2bxy+c}{3ay^2+bx^2} \end{aligned}$$

So the stationary points satisfy $2bxy + c = 0$. Substituting $y = -\dfrac{c}{2bx}$ gives:

$$\begin{aligned} a\left(-\frac{c}{2bx}\right)^3 + bx^2\left(-\frac{c}{2bx}\right) + cx &= 1 \\ -ac^3 - 4b^3x^4c + 8b^3cx^4 &= 8b^3x^3 \\ 4b^3cx^4 - 8b^3x^3 - ac^3 &= 0 \end{aligned}$$

This equation is a quartic, so could have 0, 1, 2, 3 or 4 roots (which would correspond to the same number of turning points of C_2).

To work out how many roots there are we can consider the curve $y = 4b^3cx^4 - 8b^3x^3 - ac^3$. The stationary points of this curve satisfy:

$$\begin{aligned} 16b^3cx^3 - 24b^3x^2 &= 0 \\ 8b^3x^2(2cx - 3) &= 0 \end{aligned}$$

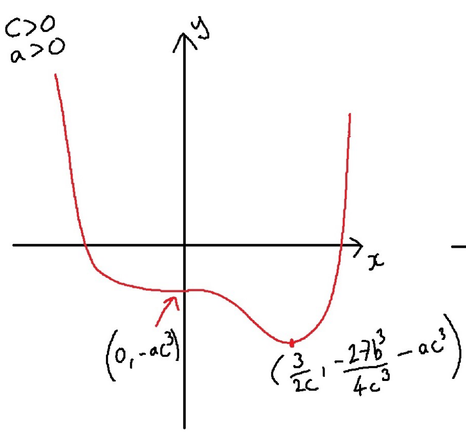
So there are two stationary points, when $x = 0$ and when $x = \dfrac{3}{2c}$.

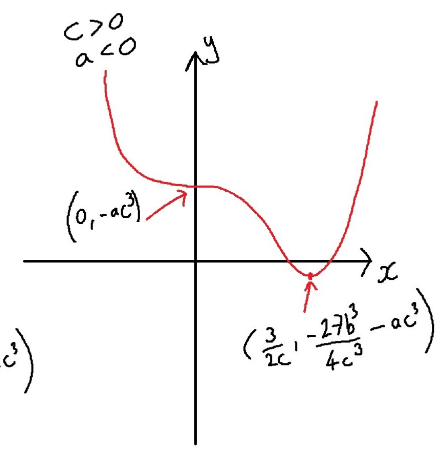
The point $\left(0, -ac^3\right)$ is a point of inflection, as the gradient slightly to each side of $x = 0$ will have the same sign. This is due to the "x^2" factor in the expression for the gradient. Note that since $abc \neq 0$ this point cannot be at the origin.

The point $\left(\dfrac{3}{2c}, -\dfrac{27b^3}{4c^3} - ac^3\right)$ is a turning point as the sign of the $(2cx - 3)$ factor will change either side of $x = \dfrac{3}{2c}$, and so the gradient either changes from positive to negative or from negative to positive as you travel along the curve.

Therefore we have a point of inflection when $x = 0$ and a turning point when $x = \dfrac{3}{2c}$. The behaviour of the quartic depends on the sign of c (we are told in this part that $b > 0$).

If $c > 0$ then there are two different cases, depending on the sign of a, that will give 2 roots of $4b^3cx^4 - 8b^3x^3 - ac^3 = 0$. In both cases the turning point has to be a minimum below the x axis.



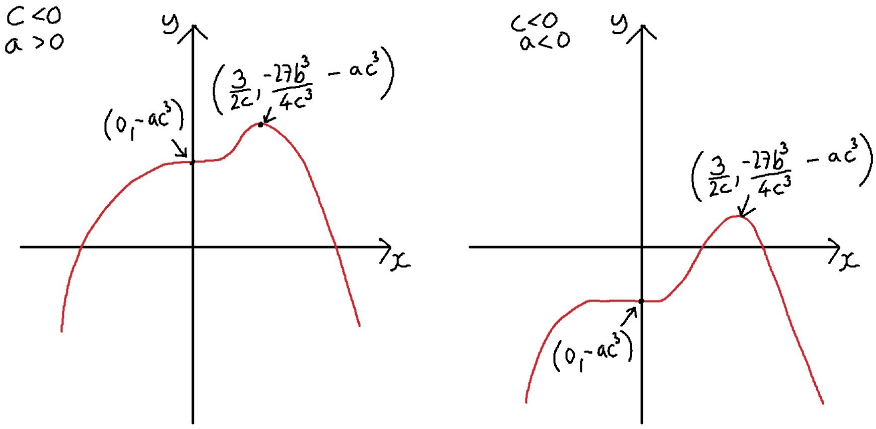


In these cases we need:

$$-\frac{27b^3}{4c^3} - ac^3 < 0$$
$$\frac{27b^3}{4c^3} + ac^3 > 0$$
$$27b^3 + 4ac^6 > 0$$

where the last line of true as here we have $c > 0$.

If we have $c < 0$ then there are another two cases. Here the turning point has to be a maximum above the x axis.



In these cases we need:

$$-\frac{27b^3}{4c^3} - ac^3 > 0$$
$$\frac{27b^3}{4c^3} + ac^3 < 0$$
$$27b^3 + 4ac^6 > 0$$

where the last line of true as here we have $c < 0$, and so multiplying by c^3 flips the sign.

(iii) Looking at the last equation, if we have $a > 0$ then the only solution of $3ay^2 + bx^2 = 0$ is $x = y = 0$ (we are given that $b > 0$). This doesn't satisfy the first equation.

Hence we must have $a < 0$.

The first two equations look familiar from part **(ii)**, but the quartic that results from solving those two doesn't look easily solvable. Instead consider the last two equations.

Rearranging the second equation to get $x = \dfrac{-c}{2by}$ we can substitute to get:

$$3ay^2 + bx^2 = 0$$
$$3ay^2 + b\left(\frac{-c}{2by}\right)^2 = 0$$
$$12aby^4 + c^2 = 0$$

Therefore we have $y = \pm\sqrt[4]{\dfrac{-c^2}{12ab}}$ (Note that as $a < 0$, $b > 0$ we are taking the fourth root of a positive value).

This gives $x = \dfrac{-c}{2by} = \mp\dfrac{c}{2b} \times \sqrt[4]{\dfrac{12ab}{-c^2}} = \mp\sqrt[4]{\dfrac{-3ac^2}{4b^3}}$.

The first equation gives:

$$\begin{aligned}
ay^3 + bx^2y + cx &= 1 \\
ay^4 + bx^2y^2 + cxy &= y \\
\frac{-ac^2}{12ab} + b\sqrt{\frac{-3ac^2}{4b^3} \times \frac{-c^2}{12ab}} - c\sqrt[4]{\frac{-3ac^2}{4b^3} \times \frac{-c^2}{12ab}} &= \pm\sqrt[4]{\frac{-c^2}{12ab}} \\
\frac{-c^2}{12b} + b\sqrt{\frac{c^4}{16b^4}} - c\sqrt[4]{\frac{c^4}{16b^4}} &= \pm\sqrt[4]{\frac{-c^2}{12ab}} \\
\frac{-c^2}{12b} + \frac{c^2}{4b} - \frac{c^2}{2b} &= \pm\sqrt[4]{\frac{-c^2}{12ab}} \\
\frac{-c^2}{3b} &= \pm\sqrt[4]{\frac{-c^2}{12ab}} \\
\frac{c^8}{81b^4} &= \frac{-c^2}{12ab} \\
12abc^8 &= -81c^2b^4 \\
4ac^6 &= -27b^3 \\
4ac^6 + 27b^3 &= 0
\end{aligned}$$

as required (phew!).

Question 4

4 *You may assume that all infinite sums and products in this question converge.*

(i) Prove by induction that for all positive integers n,

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\sinh\left(\frac{x}{2^n}\right)$$

and deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x}\,\frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right).$$

(ii) You are given that the Maclaurin series for $\sinh x$ is

$$\sinh x = \sum_{r=0}^{\infty}\frac{x^{2r+1}}{(2r+1)!}.$$

Use this result to show that, as y tends to 0, $\dfrac{y}{\sinh y}$ tends to 1.

Deduce that, for $x \neq 0$,

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\cdots.$$

(iii) Let $x = \ln 2$. Evaluate $\cosh\left(\dfrac{x}{2}\right)$ and show that

$$\cosh\left(\frac{x}{4}\right) = \frac{1+2^{\frac{1}{2}}}{2\times 2^{\frac{1}{4}}}.$$

Use part **(ii)** to show that

$$\frac{1}{\ln 2} = \frac{1+2^{\frac{1}{2}}}{2}\times\frac{1+2^{\frac{1}{4}}}{2}\times\frac{1+2^{\frac{1}{8}}}{2}\cdots.$$

(iv) Show that

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2}\times\frac{\sqrt{2+\sqrt{2}}}{2}\times\frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots.$$

Examiner's report

This was the fourth most popular question being attempted by more than four fifths of the candidates, with a moderate degree of success scoring a mean of 9/20.

Part **(i)** suffered from incorrect flows of logic in the inductive and base cases, as well as failure to mention anything about not dividing by zero.

In part **(ii)** many ignored the instruction to use the Maclaurin series, and used de L'Hopital's Rule to their cost, and some ignored the higher order terms.

Part **(iii)** was generally well done, though the most common error was not justifying the evaluation of the product using a geometric series in the exponent.

For part **(iv)**, the best attempted route was to use an imaginary substitution which led to mostly successful solutions. Some candidates attempted to prove an analogous trigonometric identity using similar arguments to the previous parts, however losing marks for not sufficiently fleshing out the details, and some attempted to use Osborn's Rule, often with insufficient justification or stating that it was being used. Once the identity was achieved, the calculation was generally done well if the candidate progressed this far.

Solution

(i) When $n = 1$ we have:

$$\sinh x = 2^1 \cosh\left(\frac{x}{2}\right) \sinh\left(\frac{x}{2}\right)$$

and hence the result is true when $n = 1$.

Assume the result is true when $n = k$. We then have:

$$\sinh x = 2^k \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^k}\right) \sinh\left(\frac{x}{2^k}\right)$$

Considering the RHS when $n = k+1$ we have:

$$\begin{aligned}
&2^{k+1} \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^k}\right) \cosh\left(\frac{x}{2^{k+1}}\right) \sinh\left(\frac{x}{2^{k+1}}\right)\\
=&2 \sinh x \frac{\cosh\left(\frac{x}{2^{k+1}}\right) \sinh\left(\frac{x}{2^{k+1}}\right)}{\sinh\left(\frac{x}{2^k}\right)}\\
=&\sinh x \frac{2\cosh\left(\frac{x}{2^{k+1}}\right) \sinh\left(\frac{x}{2^{k+1}}\right)}{\sinh\left(\frac{x}{2^k}\right)}\\
=&\sinh x \frac{\sinh\left(\frac{x}{2^k}\right)}{\sinh\left(\frac{x}{2^k}\right)}\\
=&\sinh x
\end{aligned}$$

(Note that the division by $\sinh\left(\frac{x}{2^k}\right)$ is fine as we have $x \neq 0$).

Therefore we have:

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right) \cosh\left(\frac{x}{4}\right) \cdots \cosh\left(\frac{x}{2^n}\right) \sinh\left(\frac{x}{2^n}\right)$$

for all positive integers n by induction.

We can rearrange this to get:

$$\sinh x = 2^n \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\sinh\left(\frac{x}{2^n}\right)$$
$$\frac{\sinh x}{\sinh\left(\frac{x}{2^n}\right)} \times \frac{1}{2^n} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)$$
$$\frac{\sinh x}{x} \times \frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)$$

(ii) Using the Maclaurin series we have:

$$\begin{aligned}\frac{y}{\sinh y} &= \frac{y}{y + \frac{y^3}{3!} + \frac{y^5}{5!} + \cdots}\\ &= \frac{1}{1 + \frac{y^2}{3!} + \frac{y^4}{5!} + \cdots}\end{aligned}$$

and so we have $\dfrac{y}{\sinh y} \to 1$ as $y \to 0$.

From part **(i)** we have:

$$\frac{\sinh x}{x} \times \frac{\frac{x}{2^n}}{\sinh\left(\frac{x}{2^n}\right)} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)$$

As $n \to \infty$ we have $\dfrac{x}{2^n} \to 0$, so using the earlier result in this part we have $\dfrac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)} \to 1$.

This gives:

$$\frac{\sinh x}{x} = \cosh\left(\frac{x}{2}\right)\cosh\left(\frac{x}{4}\right)\cdots\cosh\left(\frac{x}{2^n}\right)\cdots$$

(iii) Using $x = \ln 2$ and the formula for $\cosh x$ we have:

$$\begin{aligned}\cosh\left(\frac{x}{2}\right) &= \frac{e^{0.5\ln 2} + e^{-0.5\ln 2}}{2}\\ &= \frac{\sqrt{2} + \frac{1}{\sqrt{2}}}{2}\\ &= \frac{3}{2\sqrt{2}}\end{aligned}$$

Similarly we have:

$$\begin{aligned}\cosh\left(\frac{x}{4}\right) &= \frac{e^{0.25\ln 2} + e^{-0.25\ln 2}}{2}\\ &= \frac{\sqrt{\sqrt{2}} + \frac{1}{\sqrt{\sqrt{2}}}}{2}\\ &= \frac{\sqrt{2} + 1}{2\sqrt{\sqrt{2}}}\end{aligned}$$

We also have

$$\begin{aligned}\sinh x &= \frac{\mathrm{e}^{\ln 2} - \mathrm{e}^{-\ln 2}}{2}\\ &= \frac{2-\frac{1}{2}}{2}\\ &= \frac{3}{4}\end{aligned}$$

Substituting $x = \ln 2$ into the last result from part **(ii)** gives:

$$\begin{aligned}\frac{\sinh \ln 2}{\ln 2} &= \cosh\left(\frac{\ln 2}{2}\right)\cosh\left(\frac{\ln 2}{4}\right)\cdots\\ \frac{\frac{3}{4}}{\ln 2} &= \frac{3}{2\sqrt{2}} \times \frac{\sqrt{2}+1}{2\sqrt{\sqrt{2}}} \times \frac{\sqrt{\sqrt{2}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots\\ \frac{1}{\ln 2} &= \frac{4}{2\sqrt{2}} \times \frac{2^{\frac{1}{2}}+1}{2\sqrt{\sqrt{2}}} \times \frac{2^{\frac{1}{4}}+1}{2\sqrt{\sqrt{\sqrt{2}}}}\cdots\\ \frac{1}{\ln 2} &= \frac{4}{2\times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{8}}\cdots} \times \frac{1+2^{\frac{1}{2}}}{2} \times \frac{1+2^{\frac{1}{4}}}{2}\cdots\end{aligned}$$

We have:

$$2 \times 2^{\frac{1}{2}} \times 2^{\frac{1}{4}} \times 2^{\frac{1}{8}}\cdots = 2^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots} = 4$$

and so we have:

$$\frac{1}{\ln 2} = \frac{1+2^{\frac{1}{2}}}{2} \times \frac{1+2^{\frac{1}{4}}}{2} \times \frac{1+2^{\frac{1}{8}}}{2}\cdots$$

(iv) Comparing this part to the previous part it looks like a substitution of $x = \dfrac{\pi}{2}$ might be useful. However, this doesn't give a particularly nice value of $\cosh x$ or $\sinh x$. However it would give nice values for $\sin x$ and $\cos x$, and we know that $\sinh \mathrm{i}x = \mathrm{i}\sin x$ and $\cosh \mathrm{i}x = \cos x$.

Let $x = \dfrac{\mathrm{i}\pi}{2}$ and substitute into the part **(ii)** result:

$$\begin{aligned}\frac{\sinh\left(\frac{\mathrm{i}\pi}{2}\right)}{\frac{\mathrm{i}\pi}{2}} &= \cosh\left(\frac{\mathrm{i}\pi}{4}\right)\cosh\left(\frac{\mathrm{i}\pi}{8}\right)\cosh\left(\frac{\mathrm{i}\pi}{16}\right)\cdots\\ \frac{i\sin\left(\frac{\pi}{2}\right)}{\frac{\mathrm{i}\pi}{2}} &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cdots\\ \frac{2}{\pi} &= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)\cos\left(\frac{\pi}{16}\right)\cdots\end{aligned}$$

We know that $\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, and also that

$$\begin{aligned}\cos\left(\frac{\pi}{4}\right) &= 2\cos^2\left(\frac{\pi}{8}\right) - 1\\ \implies \cos\left(\frac{\pi}{8}\right) &= \sqrt{\frac{1+\frac{\sqrt{2}}{2}}{2}}\\ &= \frac{\sqrt{2+\sqrt{2}}}{2}\end{aligned}$$

Similarly we have:

$$\begin{aligned}\cos\left(\frac{\pi}{16}\right) &= \sqrt{\frac{1+\frac{\sqrt{2+\sqrt{2}}}{2}}{2}}\\ &= \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\end{aligned}$$

Hence we have:

$$\frac{2}{\pi} = \frac{\sqrt{2}}{2} \times \frac{\sqrt{2+\sqrt{2}}}{2} \times \frac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}$$

as required.

Question 5

5 **(i)** Show that

$$\int_{-a}^{a} \frac{1}{1+\mathrm{e}^x}\,\mathrm{d}x = a \ \text{ for all } a \geqslant 0.$$

(ii) Explain why, if g is a continuous function and

$$\int_0^a \mathrm{g}(x)\,\mathrm{d}x = 0 \ \text{ for all } a \geqslant 0,$$

then $\mathrm{g}(x) = 0$ for all $x \geqslant 0$.

Let f be a continuous function with $\mathrm{f}(x) \geqslant 0$ for all x. Show that

$$\int_{-a}^{a} \frac{1}{1+\mathrm{f}(x)}\,\mathrm{d}x = a \ \text{ for all } a \geqslant 0$$

if and only if

$$\frac{1}{1+\mathrm{f}(x)} + \frac{1}{1+\mathrm{f}(-x)} - 1 = 0 \ \text{ for all } x \geqslant 0,$$

and hence if and only if $\mathrm{f}(x)\mathrm{f}(-x) = 1$ for all x.

(iii) Let f be a continuous function such that, for all x, $\mathrm{f}(x) \geqslant 0$ and $\mathrm{f}(x)\mathrm{f}(-x) = 1$. Show that, if h is a continuous function with $\mathrm{h}(x) = \mathrm{h}(-x)$ for all x, then

$$\int_{-a}^{a} \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x = \int_0^a \mathrm{h}(x)\ \mathrm{d}x\,.$$

(iv) Hence find the exact value of

$$\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\mathrm{e}^{-x}\cos x}{\cosh x}\,\mathrm{d}x\,.$$

Examiner's report

This was only very slightly less popular than question **3**, but it was the third most successful with a mean of just under half marks.

Part **(i)** was well done, with a variety of methods used, the most common being by a substitution of e^x . In this part, the most frequent errors were showing insufficient working to fully justify the given result, not spotting how to simplify $\dfrac{1+\mathrm{e}^a}{1+\mathrm{e}^{-a}}$, and incorrectly integrating $\int \dfrac{1}{1+\mathrm{e}^x}\,\mathrm{d}x$ to get $\ln(1+\mathrm{e}^x)$.

Part **(ii)** was generally found to be the hardest. There was a range of responses to the first requirement from concise use of the Fundamental Theorem of Calculus to long, often imprecise, paragraphs of text. Candidates attempting proof by contradiction tended to be more successful if they used a sketch to back up their argument. The second result saw many different methods used. The most common mistakes were not showing enough working when using a $u = -x$ substitution, not showing that the argument can be reversed, and using an incorrect argument such as $\int_{-a}^{a} \mathrm{g}(x)\,\mathrm{d}x = 0 \to \mathrm{g}(x) = 0$ (to which $\mathrm{g}(x) = \sin x$ is a counterexample). Many candidates did not see the link with the first requirement of the part. However, the final result of this part was usually done well.

Candidates found part **(iii)** easier than part **(ii)**, the most common mistake was not realising that $\mathrm{h}(x) = \mathrm{h}(-x)$ holds, even though this was stated in the question.

Part **(iv)** was generally done well, with the most common mistakes being neglecting to show that the functions satisfied the conditions in part 3 or omitting a factor of 2. A few candidates did not use the results from the previous parts, instead using other methods, which, as the question stated "hence", gained no credit for this part.

Solution

(i) There are many different ways of doing this first part! Here is one possible way:

$$\begin{aligned}
\text{Let} \quad I &= \int_{-a}^{a} \frac{1}{1+\mathrm{e}^x}\,\mathrm{d}x \\
\text{Substitute } t = -x \implies I &= \int_{a}^{-a} \frac{1}{1+\mathrm{e}^{-t}} \times -1\,\mathrm{d}t \\
&= \int_{-a}^{a} \frac{1}{1+\mathrm{e}^{-t}}\,\mathrm{d}t \\
&= \int_{-a}^{a} \frac{\mathrm{e}^t}{\mathrm{e}^t+1}\,\mathrm{d}t \\
&= \int_{-a}^{a} \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\,\mathrm{d}x
\end{aligned}$$

Adding together the two different expressions for I gives:

$$\begin{aligned}
2I &= \int_{-a}^{a} \frac{1}{1+\mathrm{e}^x}\,\mathrm{d}x + \int_{-a}^{a} \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\,\mathrm{d}x \\
&= \int_{-a}^{a} \frac{1}{1+\mathrm{e}^x} + \frac{\mathrm{e}^x}{\mathrm{e}^x+1}\,\mathrm{d}x \\
&= \int_{-a}^{a} \frac{1+\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x \\
&= \int_{-a}^{a} 1\,\mathrm{d}x \\
&= 2a \\
\therefore I &= a
\end{aligned}$$

Some of the possible alternative methods are shown in the mark scheme for this paper, which can be downloaded from here.

This question is a "show that", which means that you must show enough working to fully justify the answer.

For example, if you had ended up with the expression $\ln(e^a + 1) - \ln(e^{-a} + 1)$, you could not jump straight from there to the answer a, as it is impossible to differentiate between those who can do the necessary algebraic manipulations in their heads, and those who could not see how to do it and just wrote down the answer they were trying to get to.

(ii) For the first part let:

$$\begin{aligned} G(t) &= \int_0^t g(x)\,dx \\ \implies G'(t) &= g(t) \end{aligned}$$

This is sometimes known as the "Fundamental Theorem of Calculus".

Therefore we have $G(t) = 0$ for all $t \geqslant 0$, and so $g(t) = G'(t) = 0$ for all $t \geqslant 0$.

Another way to tackle this part is using proof by contradiction. Assume there is a small region where we do not have $g(x) = 0$. WLOG assume that $g(x) > 0$ for $b < x < b + \delta$ (it doesn't actually matter if $g(x)$ is negative or positive in this range, the argument is the same). This means that we have:

$$\int_b^{b+\delta} g(x)\,dx > 0$$

We also have:

$$\int_0^b g(x)\,dx + \int_b^{b+\delta} g(x)\,dx = \int_0^{b+\delta} g(x)\,dx$$

Since $\displaystyle\int_b^{b+\delta} g(x)\,dx > 0$, then we cannot have both $\displaystyle\int_0^b g(x)\,dx = 0$ and $\displaystyle\int_0^{b+\delta} g(x)\,dx = 0$. Hence if $\displaystyle\int_0^a g(x)\,dx = 0$ for all a then we must have $g(x) = 0$.

For the next part, it is probably a good idea if we can rearrange the integral so that we can use the first part of **(ii)**.

We have:

$$\begin{aligned} \int_{-a}^a \frac{1}{1+f(x)}\,dx &= \int_{-a}^0 \frac{1}{1+f(x)}\,dx + \int_0^a \frac{1}{1+f(x)}\,dx \\ &= \int_a^0 \frac{1}{1+f(-t)} \times -1\,dt + \int_0^a \frac{1}{1+f(x)}\,dx \\ &= \int_0^a \frac{1}{1+f(-t)}\,dt + \int_0^a \frac{1}{1+f(x)}\,dx \\ &= \int_0^a \left[\frac{1}{1+f(x)} + \frac{1}{1+f(-x)}\right] dx \end{aligned}$$

Then if we have $\displaystyle\int_{-a}^{a} \frac{1}{1+\mathrm{f}(x)}\,\mathrm{d}x = a$ we can write this as:

$$\begin{aligned}\int_0^a \left[\frac{1}{1+\mathrm{f}(x)} + \frac{1}{1+\mathrm{f}(-x)}\right] \mathrm{d}x &= a \\ \int_0^a \left[\frac{1}{1+\mathrm{f}(x)} + \frac{1}{1+\mathrm{f}(-x)} - 1\right] \mathrm{d}x &= 0\end{aligned}$$

and so by the first part we have $\dfrac{1}{1+\mathrm{f}(x)} + \dfrac{1}{1+\mathrm{f}(-x)} - 1 = 0$ for all $x \geqslant 0$.

All of the steps above as reversible, and so we have:

$$\begin{aligned}&\int_{-a}^{a} \frac{1}{1+\mathrm{f}(x)}\,\mathrm{d}x = a \qquad &&\text{for all } a \geqslant 0 \\ \iff &\frac{1}{1+\mathrm{f}(x)} + \frac{1}{1+\mathrm{f}(-x)} - 1 = 0 &&\text{for all } x \geqslant 0\end{aligned}$$

We also have:

$$\begin{aligned}&\frac{1}{1+\mathrm{f}(x)} + \frac{1}{1+\mathrm{f}(-x)} - 1 = 0 \\ \iff &\frac{[1+\mathrm{f}(-x)] + [1+\mathrm{f}(x)]}{[1+\mathrm{f}(x)][1+\mathrm{f}(-x)]} = 1 \\ \iff &2 + \mathrm{f}(x) + \mathrm{f}(-x) = 1 + \mathrm{f}(x) + \mathrm{f}(-x) + \mathrm{f}(x)\mathrm{f}(-x) \\ \iff &\mathrm{f}(x)\mathrm{f}(-x) = 1\end{aligned}$$

(iii) Splitting the integral gives:

$$\begin{aligned}\int_{-a}^{a} \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x &= \int_{-a}^{0} \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x + \int_0^a \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x \\ &= \int_a^0 \frac{\mathrm{h}(-x)}{1+\mathrm{f}(-x)} \times -1\,\mathrm{d}x + \int_0^a \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x \\ &= \int_0^a \frac{\mathrm{h}(-x)}{1+\mathrm{f}(-x)}\,\mathrm{d}x + \int_0^a \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x \\ &= \int_0^a \left[\frac{\mathrm{h}(x)}{1+\mathrm{f}(-x)} + \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\right] \mathrm{d}x \qquad \text{as } \mathrm{h}(-x) = \mathrm{h}(x) \\ &= \int_0^a \mathrm{h}(x)\,\mathrm{d}x \qquad \text{using last result from part } \textbf{(ii)}\end{aligned}$$

(iv) This part is a "Hence" which means that you need the use the previous part (or parts!). You might be able to find a different way of evaluating the integral, but the point of this part of the question is whether you can link the previous work to this part.

Comparing this to part **(iii)** we could take $\mathrm{h}(x) = \cos x$, which satisfies $\mathrm{h}(x) = \mathrm{h}(-x)$ (and we also know that $\cos x$ is a continuous function).

Finding $\mathrm{f}(x)$ is a little trickier. We have:

$$\begin{aligned}\frac{\mathrm{e}^{-x}}{\cosh x} &= \frac{\mathrm{e}^{-x}}{\frac{1}{2}(\mathrm{e}^{x} + \mathrm{e}^{-x})} \\ &= \frac{2}{\mathrm{e}^{2x} + 1}\end{aligned}$$

Therefore we have:

$$\int_{-a}^{a} \frac{\mathrm{h}(x)}{1+\mathrm{f}(x)}\,\mathrm{d}x = \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{2\cos x}{1+\mathrm{e}^{2x}}\,\mathrm{d}x$$

We take:

$$\begin{aligned} a &= \tfrac{1}{2}\pi \\ \mathrm{h}(x) &= 2\cos x \\ \mathrm{f}(x) &= \mathrm{e}^{2x} \end{aligned}$$

Then $\mathrm{h}(x)$ satisfies $\mathrm{h}(x) = \mathrm{h}(-x)$ and $\mathrm{h}(x)$ is continuous. We also know that $\mathrm{f}(x) = \mathrm{e}^{2x}$ is continuous, satisfies $\mathrm{f}(x) \geqslant 0$ and we also have $\mathrm{f}(x)\mathrm{f}(-x) = \mathrm{e}^{2x}\mathrm{e}^{-2x} = 1$. Hence $\mathrm{f}(x)$ and $\mathrm{h}(x)$ satisfy the conditions in part **(iii)** and so we have:

$$\begin{aligned} \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \frac{\mathrm{e}^{-x}\cos x}{\cosh x}\,\mathrm{d}x &= \int_{0}^{\frac{1}{2}\pi} 2\cos x\,\mathrm{d}x \\ &= \Big[2\sin x\Big]_{0}^{\frac{1}{2}\pi} \\ &= 2 \end{aligned}$$

You could have used $\mathrm{h}(x) = \cos x$, but you would have had to remember to multiply by 2 at the end.

Question 6

6 **(i)** Show that when α is small, $\cos(\theta+\alpha)-\cos\theta \approx -\alpha\sin\theta - \frac{1}{2}\alpha^2\cos\theta$.

Find the limit as $\alpha \to 0$ of

$$\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta} \qquad (*)$$

in the case $\sin\theta \neq 0$.

In the case $\sin\theta = 0$, what happens to the value of expression $(*)$ when $\alpha \to 0$?

(ii) A circle C_1 of radius a rolls without slipping in an anti-clockwise direction on a fixed circle C_2 with centre at the origin O and radius $(n-1)a$, where n is an integer greater than 2. The point P is fixed on C_1. Initially the centre of C_1 is at $(na, 0)$ and P is at $\big((n+1)a, 0\big)$.

(a) Let Q be the point of contact of C_1 and C_2 at any time in the rolling motion.
Show that when OQ makes an angle θ, measured anticlockwise, with the positive x-axis, the x-coordinate of P is $x(\theta) = a\,(n\cos\theta + \cos n\theta)$, and find the corresponding expression for the y-coordinate, $y(\theta)$, of P.

(b) Find the values of θ for which the distance OP is $(n-1)a$.

(c) Let $\theta_0 = \dfrac{1}{n-1}\pi$. Find the limit as $\alpha \to 0$ of

$$\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}\,.$$

Hence show that, at the point $\big(x(\theta_0)\,,\ y(\theta_0)\big)$, the tangent to the curve traced out by P is parallel to OP.

Examiner's report

About half the candidates attempted this, but it was one of the least successful with a mean score of one quarter marks.

Many candidates managed the opening 'show that' in part **(i)** but the limit attempt had varying levels of success, and a common error was division by a quantity that was not necessarily nonzero.

In part **(ii)**, diagrams were regularly lacking, often being drawn extremely small with the most salient details omitted.

In part **(a)**, very few indicated from where the second term in the expression for x arose. Most attempts appealed to a diagram but did not indicate the pertinent angles.

Many formed the correct equation in **(b)**, but a large number forgot to account for the periodicity; those that remembered to do so largely did so correctly.

Many who got to **(c)**, erroneously evaluated a 0/0 limit and then argued that the cotangent was the answer they wanted. However, pleasingly others did spot the zeros and manipulated the trigonometry effectively.

Solution

(i) We have:

$$\begin{aligned}\cos(\theta+\alpha)-\cos\theta &= \cos\theta\cos\alpha-\sin\theta\sin\alpha-\cos\theta\\ &\approx \cos\theta\left(1-\tfrac{1}{2}\alpha^2\right)-\alpha\sin\theta-\cos\theta\\ &= -\alpha\sin\theta-\tfrac{1}{2}\alpha^2\cos\theta\end{aligned}$$

Similarly:

$$\begin{aligned}\sin(\theta+\alpha)-\sin\theta &= \sin\theta\cos\alpha+\sin\alpha\cos\theta-\sin\theta\\ &\approx \sin\theta\left(1-\tfrac{1}{2}\alpha^2\right)+\alpha\cos\theta-\sin\theta\\ &= \alpha\cos\theta-\tfrac{1}{2}\alpha^2\sin\theta\end{aligned}$$

Using these two approximations we have:

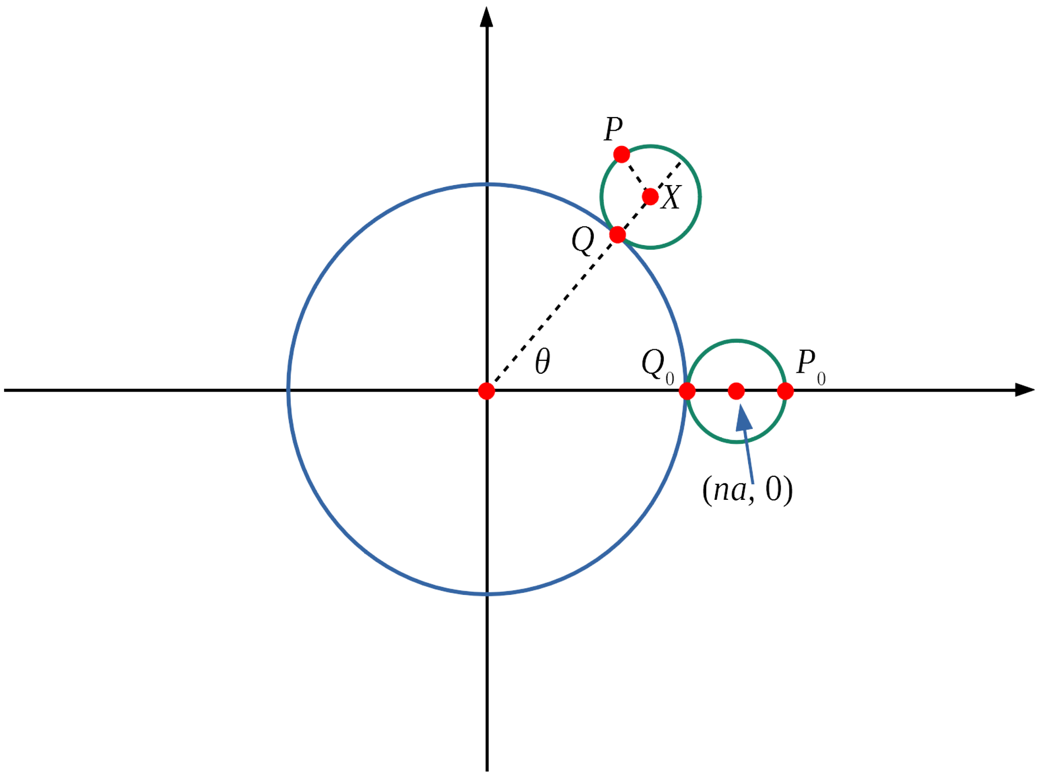
$$\begin{aligned}\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta} &= \lim_{\alpha\to 0}\frac{\alpha\cos\theta-\frac{1}{2}\alpha^2\sin\theta}{-\alpha\sin\theta-\frac{1}{2}\alpha^2\cos\theta}\\ &= \lim_{\alpha\to 0}\frac{\cos\theta-\frac{1}{2}\alpha\sin\theta}{-\sin\theta-\frac{1}{2}\alpha\cos\theta}\\ &= -\cot\theta\end{aligned}$$

If instead we have $\sin\theta=0$ then the approximation becomes:

$$\begin{aligned}\lim_{\alpha\to 0}\frac{\sin(\theta+\alpha)-\sin\theta}{\cos(\theta+\alpha)-\cos\theta} &= \lim_{\alpha\to 0}\frac{\cos\theta}{-\frac{1}{2}\alpha\cos\theta}\\ &= \lim_{\alpha\to 0}\frac{-2}{\alpha}\end{aligned}$$

which tends to $-\infty$ as $\alpha\to 0_+$ and tends to ∞ as $\alpha\to 0_-$ (where $\alpha\to 0_+$ means α tends to 0 through positive values).

(ii) A really good first step here is to draw a picture with the information on it!



In this diagram Q_0 and P_0 are the initial positions of Q and P and X is the centre of circle C_1.

(a) When OQ makes an angle of θ with the positive x axis then the centre of C_1 will now be at $(an\cos\theta, an\sin\theta)$.

We can also consider the position of P relative to the centre of C_1. When OQ makes an angle θ, the point of contact between the two circles has travelled an arc length of $a(n-1)\theta$ around C_2 (remember that the radius of C_2 is $a(n-1)$. This will also be the same as the arc length travelled by C_1 (as there is no slipping), and so the angle C_1 has turned through satisfies:

$$a\phi = a(n-1)\theta$$

and so the point P has turned through an angle of $(n-1)\theta$ with respect to the line QX. We also have that the line QX has turned through an angle θ, which means that the angle PX has with the x axis is $n\theta$. Therefore the position of P relative to X is given by $(a\cos n\theta, a\sin n\theta)$.

Putting these together gives the x-coordinate of P as $x(\theta) = a(n\cos\theta + \cos n\theta)$, and the y coordinate as $y(\theta) = a(n\sin\theta + \sin n\theta)$.

(b) Using Pythagoras' theorem, the distance OP is $(n-1)a$ when:

$$\begin{aligned} a^2(n\cos\theta+\cos n\theta)^2+a^2(n\sin\theta+\sin n\theta)^2 &= a^2(n-1)^2 \\ n^2(\cos^2\theta+\sin^2\theta)+2n(\cos\theta\cos n\theta+\sin\theta\sin n\theta)+(\sin^2 n\theta+\cos^2 n\theta) &= n^2-2n+1 \\ 2n(\cos\theta\cos n\theta+\sin\theta\sin n\theta) &= -2n \\ \cos\left[(n-1)\theta\right] &= -1 \end{aligned}$$

This has solutions $(n-1)\theta = \pi, 3\pi, 5\pi, \cdots$, so we have $\theta = \dfrac{(2k+1)\pi}{n-1}$ where $k = 0, 1, 2, \cdots$.

(c)

$$\begin{aligned} &\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)} \\ &= \lim_{\alpha\to 0}\frac{a\left[n\sin(\theta_0+\alpha)+\sin n(\theta_0+\alpha)\right]-a\left[n\sin(\theta_0)+\sin n(\theta_0)\right]}{a\left[n\cos(\theta_0+\alpha)+\cos n(\theta_0+\alpha)\right]-a\left[n\cos(\theta_0)+\cos n(\theta_0)\right]} \\ &= \lim_{\alpha\to 0}\frac{\left[n\sin(\theta_0+\alpha)-n\sin(\theta_0)\right]+\left[\sin n(\theta_0+\alpha)-\sin n(\theta_0)\right]}{\left[n\cos(\theta_0+\alpha)-n\cos(\theta_0)\right]+\left[\cos n(\theta_0+\alpha)-\cos n(\theta_0)\right]} \\ &= \lim_{\alpha\to 0}\frac{n\left(\alpha\cos\theta_0-\frac{1}{2}\alpha^2\sin\theta_0\right)+\left(n\alpha\cos n\theta_0-\frac{1}{2}(n\alpha)^2\sin n\theta_0\right)}{n\left(-\alpha\sin\theta_0-\frac{1}{2}\alpha^2\cos\theta_0\right)+\left(-n\alpha\sin n\theta_0-\frac{1}{2}(n\alpha)^2\cos n\theta_0\right)} \\ &= \lim_{\alpha\to 0}\frac{\cos\theta_0+\cos n\theta_0-\frac{\alpha}{2}\left(\sin\theta_0+n\sin n\theta_0\right)}{-(\sin\theta_0+\sin n\theta_0)-\frac{\alpha}{2}\left(\cos\theta_0+n\cos n\theta_0\right)} \end{aligned}$$

There is a temptation at this stage to ignore the α terms, which would be fine as long as the other terms were non-zero. However the request in part **(i)** to consider what happens when $\sin\theta = 0$, and the given value of θ_0 suggests that we might need to be a little cautious.

We know that $(n-1)\theta_0 = \pi$. This means that $\cos(n-1)\theta_0 = -1$ and $\sin(n-1)\theta_0 = 0$. Considering $\cos n\theta_0$ we have:

$$\begin{aligned} \cos n\theta_0 &= \cos\left[(n-1)\theta_0+\theta_0\right] \\ &= \cos(n-1)\theta_0\cos\theta_0-\sin(n-1)\theta_0\sin\theta_0 \\ &= -\cos\theta_0 \end{aligned}$$

Hence we have $\cos n\theta_0 + \cos\theta_0 = 0$.

Similarly:

$$\begin{aligned} \sin n\theta_0 &= \sin\left[(n-1)\theta_0+\theta_0\right] \\ &= \sin(n-1)\theta_0\cos\theta_0+\cos(n-1)\theta_0\sin\theta_0 \\ &= -\sin\theta_0 \end{aligned}$$

So we have $\sin n\theta_0 + \sin\theta_0 = 0$.

Going back to our limit we have:

$$\begin{aligned}
&\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}\\
=&\lim_{\alpha\to 0}\frac{-\frac{\alpha}{2}(\sin\theta_0+n\sin n\theta_0)}{-\frac{\alpha}{2}(\cos\theta_0+n\cos n\theta_0)}\\
=&\frac{\sin\theta_0+n\sin n\theta_0}{\cos\theta_0+n\cos n\theta_0}\\
=&\frac{\sin\theta_0-n\sin\theta_0}{\cos\theta_0-n\cos\theta_0}\\
=&\frac{(1-n)\sin\theta_0}{(1-n)\cos\theta_0}\\
=&\tan\theta_0
\end{aligned}$$

Where the fourth line down used $\sin n\theta_0 = -\sin\theta_0$ and $\cos n\theta_0 = -\cos\theta_0$.
Since $\theta_0 = \dfrac{1}{n-1}\pi$, this is one of the values for which the distance OP is $(n-1)a$, i.e. P and Q are at the same point here. Hence the gradient of the line OP is equal to $\tan\theta_0$.
We also have:

$$\lim_{\alpha\to 0}\frac{y(\theta_0+\alpha)-y(\theta_0)}{x(\theta_0+\alpha)-x(\theta_0)}=\tan\theta_0$$

Therefore the gradient of the tangent to the curve at $\big(x(\theta_0), y(\theta_0)\big)$ is equal to the gradient of OP, as required.

You might like to investigate this Desmos graph, which shows the path of P and also where the distance OP is equal to $(n-1)a$.

Question 7

7 Let $\mathbf{n}$ be a vector of unit length in three dimensions. For each vector $\mathbf{r}$, $\mathrm{f}(\mathbf{r})$ is defined by $\mathrm{f}(\mathbf{r}) = \mathbf{n} \times \mathbf{r}$.

(i) Given that

$$\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ and } \mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

show that the x-component of $\mathrm{f}(\mathrm{f}(\mathbf{r}))$ is $-x(b^2+c^2)+aby+acz$. Show further that

$$\mathrm{f}(\mathrm{f}(\mathbf{r})) = (\mathbf{n}.\mathbf{r})\mathbf{n} - \mathbf{r}.$$

Explain, by means of a diagram, how $\mathrm{f}(\mathrm{f}(\mathbf{r}))$ is related to $\mathbf{n}$ and $\mathbf{r}$.

(ii) Let R be the point with position vector $\mathbf{r}$ and P be the point with position vector $\mathrm{g}(\mathbf{r})$, where g is defined by

$$\mathrm{g}(\mathbf{s}) = \mathbf{s} + \sin\theta\, \mathrm{f}(\mathbf{s}) + (1-\cos\theta)\, \mathrm{f}(\mathrm{f}(\mathbf{s}))\,.$$

By considering $\mathrm{g}(\mathbf{n})$ and $\mathrm{g}(\mathbf{r})$ when $\mathbf{r}$ is perpendicular to $\mathbf{n}$, state, with justification, the geometric transformation which maps R onto P.

(iii) Let R be the point with position vector $\mathbf{r}$ and Q be the point with position vector $\mathrm{h}(\mathbf{r})$, where h is defined by

$$\mathrm{h}(\mathbf{s}) = -\mathbf{s} - 2\,\mathrm{f}(\mathrm{f}(\mathbf{s}))\,.$$

State, with justification, the geometric transformation which maps R onto Q.

Examiner's report

More than a third attempted this, marginally more successfully than questions **3** and **6**. Many attempts were restricted to part **(i)**. The first result was generally achieved, and whilst the second result was often obtained, quite a few had difficulties doing so because they overlooked that n was a unit vector and what this implied.

Far fewer correctly drew and labelled the diagram required in part **(i)** because they failed to appreciate the magnitudes of the three vectors and that two were perpendicular. Parts **(ii)** and **(iii)**, when attempted, saw candidates fall into two camps. A small number could see what both transformations were and using the considerations suggested in **(ii)** in part **(iii)** as well, could justify their answers. However, a larger number had some idea what the transformations might be, but often failed to define them precisely, and likewise failed to justify their conclusions, even given the approach to use in **(ii)**.

Solution

(i) Start by evaluating $\mathrm{f}(\mathbf{r})$:

$$\begin{aligned}\mathrm{f}(\mathbf{r}) &= \mathbf{n}\times\mathbf{r}\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & b & c\\ x & y & z\end{vmatrix}\\ &= \begin{pmatrix} bz-cy\\ cx-az\\ ay-bx\end{pmatrix}\end{aligned}$$

We then have:

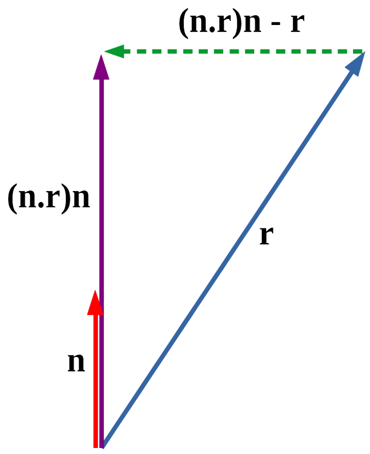
$$\begin{aligned}\mathrm{f}\big(\mathrm{f}(\mathbf{r})\big) &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a & b & c\\ bz-cy & cx-az & ay-bx\end{vmatrix}\\ &= \begin{pmatrix} b(ay-bx)-c(cx-az)\\ c(bz-cy)-a(ay-bx)\\ a(cx-az)-b(bz-cy)\end{pmatrix}\\ &= \begin{pmatrix} aby-b^2x-c^2x+acz\\ bcz-c^2y-a^2y+abx\\ acx-a^2z-b^2z+bcy\end{pmatrix}\\ &= \begin{pmatrix} -x(b^2+c^2)+aby+acz\\ -y(a^2+c^2)+abx+bcz\\ -z(a^2+b^2)+acx+bcy\end{pmatrix}\end{aligned}$$

Considering $(\mathbf{n}.\mathbf{r})\mathbf{n}-\mathbf{r}$ we have:

$$\begin{aligned}(\mathbf{n}.\mathbf{r})\mathbf{n}-\mathbf{r} &= \big(ax+by+cz\big)\begin{pmatrix}a\\ b\\ c\end{pmatrix} - \begin{pmatrix}x\\ y\\ z\end{pmatrix}\\ &= \begin{pmatrix}(a^2-1)x+aby+acz\\ abx+(b^2-1)y+bcz\\ acx+bcy+(c^2-1)z\end{pmatrix}\\ &= \begin{pmatrix} -x(b^2+c^2)+aby+acz\\ -y(a^2+c^2)+abx+bcz\\ -z(a^2+b^2)+acx+bcy\end{pmatrix}\end{aligned}$$

where the last line uses the fact that $\mathbf{n}$ is a unit vector, and so we have $a^2+b^2+c^2=1$. Therefore the two vectors are the same and we have $\mathrm{f}\big(\mathrm{f}(\mathbf{r})\big) = (\mathbf{n}.\mathbf{r})\mathbf{n}-\mathbf{r}$.

The $(\mathbf{n}.\mathbf{r})\mathbf{n}$ part of $\mathrm{f}\big(\mathrm{f}(\mathbf{r})\big)$ is the projection of $\mathbf{r}$ onto a vector with direction $\mathbf{n}$. A diagram showing $\mathrm{f}\big(\mathrm{f}(\mathbf{r})\big)$ might look like:



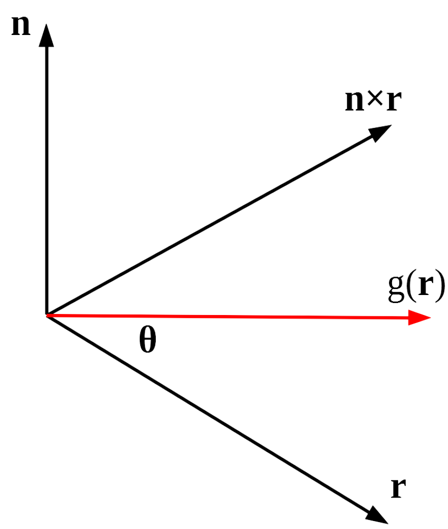
(ii) Start by considering $\mathrm{f}(\mathbf{n}) = \mathbf{n} \times \mathbf{n} = \mathbf{0}$. This means that $\mathrm{g}(\mathbf{n}) = \mathbf{n}$. We are told that $\mathbf{r}$ is perpendicular to $\mathbf{n}$, so we have $\mathbf{n} \cdot \mathbf{r} = 0$, and so $\mathrm{f}\big(\mathrm{f}(\mathbf{r})\big) = -\mathbf{r}$.

Considering $\mathrm{g}(\mathbf{r})$ we have:

$$\begin{aligned} \mathrm{g}(\mathbf{r}) &= \mathbf{r} + \sin\theta\,\mathrm{f}(\mathbf{r}) + (1-\cos\theta)\mathrm{f}\big(\mathrm{f}(\mathbf{r})\big) \\ &= \mathbf{r} + \sin\theta\,\mathbf{n}\times\mathbf{r} + (1-\cos\theta)(-\mathbf{r}) \\ &= \sin\theta\,\mathbf{n}\times\mathbf{r} + \cos\theta\,\mathbf{r} \end{aligned}$$

In this part we are told that $\mathbf{r}$ is perpendicular to $\mathbf{n}$, so $\mathbf{r}$, $\mathbf{n}$ and $\mathbf{n}\times\mathbf{r}$ form a mutually perpendicular set of vectors.

Since $\mathbf{n}$ is a unit vector, we know that $\mathbf{r}$ and $\mathbf{n}\times\mathbf{r}$ have the same magnitude. We therefore have a picture that looks like this:

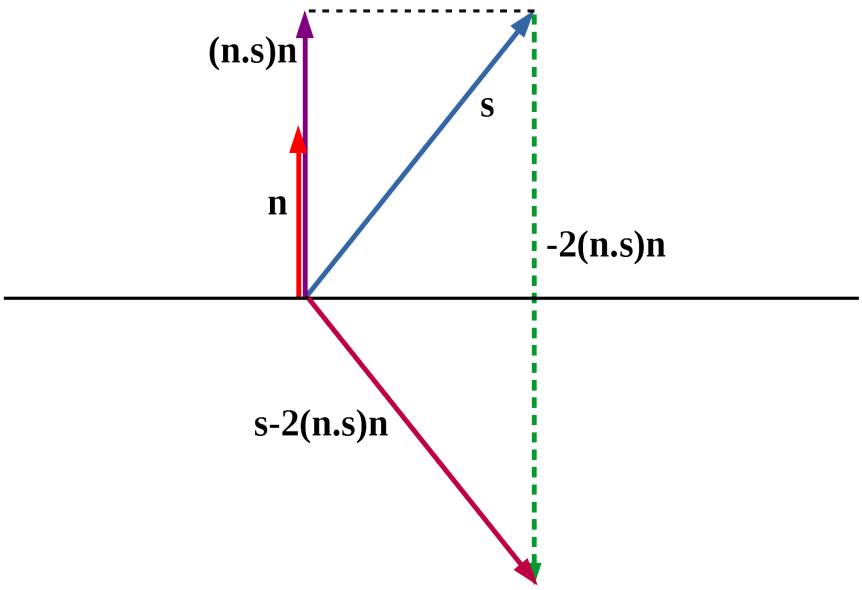


Therefore $\mathrm{g}(\mathbf{r})$ is the rotation of $\mathbf{r}$ by angle θ anticlockwise about the direction of $\mathbf{n}$.

(iii) We have:

$$\begin{aligned} \mathrm{h}(\mathbf{s}) &= -\mathbf{s} - 2\mathrm{f}\big(\mathrm{f}(\mathbf{s})\big) \\ &= -\mathbf{s} - 2\big((\mathbf{n}\cdot\mathbf{s})\mathbf{n} - \mathbf{s}\big) \\ &= \mathbf{s} - 2(\mathbf{n}\cdot\mathbf{s})\mathbf{n} \end{aligned}$$

The relationship between the vectors can be seen below:



So it looks as if $\mathrm{h}(\mathbf{s})$ represents a reflection in the plane perpendicular to $\mathbf{n}$.

Write a vector $\mathbf{s}$ in the form $\mathbf{s} = \alpha\mathbf{n} + \beta\mathbf{r} + \gamma(\mathbf{n}\times\mathbf{r})$ (which we can do as $\mathbf{n}$, $\mathbf{r}$ and $\mathbf{n}\times\mathbf{r}$ are mutually perpendicular, so form a *basis* in 3D).

We have:

$$\begin{aligned} \mathrm{h}(\mathbf{n}) &= \mathbf{n} - 2(\mathbf{n}\cdot\mathbf{n})\mathbf{n} = -\mathbf{n} \\ \mathrm{h}(\mathbf{r}) &= \mathbf{r} - 2(\mathbf{n}\cdot\mathbf{r})\mathbf{n} = \mathbf{r} \\ \mathrm{h}(\mathbf{n}\times\mathbf{r}) &= \mathbf{n}\times\mathbf{r} - 2\big(\mathbf{n}\cdot(\mathbf{n}\times\mathbf{r})\big)\mathbf{n} = \mathbf{n}\times\mathbf{r} \end{aligned}$$

In the above working we have used the facts that $\mathbf{n}\cdot\mathbf{n} = 1$, $\mathbf{n}\cdot\mathbf{r} = 0$ and $\mathbf{n}\cdot(\mathbf{n}\times\mathbf{r}) = 0$.

Hence we have the $\mathbf{n}$ component of the vector reversing direction, and the other components staying the same, therefore the transformation is a reflection in a plane perpendicular to $\mathbf{n}$.

Question 8

8 **(i)** Use De Moivre's theorem to prove that for any positive integer $k > 1$,

$$\sin(k\theta) = \sin\theta\cos^{k-1}\theta\left(k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right)$$

and find a similar expression for $\cos(k\theta)$.

(ii) Let $\theta = \cos^{-1}(\frac{1}{a})$, where θ is measured in degrees, and a is an odd integer greater than 1.

Suppose that there is a positive integer k such that $\sin(k\theta) = 0$ and $\sin(m\theta) \neq 0$ for all integers m with $0 < m < k$.

Show that it would be necessary to have k even and $\cos(\frac{1}{2}k\theta) = 0$.

Deduce that θ is irrational.

(iii) Show that if $\phi = \cot^{-1}(\frac{1}{b})$, where ϕ is measured in degrees, and b is an even integer greater than 1, then ϕ is irrational.

Examiner's report

This was the least popular Pure question, being attempted by marginally fewer than question **7**, but by more than any of the Mechanics or Probability and Statistics. The mean mark was 6/20.

Generally, part **(i)** was done well and candidates used binomial expansions accurately, manipulating their results to find the two required expressions. A few did not gain full credit through providing insufficient working for the result given in the question.

More than half the candidates progressed no further than attempting part **(ii)** and, of those who did attempt it, often stopped part of the way through, although there were some very well-reasoned attempts. Most candidates attempting part **(ii)** substituted $a = \sec(\theta)$ into their sin expansion but found it difficult to complete the argument to explain why k had to be even.

Of those who got further and successfully managed to show the given results, often the relevance of those results was not appreciated, and some candidates attempted to prove irrationality by quoting the irrationality of π, despite the fact that the question stated θ was measured in degrees. Very few candidates gained full credit for this part. Those candidates who gained full credit in part **(ii)** also did well in **(iii)**.

Solution

(i) By De Moivre's theorem we have:

$$\cos(k\theta) + \mathrm{i}\sin(k\theta) = (\cos\theta + \mathrm{i}\sin\theta)^k$$
$$= \left[\cos^k\theta - \binom{k}{2}\cos^{k-2}\theta\sin^2\theta + \binom{k}{4}\cos^{k-4}\theta\sin^4\theta - \cdots\right]$$
$$+\, i\left[\binom{k}{1}\cos^{k-1}\theta\sin\theta - \binom{k}{3}\cos^{k-3}\theta\sin^3\theta + \binom{k}{5}\cos^{k-5}\theta\sin^5\theta - \cdots\right]$$

Equating imaginary parts gives:

$$\sin(k\theta) = k\cos^{k-1}\theta\sin\theta - \binom{k}{3}\cos^{k-3}\theta\sin^3\theta + \binom{k}{5}\cos^{k-5}\theta\sin^5\theta - \cdots$$
$$= \sin\theta\cos^{k-1}\theta\left[k - \binom{k}{3}\tan^2\theta + \binom{k}{5}\tan^4\theta - \cdots\right]$$
$$= \sin\theta\cos^{k-1}\theta\left[k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right]$$

Equating real parts gives:

$$\cos(k\theta) = \left[\cos^k\theta - \binom{k}{2}\cos^{k-2}\theta\sin^2\theta + \binom{k}{4}\cos^{k-4}\theta\sin^4\theta - \cdots\right]$$
$$= \cos^k\theta\left[1 - \binom{k}{2}(\sec^2\theta - 1) + \binom{k}{4}(\sec^2\theta - 1)^2 - \cdots\right]$$

Note that in the expression for $\sin(k\theta)$ I put in an extra line of working, which is because in this case the question said "show that", so I made sure the derivation was fully justified.

(ii) If $\theta = \cos^{-1}\left(\frac{1}{a}\right)$, where $a > 1$ then we have $0° < \theta < 90°$ (noting that the question tells us that θ is measured in degrees). We also have $\sec\theta = a$.

If we have $\sin(k\theta) = 0$ then we have:

$$\sin\theta\cos^{k-1}\theta\left[k - \binom{k}{3}(\sec^2\theta - 1) + \binom{k}{5}(\sec^2\theta - 1)^2 - \cdots\right] = 0$$
$$\sin\theta\frac{1}{a^{k-1}}\left[k - \binom{k}{3}(a^2 - 1) + \binom{k}{5}(a^2 - 1)^2 - \cdots\right] = 0$$

For this expression to be equal to 0, the part in the square brackets must be equal to 0 (as $\sin\theta \neq 0$). Looking back at the information given in the question, we are told that a is an odd integer, which means that $a^2 - 1$ is even.

Considering:

$$\left[k - \binom{k}{3}(a^2 - 1) + \binom{k}{5}(a^2 - 1)^2 - \cdots\right] = 0$$

we can see that all the terms after the first one are even (as they have a factor of $a^2 - 1$), and so if this sum is equal to 0 the first term must also be even. Hence we have k is even.

We also have $\sin k\theta = 2\sin\left(\frac{1}{2}k\theta\right)\cos\left(\frac{1}{2}k\theta\right)$, and if k is even then $\sin\left(\frac{1}{2}k\theta\right) \neq 0$ (as $\sin(m\theta) \neq 0$ for all integers $0 < m < k$ and k even means $\frac{1}{2}k$ is an integer).

Therefore $\sin(k\theta) = 0 \implies \cos\left(\frac{1}{2}k\theta\right) = 0$.

Let $k = 2n$, so we have $\cos(n\theta) = 0$. Using the expression for $\cos(k\theta)$ from part **(i)** we have:

$$\begin{aligned}\cos(n\theta) &= \cos^n\theta\left[1 - \binom{n}{2}(\sec^2\theta - 1) + \binom{n}{4}(\sec^2\theta - 1)^2 - \cdots\right]\\ &= \frac{1}{a^n}\left[1 - \binom{n}{2}(a^2 - 1) + \binom{n}{4}(a^2 - 1)^2 - \cdots\right]\end{aligned}$$

We know that $\cos\theta = \frac{1}{a} \neq 0$. In the square bracket we have 1 and then all the other terms are even, so it is not possible for this bracket to be equal to zero. Hence we have $\cos(n\theta) \neq 0$ which is a contradiction, and so there is no integer k for which $\sin(k\theta) = 0$, but $\sin(m\theta) \neq 0$ for $0 < m < k$.

Note that we have $\sin\theta = \sqrt{1 - \left(\frac{1}{a}\right)^2} \neq 0$.

If θ is rational then we can write $\theta = \dfrac{p}{q}$ for some integers p and q where p and q have no common factors. This means we have $\theta = \dfrac{180p}{180q}$, which means $180q\theta = k\theta = 180p$. However this would imply that $\sin(k\theta) = 0$ hence we have a contradiction, and so θ cannot be rational.

(iii) Like in part **(ii)** start by assuming that there exists a positive integer k such that $\sin(k\phi) = 0$ and $\sin(m\phi) \neq 0$ for all integers m with $0 < m < k$.

We have:

$$\begin{aligned}\sin(k\phi) &= \sin\phi\cos^{k-1}\phi\left[k - \binom{k}{3}\tan^2\phi + \binom{k}{5}\tan^4\phi - \cdots\right]\\ &= \sin\phi\cos^{k-1}\phi\left[k - \binom{k}{3}b^2 + \binom{k}{5}b^4 - \cdots\right]\end{aligned}$$

Since $\phi = \cot^{-1}\left(\frac{1}{b}\right)$, so $\tan\phi = b$, and so $\sin\phi \neq 0$ and $\cos\phi \neq 0$. Therefore if $\sin(k\phi) = 0$ then the expression in the square bracket must be equal to 0. Since all of the terms after the first one have a factor of b they are even, and so we need k to be even.

If $\sin(k\phi) = 0$ then $2\sin\left(\frac{k\phi}{2}\right)\cos\left(\frac{k\phi}{2}\right) = 0$, and as $\sin\left(\frac{k\phi}{2}\right) \neq 0$ we have $\cos\left(\frac{k\phi}{2}\right) = 0$.

Letting $\frac{k}{2} = n$ we have:

$$\cos(n\phi) = \cos^n\phi\left[1 - \binom{n}{2}b^2 + \binom{n}{4}b^4 - \cdots\right]$$

Like before we have $\cos(n\phi) \neq 0$, which is a contradiction. Therefore there does not exist a k for which $\sin(k\phi) = 0$. Therefore we cannot have $\phi = \dfrac{180p}{k}$, and ϕ is irrational.

Question 9

9 **(i)** Two particles A and B, of masses m and km respectively, lie at rest on a smooth horizontal surface. The coefficient of restitution between the particles is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Let v_1 and v_2 be the velocities of particles A and B, respectively, after the collision, in the direction of the initial velocity of A.

Show that $v_1 = \alpha u$ and $v_2 = \beta u$, where $\alpha = \dfrac{1-ke}{k+1}$ and $\beta = \dfrac{1+e}{k+1}$.

Particle B strikes a vertical wall which is perpendicular to its direction of motion and a distance D from the point of collision with A, and rebounds. The coefficient of restitution between particle B and the wall is also e.

Show that, if A and B collide for a second time at a point $\frac{1}{2}D$ from the wall, then $k = \dfrac{1+e-e^2}{e(2e+1)}$.

(ii) Three particles A, B and C, of masses m, km and k^2m respectively, lie at rest on a smooth horizontal surface in a straight line, with B between A and C. A vertical wall is perpendicular to this line and lies on the side of C away from A and B. The distance between B and C is equal to d and the distance between C and the wall is equal to $3d$. The coefficient of restitution between each pair of particles, and between particle C and the wall, is e, where $0 < e < 1$. Particle A is then projected directly towards particle B with speed u.

Show that, if all three particles collide simultaneously at a point $\frac{3}{2}d$ from the wall, then $e = \frac{1}{2}$.

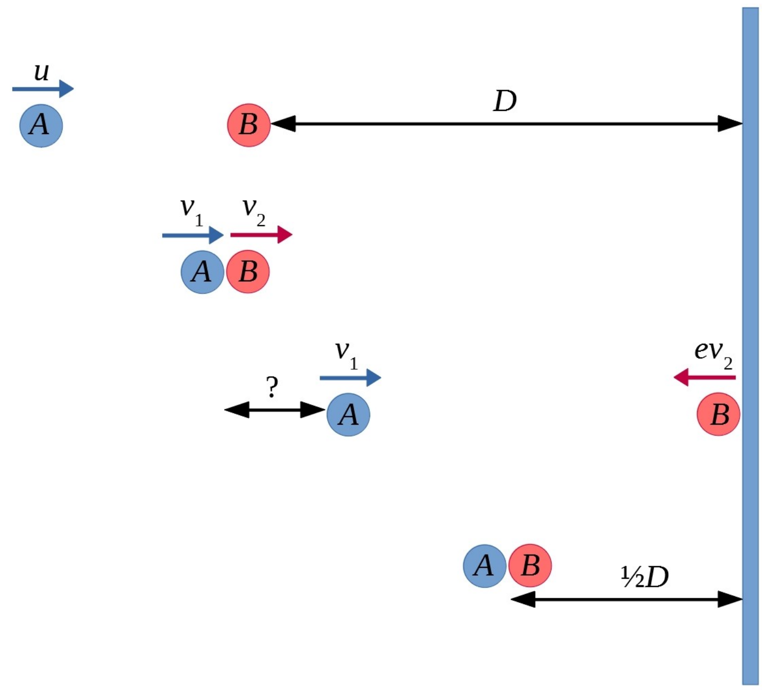
Examiner's report

The most popular of the Applied questions, with a third of candidates attempting it; it was the second-best scoring question on the paper with a mean score of just above half marks.

The question relied mainly on the use of conservation of momentum and Newton's experimental law of impact. Most candidates made a very good start with several scoring full or close to full marks in the first part. The difficulties arose later when dealing with the three-particle situation in part **(ii)**. Very few candidates were able to take a step back and see how this problem linked to part **(i)**, resulting in long pages of algebraic manipulation which were inefficient and rarely correct. A good diagram would have made the link so much more obvious!

Solution

(i) The first thing to do is to read **all** of the question! Once you have done this, then the second thing to do is to draw a diagram showing the information given in part **(i)**. You can also see that finding out how far A has moved at the point when B hits the wall might be useful. There is some artistic licence taken here - the "Particles" are rather large!



Using conservation of momentum and Newton's law of restitution gives:

$$mu = mv_1 + kmv_2 \implies u = v_1 + kv_2 \quad (1)$$
$$eu = v_2 - v_1 \quad (2)$$

We then have:

$$\begin{aligned}(1)+(2) &\implies u(1+e) = (k+1)v_2 \\ &\implies v_2 = \left(\frac{1+e}{k+1}\right)u \\ (1)-k(2) &\implies u(1-ke) = (1+k)v_1 \\ &\implies v_1 = \left(\frac{1-ke}{1+k}\right)u\end{aligned}$$

After the collision, B hits the wall after a time of $\dfrac{D}{v_2}$. In this time, A travels a distance of:

$$\begin{aligned}\frac{v_1}{v_2}D &= \frac{1-ke}{1+k} \times \frac{1+k}{1+e} \times D \\ &= \frac{1-ke}{1+e}D\end{aligned}$$

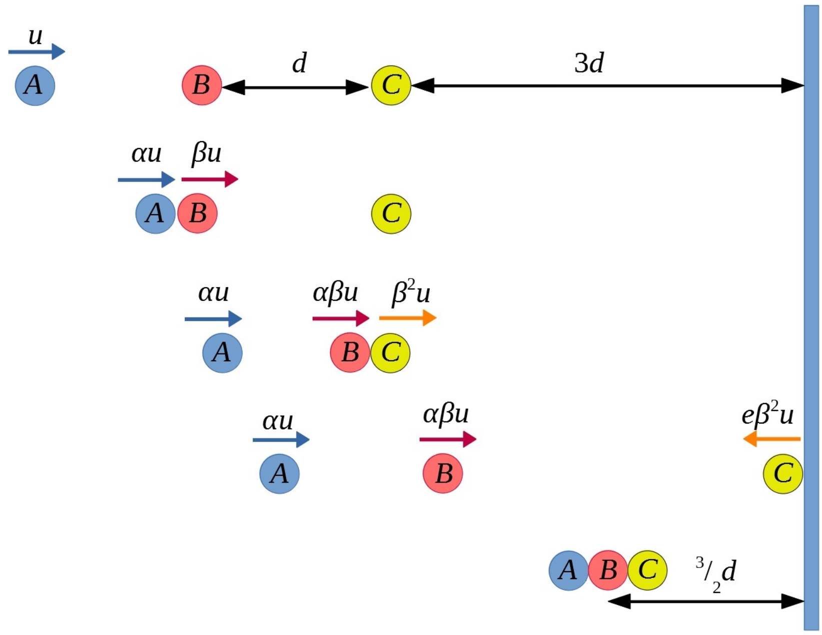
If the particles are going to meet at a distance of $\frac{1}{2}D$ from the wall, A still needs to travel:

$$\begin{aligned}\frac{D}{2} - \frac{1-ke}{1+e}D &= \frac{D}{2(1+e)}\left[(1+e) - 2(1-ke)\right] \\ &= \frac{D}{2(1+e)}\left[e + 2ke - 1\right]\end{aligned}$$

After rebounding from the wall, B has a velocity of ev_2 back towards A. Equating the time it takes B to travel $\frac{1}{2}D$, and A to travel the time above we have:

$$\begin{aligned}\frac{D}{2ev_2} &= \frac{D\left[e+2ke-1\right]}{2(1+e)v_1} \\ (1+e)v_1 &= (e+2ke-1)ev_2 \\ \frac{(1+e)(1-ke)u}{1+k} &= \frac{(e+2ke-1)e(1+e)u}{1+k} \\ (1-ke) &= e(e+2ke-1) \\ 2ke^2 + ke &= 1 + e - e^2 \\ k &= \frac{1+e-e^2}{e(2e+1)}\end{aligned}$$

(ii) This time there are three particles to consider. Since the ratio of the masses of the particles is $1 : k : k^2$, particles B and C act like particles A and B on collisions. Using the same α and β as in part **(i)** we can write down fairly simple expressions for the velocities at each stage.



Ignoring A for a moment, B and C act in exactly the same way as A and B in the first part, with $u \to \beta u$ and $D \to 3d$. This means that if B and C collide at the point $\frac{3}{2}d$ from the wall then the result from part **(i)** holds, i.e. k satisfies:

$$k = \frac{1+e-e^2}{e(2e+1)}$$

We also need A to be at this point at the same time. A needs to travel a total of $\frac{3}{2}d + d = \frac{5}{2}d$, and it does this at a constant speed of αu, so the time taken is $\dfrac{5d}{2\alpha u}$.

Particle B travels a distance of d at a speed of βu, and then a further distance of $\frac{3}{2}d$ at speed $\alpha\beta u$. The total time taken is $\dfrac{d}{\beta u} + \dfrac{3d}{2\alpha\beta u}$. Equating the times gives:

$$\begin{aligned}
\frac{5d}{2\alpha u} &= \frac{d}{\beta u} + \frac{3d}{2\alpha\beta u} \\
\frac{5}{2\alpha} &= \frac{1}{\beta} + \frac{3}{2\alpha\beta} \\
5\beta &= 2\alpha + 3 \\
5 \times \frac{1+e}{k+1} &= 2 \times \frac{1-ke}{k+1} + 3 \\
5(1+e) &= 2(1-ke) + 3(k+1) \\
5 + 5e &= 2 - 2ke + 3k + 3 \\
k &= \frac{5e}{3-2e}
\end{aligned}$$

Equating the two expressions for k gives:

$$\begin{aligned}
\frac{5e}{3-2e} &= \frac{1+e-e^2}{e(2e+1)} \\
5e^2(2e+1) &= (1+e-e^2)(3-2e) \\
10e^3 + 5e^2 &= 3 + 3e - 3e^2 - 2e - 2e^2 + 2e^3 \\
8e^3 + 10e^2 - e - 3 = 0
\end{aligned}$$

By inspection, $e = -1$ is a root so we have:

$$\begin{aligned}
(e+1)(8e^2 + 2e - 3) &= 0 \\
(e+1)(2e-1)(4e+3) &= 0
\end{aligned}$$

Since we need $0 \leqslant e \leqslant 1$ we have $e = \frac{1}{2}$ as required.

Question 10

10 Two light elastic springs each have natural length a. One end of each spring is attached to a particle P of weight W. The other ends of the springs are attached to the end-points, B and C, of a fixed horizontal bar BC of length $2a$. The moduli of elasticity of the springs PB and PC are s_1W and s_2W respectively; these values are such that the particle P hangs in equilibrium with angle BPC equal to $90°$.

(i) Let angle $PBC = \theta$. Show that $s_1 = \dfrac{\sin\theta}{2\cos\theta - 1}$ and find s_2 in terms of θ.

(ii) Take the zero level of gravitational potential energy to be the horizontal bar BC and let the total potential energy of the system be $-paW$. Show that p satisfies

$$\frac{1}{2}\sqrt{2} \geqslant p > \frac{1}{4}(1+\sqrt{3})$$

and hence that $p = 0.7$, correct to one significant figure.

Examiner's report

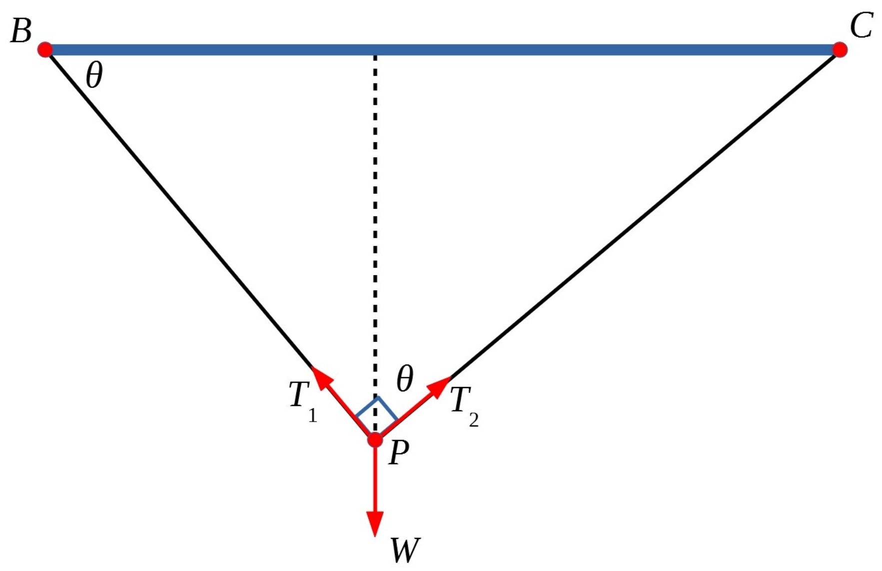
Along with question **12**, this was the least popular question on the paper with a sixth of candidates trying it, and scoring one third marks, on average.

Part **(i)** was done well, demonstrating good use of Hooke's law, and resolving forces. It was failing to think about right angled triangle trigonometry that created most problems.

In part **(ii)**, many candidates got the signs of their potential energies wrong. Of those candidates who got to the correct expression for p most were able to find the maximum value correctly but very few were able to explain why the physical situation resulted in a restricted domain for the function. Showing the value of p must be 0.7 to one significant figure was rarely done well as many candidates used known approximations to the given surds without justifying the accuracy of these approximations.

Solution

It's a mechanics question, so after reading through the first thing to do is to draw a large, clear diagram!



(i) Resolving would seem to be a good idea, and we could resolve horizontally and vertically. However, there is another pair of perpendicular directions we could use, and these will only involve one of the tensions each time. It feels a bit unusual to be resolving in this way, but it does simplify the algebra a bit.

If we consider the direction PB, we know that the length of PB is given by $2a\cos\theta$ (using triangle PBC). Therefore the extension in this spring is $2a\cos\theta - a = a(2\cos\theta - 1)$. The component of weight in this direction is given by $W\cos(90° - \theta) = W\sin\theta$. Equating forces gives:

$$\begin{aligned}\frac{s_1 W \cancel{a}(2\cos\theta - 1)}{\cancel{a}} &= W\sin\theta \\ \implies s_1 &= \frac{\sin\theta}{2\cos\theta - 1}\end{aligned}$$

Similarly we can consider the direction PC to get:

$$\begin{aligned}\frac{s_2 W \cancel{a}(2\sin\theta - 1)}{\cancel{a}} &= W\cos\theta \\ \implies s_2 &= \frac{\cos\theta}{2\sin\theta - 1}\end{aligned}$$

You can just write down the second result "by symmetry". To see why this is the case, if we let angle $PCB = \phi$, then we have $s_2 = \frac{\sin\phi}{2\cos\phi - 1}$, and then since $\phi = 90° - \theta$ we have $\sin\phi = \cos\theta$ and $\cos\phi = \sin\theta$.

Note that since both extensions of the springs are positive we have $2\cos\theta - 1 > 0$ and $2\sin\theta - 1 > 0$, and so the divisions above are not problematic.

(ii) The GPE of the particle will be negative (as it is below the zero level). It is given by:

$$-W \times BP \sin\theta = -2Wa\sin\theta\cos\theta$$

The EPE of spring BP is given by:

$$\frac{s_1 W\left[a(2\cos\theta - 1)\right]^2}{2a}$$

and the EPE of spring CP is given by:

$$\frac{s_2 W\left[a(2\sin\theta - 1)\right]^2}{2a}$$

The total potential energy is therefore:

$$\begin{aligned}
&-2Wa\sin\theta\cos\theta + \frac{s_1 W\left[a(2\cos\theta - 1)\right]^2}{2a} + \frac{s_2 W\left[a(2\sin\theta - 1)\right]^2}{2a}\\
=& -\frac{Wa}{2}\left[4\sin\theta\cos\theta - s_1(2\cos\theta - 1)^2 - s_2(2\sin\theta - 1)^2\right]\\
=& -\frac{Wa}{2}\left[4\sin\theta\cos\theta - \frac{\sin\theta}{2\cos\theta - 1} \times (2\cos\theta - 1)^2 - \frac{\cos\theta}{2\sin\theta - 1} \times (2\sin\theta - 1)^2\right]\\
=& -\frac{Wa}{2}\left[4\sin\theta\cos\theta - \sin\theta(2\cos\theta - 1) - \cos\theta(2\sin\theta - 1)\right]\\
=& -\frac{Wa}{2}(\sin\theta + \cos\theta)
\end{aligned}$$

Therefore we have $p = \frac{1}{2}(\sin\theta + \cos\theta)$.

To find the maximum and minimum values of $\sin\theta + \cos\theta$ we could differentiate, alternatively we can use $\sin\theta + \cos\theta = R\cos(\theta - \alpha)$, i.e.:

$$\begin{aligned}
\sin\theta + \cos\theta &= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha\\
\implies R^2(\cos^2\alpha + \sin^2\alpha) &= 2\\
\tan\alpha &= 1
\end{aligned}$$

Therefore we have $\sin\theta + \cos\theta = \sqrt{2}\,(\cos\theta - 45^\circ)$. The maximum value of this is $\sqrt{2}$ when $\theta = 45^\circ$, and the minimum value is $-\sqrt{2}$, which occurs when $\theta = 180^\circ + 45^\circ$ — which is not possible! Hence we have $p \leqslant \frac{1}{2}\sqrt{2}$, (with the maximum being obtained when $\theta = 45^\circ$) but we need to think a little more about the minimum.

Since both of the springs are extended we have $\cos\theta > \frac{1}{2}$ and $\sin\theta > \frac{1}{2}$. This means that we need $30^\circ < \theta < 60^\circ$. We know that $\cos(\theta - 45^\circ)$ will decrease as θ gets further from 45°, and so the minimum value of p will be when $\theta = 60^\circ$ or 30°. This minimum cannot be attained as the restrictions on θ are strict. It is easier to substitute into to original for p (rather than having to work out $\cos 15^\circ$), so we have:

$$\begin{aligned}
p_{\min} &= \frac{1}{2}(\sin 60^\circ + \cos 60^\circ)\\
&= \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)\\
&= \frac{1}{4}(1 + \sqrt{3})
\end{aligned}$$

Hence we have $\frac{1}{4}(1+\sqrt{3}) < p \leqslant \frac{1}{2}\sqrt{2}$ as required.

To show that $p = 0.7$ to one significant figure we need to show that $0.65 \leqslant p < 0.75$, i.e. $\frac{13}{20} \leqslant p < \frac{3}{4}$. Considering the top limit we have:

$$\begin{aligned} \frac{1}{2} &< \frac{9}{16} \\ \implies \frac{1}{\sqrt{2}} &< \frac{3}{4} \\ \implies \frac{\sqrt{2}}{2} &< \frac{3}{4} \end{aligned}$$

and since we know that $p \leqslant \frac{\sqrt{2}}{2}$, we must have $p < \frac{3}{4}$.

For the lower limit we want to show that $\frac{13}{20} < \frac{1}{4}(1+\sqrt{3})$, or equivalently that $\frac{8}{20} < \frac{\sqrt{3}}{4}$. We have:

$$\begin{aligned} 256 &< 300 \\ \implies \frac{16}{100} &< \frac{3}{16} \\ \implies \frac{4}{10} &< \frac{\sqrt{3}}{4} \end{aligned}$$

and so we have $\frac{8}{20} < \frac{\sqrt{3}}{4}$, and therefore we have $p > 0.65$.

Therefore we have $0.65 < p < 0.75$, and so $p = 0.7$ to one significant figure.

Question 11

11 A fair coin is tossed N times and the random variable X records the number of heads. The mean deviation, δ, of X is defined by

$$\delta = \mathrm{E}\big(|X - \mu|\big)$$

where μ is the mean of X.

(i) Let $N = 2n$ where n is a positive integer.

(a) Show that $\mathrm{P}(X \leqslant n-1) = \frac{1}{2}\big(1 - \mathrm{P}(X = n)\big)$.

(b) Show that

$$\delta = \sum_{r=0}^{n-1} (n-r)\binom{2n}{r}\frac{1}{2^{2n-1}}\,.$$

(c) Show that for $r > 0$,

$$r\binom{2n}{r} = 2n\binom{2n-1}{r-1}.$$

Hence show that

$$\delta = \frac{n}{2^{2n}}\binom{2n}{n}\,.$$

(ii) Find a similar expression for δ in the case $N = 2n+1$.

Examiner's report

A quarter of the candidates attempted this, scoring a mean of one third marks. Of these, about a quarter made little or no progress. However, there were also several very good attempts achieving most or all of the marks available.

In part **(i)(a)** the majority of candidates noticed the symmetry of the distribution and were therefore able to answer this part well, although errors such as omitting the binomial coefficients and forgetting that X could take the value 0 were made in some cases.

In part **(i)(b)** most candidates were able to see that the modulus sign in the sum effectively meant that the calculation of δ should be split into two sums. However, in several cases candidates simply observed that the given result followed from the two sums by symmetry without sufficient justification to earn the marks.

Almost all candidates who attempted part **(i)(c)** were able to show the first result by applying the definition of $\binom{2n}{r}$ and then cancelling terms. A small number of candidates argued the result by viewing the two expressions as representing different ways of counting the same total number of things. For the next part of **(i)(c)** the majority of candidates split the sum and then applied the previous result to the second term. Many candidates, however, did not pay sufficient attention to the case where $r = 0$ and ended up with an incorrect term in the sum. Many candidates jumped straight to the given answer at this point and therefore did not show sufficient detail to earn the remaining marks for this part. Many of the candidates who progressed further with this part dealt with the two sums separately, but some used the fact that $\binom{2n}{r} = \binom{2n-1}{r-1} + \binom{2n-1}{r}$ to rearrange into a sum of differences, most of which then cancelled out.

Candidates who had completed part **(i)(c)** well were able to apply the same methods to the case in part **(ii)** and this part was generally completed well, although a small number of candidates failed to notice that the expression for the mean in terms of n had changed.

Solution

(i) **(a)** Since we are tossing a fair coin we know that the probability of getting r heads is the same as the probability of getting r tails, i.e. the distribution is symmetric. Hence we have $\mathrm{P}(X = r) = \mathrm{P}(X = 2n - r)$. There are $2n + 1$ possible number of heads (from 0 to $2n$ inclusive), so they occur in equal pairs apart from the middle one. We have:

$$\begin{aligned} \mathrm{P}(X=0) + \mathrm{P}(X=1) + \cdots + \mathrm{P}(X=n) + \cdots + \mathrm{P}(X=2n-1) + \mathrm{P}(X=2n) &= 1 \\ 2\mathrm{P}(X=0) + 2\mathrm{P}(X=1) + \cdots + 2\mathrm{P}(X=n-1) + \mathrm{P}(X=n) &= 1 \\ 2\mathrm{P}(X \leqslant n-1) + \mathrm{P}(X=n) &= 1 \end{aligned}$$

And so we have $\mathrm{P}(X \leqslant n-1) = \frac{1}{2}\left[1 - \mathrm{P}(X = n)\right]$ as required.

(b) We have $\mu = 2n \times \frac{1}{2} = n$. Using $\delta = \mathrm{E}(|X - \mu|)$:

$$\begin{aligned} \delta &= \sum_{r=0}^{2n} |r-n| \binom{2n}{r} \left(\frac{1}{2}\right)^{2n} \\ &= \sum_{r=0}^{n-1} (n-r) \binom{2n}{r} \left(\frac{1}{2}\right)^{2n} + \sum_{r=n+1}^{2n} (r-n) \binom{2n}{r} \left(\frac{1}{2}\right)^{2n} \\ &= \sum_{r=0}^{n-1} (n-r) \binom{2n}{r} \left(\frac{1}{2}\right)^{2n} + \sum_{r=n+1}^{2n} (r-n) \binom{2n}{2n-r} \left(\frac{1}{2}\right)^{2n} \end{aligned}$$

We would like to change the limits on the second sum so that they go from 0 to $n - 1$, and it would be good if some of the other expressions looked the same. Using an index transformation $s = 2n - r$ gives:

$$\begin{aligned} \delta &= \sum_{r=0}^{n-1} (n-r) \binom{2n}{r} \left(\frac{1}{2}\right)^{2n} + \sum_{s=0}^{n-1} (2n-s-n) \binom{2n}{s} \left(\frac{1}{2}\right)^{2n} \\ &= 2\sum_{r=0}^{n-1} (n-r) \binom{2n}{r} \left(\frac{1}{2}\right)^{2n} \\ &= \sum_{r=0}^{n-1} (n-r) \binom{2n}{r} \left(\frac{1}{2^{2n-1}}\right) \end{aligned}$$

(c) We have:

$$\begin{aligned} r\binom{2n}{r} &= \frac{r(2n)!}{r!(2n-r)!} \\ &= \frac{(2n)!}{(r-1)!(2n-r)!} \\ &= \frac{(2n)\times(2n-1)!}{(r-1)!\big[(2n-1)-(r-1)\big]!} \\ &= 2n\binom{2n-1}{r-1} \end{aligned}$$

Lets try using this in our expression for δ:

$$\begin{aligned} \delta &= \sum_{r=0}^{n-1}(n-r)\binom{2n}{r}\left(\frac{1}{2^{2n-1}}\right) \\ &= n\sum_{r=0}^{n-1}\binom{2n}{r}\left(\frac{1}{2^{2n-1}}\right) - \sum_{r=0}^{n-1} r\binom{2n}{r}\left(\frac{1}{2^{2n-1}}\right) \\ &= n\sum_{r=0}^{n-1}\binom{2n}{r}\left(\frac{1}{2^{2n-1}}\right) - \sum_{r=1}^{n-1} r\binom{2n}{r}\left(\frac{1}{2^{2n-1}}\right) \\ &= n\sum_{r=0}^{n-1}\binom{2n}{r}\left(\frac{1}{2^{2n-1}}\right) - 2n\sum_{r=1}^{n-1}\binom{2n-1}{r-1}\left(\frac{1}{2^{2n-1}}\right) \end{aligned}$$

Note that we can factorise the n terms out from the sum, and that we could change the lower limit on the second sum as when $r = 0$ the term was also equal to 0.

At this point it isn't very obvious what to do next. It might help to look back at the start of the question to see if there is any help there.

In part **(a)** we considered $\mathrm{P}(X \leqslant n-1)$. As a sum this is:

$$\mathrm{P}(X \leqslant n-1) = \sum_{r=0}^{n-1}\binom{2n}{r}\left(\frac{1}{2}\right)^{2n}$$

Therefore we have:

$$\begin{aligned} \sum_{r=0}^{n-1}\binom{2n}{r}\left(\frac{1}{2}\right)^{2n} &= \tfrac{1}{2}\big[1-\mathrm{P}(X=n)\big] \\ &= \frac{1}{2}\left[1-\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right] \\ \implies \sum_{r=0}^{n-1}\binom{2n}{r} &= 2^{2n-1}\left[1-\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right] \end{aligned}$$

Using this in our expression for δ:

$$\begin{aligned} \delta &= n\sum_{r=0}^{n-1}\binom{2n}{r}\left(\frac{1}{2^{2n-1}}\right) - 2n\sum_{r=1}^{n-1}\binom{2n-1}{r-1}\left(\frac{1}{2^{2n-1}}\right) \\ &= n\left[1-\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right] - 2n\sum_{r=1}^{n-1}\binom{2n-1}{r-1}\left(\frac{1}{2^{2n-1}}\right) \end{aligned}$$

At this point it's probably a good idea to remind ourselves of the result we are aiming for. Factorising out a $\frac{n}{2^{2n}}$ term gives:

$$\begin{aligned}
\delta &= \frac{n}{2^{2n}}\left[2^{2n} - \binom{2n}{n} - 4\sum_{r=1}^{n-1}\binom{2n-1}{r-1}\right] \\
&= \frac{n}{2^{2n}}\left[2^{2n} - \binom{2n}{n} - 2\sum_{r=1}^{n-1}\binom{2n-1}{r-1} - 2\sum_{r=1}^{n-1}\binom{2n-1}{r-1}\right] \\
&= \frac{n}{2^{2n}}\left[2^{2n} - \binom{2n}{n} - 2\left(\sum_{r=1}^{n-1}\binom{2n-1}{r-1} + \sum_{s=n+1}^{2n-1}\binom{2n-1}{2n-s}\right)\right] \\
&= \frac{n}{2^{2n}}\left[2^{2n} - \binom{2n}{n} - 2\left(\sum_{r=0}^{n-2}\binom{2n-1}{r} + \sum_{r=n+1}^{2n-1}\binom{2n-1}{r}\right)\right] \\
&= \frac{n}{2^{2n}}\left[2^{2n} - \binom{2n}{n} - 2\left(\sum_{r=0}^{2n-1}\binom{2n-1}{r} - \binom{2n-1}{n-1} - \binom{2n-1}{n}\right)\right]
\end{aligned}$$

Time for another pause for reflection! We have:

$$\binom{2n-1}{n-1} = \binom{2n-1}{n}$$

which might be useful.

The sum that we are left with looks a bit like a sum of probabilities. If we consider the case when $N = 2n-1$, we know that:

$$\begin{aligned}
\sum_{r=0}^{2n-1}\binom{2n-1}{r}\left(\frac{1}{2}\right)^{2n-1} &= 1 \\
\implies \sum_{r=0}^{2n-1}\binom{2n-1}{r} &= 2^{2n-1}
\end{aligned}$$

Using these two results in δ gives:

$$\begin{aligned}
\delta &= \frac{n}{2^{2n}}\left[2^{2n} - \binom{2n}{n} - 2\left(2^{2n-1} - 2\binom{2n-1}{n}\right)\right] \\
&= \frac{n}{2^{2n}}\left[\cancel{2^{2n}} - \binom{2n}{n} - \cancel{2^{2n}} + 4\binom{2n-1}{n}\right]
\end{aligned}$$

This feels like we are almost there! We have:

$$\begin{aligned}\binom{2n-1}{n} &= \frac{(2n-1)!}{n!(n-1)!}\\ &= \frac{(2n)\times(2n-1)!}{(2n)\times n!(n-1)!}\\ &= \frac{(2n)!}{2\times n!n!}\\ &= \frac{1}{2}\binom{2n}{n}\end{aligned}$$

And so we have:

$$\begin{aligned}\delta &= \frac{n}{2^{2n}}\left[-\binom{2n}{n}+2\binom{2n}{n}\right]\\ &= \frac{n}{2^{2n}}\binom{2n}{n}\end{aligned}$$

Phew!

(ii) In this case we have $\mu = \frac{1}{2}(2n+1)$, and using the symmetry of the binomial distribution we have:

$$\begin{aligned}\delta &= \sum_{r=0}^{2n+1}\left|r-\frac{2n+1}{2}\right|\binom{2n+1}{r}\left(\frac{1}{2}\right)^{2n+1}\\ &= 2\sum_{r=0}^{n}\left(\frac{2n+1}{2}-r\right)\binom{2n+1}{r}\left(\frac{1}{2}\right)^{2n+1}\\ &= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\sum_{r=0}^{n}\binom{2n+1}{r}-\sum_{r=0}^{n}r\binom{2n+1}{r}\right]\end{aligned}$$

In a similar way to before we can look at the sum of probabilities:

$$\begin{aligned}\sum_{r=0}^{2n+1}\binom{2n+1}{r}\left(\frac{1}{2}\right)^{2n+1} &= 1\\ 2\sum_{r=0}^{n}\binom{2n+1}{r}\left(\frac{1}{2}\right)^{2n+1} &= 1\\ \sum_{r=0}^{n}\binom{2n+1}{r} &= 2^{2n}\end{aligned}$$

For the second term in the expression for δ we have:

$$\begin{aligned}\sum_{r=0}^{n}r\binom{2n+1}{r} &= \sum_{r=0}^{n-1}(2n+1)\binom{2n}{r}\\ &= 2^{2n}(2n+1)\sum_{r=0}^{n-1}\binom{2n}{r}\left(\frac{1}{2}\right)^{2n}\\ &= 2^{2n}(2n+1)\times\frac{1}{2}\left[1-\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]\end{aligned}$$

where the first step uses the first result from part **(i)(c)** and the last step uses the result in part **(i)(a)**.

Using these two we have:

$$\begin{aligned}\delta &= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\sum_{r=0}^{n}\binom{2n+1}{r} - \sum_{r=0}^{n} r\binom{2n+1}{r}\right]\\ &= \frac{1}{2^{2n}}\left[\frac{2n+1}{2}\times 2^{2n} - 2^{2n}(2n+1)\times\frac{1}{2}\left[1-\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]\right]\\ &= \frac{2n+1}{2}\left[1-1+\binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\right]\\ &= \frac{2n+1}{2^{2n+1}}\binom{2n}{n}\end{aligned}$$

Question 12

12 **(i)** The point A lies on the circumference of a circle of radius a and centre O. The point B is chosen at random on the circumference, so that the angle AOB has a uniform distribution on $[0, 2\pi]$. Find the expected length of the chord AB.

(ii) The point C is chosen at random in the interior of a circle of radius a and centre O, so that the probability that it lies in any given region is proportional to the area of the region. The random variable R is defined as the distance between C and O.

Find the probability density function of R.

Obtain a formula in terms of a, R and t for the length of a chord through C that makes an acute angle of t with OC.

Show that as C varies (with t fixed), the expected length Ł(t) of such chords is given by

$$\text{Ł}(t) = \frac{4a(1-\cos^3 t)}{3\sin^2 t}\,.$$

Show further that

$$\text{Ł}(t) = \frac{4a}{3}\left(\cos t + \tfrac{1}{2}\sec^2(\tfrac{1}{2}t)\right).$$

(iii) The random variable T is uniformly distributed on $[0, \frac{1}{2}\pi]$. Find the expected value of Ł(T).

Examiner's report

The least popular question on the paper, it was also the least successful with a mean score of just under one quarter marks. Many attempts did not make much progress beyond the first part. Candidates with a good understanding of how to calculate the expectation of a function of a random variable generally made very good progress.

In part **(i)** many candidates were able to calculate the length of the chord, although many used the cosine rule on an isosceles triangle to reach $a\sqrt{2 - 2\cos 2\theta}$, making that integration a little harder. A significant number of candidates who attempted this part omitted to include the probability density function when integrating to calculate the expected value. A small number of candidates chose to consider the length of the chord as a random variable and calculated its probability density function, from which they could then calculate the expected value. While this approach was in general successful it was a significantly more complicated approach.

In part **(ii)**, many candidates were able to work out the probability density function. Several candidates struggled to find an expression for the length of the chord and so failed to make any further progress from this point. Those that did were often able to complete the calculation of the expected value correctly. In a small number of cases, candidates attempted to calculate the probability density function for the length of the chord in order to calculate the expected value. In this case care needs to be taken with the limits of the integration as the shortest possible length for such chords needs to be calculated. A good number of candidates were able to rearrange the expected value in part **(ii)** into the requested form and many were then able to complete part **(iii)** successfully, although a number of attempts again omitted the probability density function and other attempts multiplied the function by t before integrating.

Solution

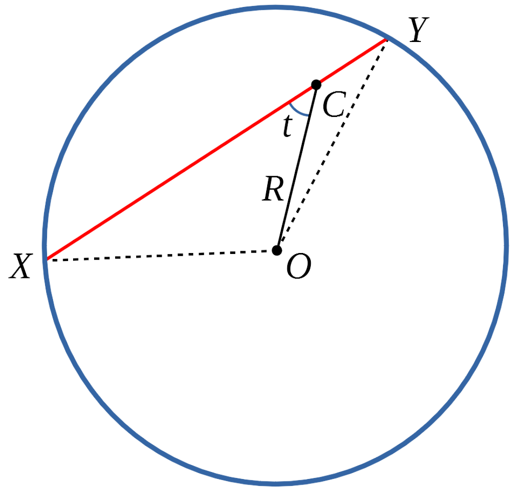
(i) We have $\mathrm{F}(\theta) = \mathrm{P}(X \leqslant \theta) = \dfrac{\theta}{2\pi}$, and so the probability distribution function is given by $\mathrm{f}(\theta) = \dfrac{1}{2\pi}$. The length of the chord is given by $AB = 2a\sin\frac{\theta}{2}$ and so the expected value of the length of the chord is given by:

$$\begin{aligned}\mathrm{E}(AB) &= \int_0^{2\pi} 2a\sin\tfrac{\theta}{2} \times \frac{1}{2\pi}\,\mathrm{d}\theta \\ &= \frac{2a}{\pi}\Big[-\cos\tfrac{\theta}{2}\Big]_0^{2\pi} \\ &= \frac{4a}{\pi}\end{aligned}$$

(ii) Start by considering $\mathrm{P}(R \leqslant x)$, which means that C lies with a circle of radius x. This means we have $\mathrm{P}(R \leqslant x) = \dfrac{\pi x^2}{\pi a^2} = \dfrac{x^2}{a^2}$ (the probability of being in a given region is proportional to the area of the region, and the area of whole region is 1).

Differentiating gives the probability density function as $\mathrm{f}_R(x) = \dfrac{2x}{a^2}$ (where $0 \leqslant x \leqslant a$).

It's probably a good idea to draw a sketch to show the location of the chord:



We know that XOY is an isosceles triangle, so we can create a right angled triangle by joining the midpoint of XY to O. Using Pythagoras' theorem we have:

$$\begin{aligned} a^2 &= \left(\tfrac{1}{2}XY\right)^2 + (R\sin t)^2 \\ \implies XY &= 2\sqrt{a^2 - R^2\sin^2 t} \end{aligned}$$

This gives:

$$\begin{aligned} \text{Ł}(t) &= \int_0^a 2\sqrt{a^2 - x^2\sin^2 t} \times \frac{2x}{a^2}\,\mathrm{d}x \\ &= -\frac{4}{3a^2\sin^2 t}\left[(a^2 - x^2\sin^2 t)^{\frac{3}{2}}\right]_0^a \\ &= -\frac{4}{3a^2\sin^2 t}\left[(a^2 - a^2\sin^2 t)^{\frac{3}{2}} - a^3\right] \\ &= -\frac{4}{3a^2\sin^2 t}\left[a^3\cos^3 t - a^3\right] \\ &= \frac{4a(1-\cos^3 t)}{3\sin^2 t} \end{aligned}$$

Note that t is fixed here, so we are integrating a function of x of the form $\int \sqrt{A - Bx^2} \times 2x\,\mathrm{d}x$.

We now need to rearrange to complete the "Show further that" part of the question. We want to force out a $\cos t$ term.

$$\begin{aligned}
\text{Ł}(t) &= \frac{4a(1-\cos^3 t)}{3\sin^2 t} \\
&= \frac{4a}{3}\left[\frac{\cos t(1-\cos^2 t) - \cos t + 1}{\sin^2 t}\right] \\
&= \frac{4a}{3}\left[\cos t + \frac{1-\cos t}{\sin^2 t}\right] \\
&= \frac{4a}{3}\left[\cos t + \frac{2\cancel{\sin^2\left(\frac{t}{2}\right)}}{4\cancel{\sin^2\left(\frac{t}{2}\right)}\cos^2\left(\frac{t}{2}\right)}\right] \\
&= \frac{4a}{3}\left[\,\cos t + \tfrac{1}{2}\sec^2\left(\tfrac{t}{2}\right)\,\right]
\end{aligned}$$

(iii) Since T is uniformly distributed on $\left[0, \frac{1}{2}\pi\right]$ we have $\mathrm{f}(t) = \frac{2}{\pi}$., The expectation is then:

$$\begin{aligned}
\mathrm{E}\big(\text{Ł}(T)\big) &= \int_0^{\frac{\pi}{2}} \frac{4a}{3}\left[\,\cos t + \tfrac{1}{2}\sec^2\left(\tfrac{t}{2}\right)\,\right] \times \frac{2}{\pi}\,\mathrm{d}t \\
&= \frac{8a}{3\pi}\int_0^{\frac{\pi}{2}} \left[\cos t + \tfrac{1}{2}\sec^2\left(\tfrac{t}{2}\right)\,\right]\mathrm{d}t \\
&= \frac{8a}{3\pi}\left[\,\sin t + \tan\left(\tfrac{t}{2}\right)\,\right]_0^{\frac{\pi}{2}} \\
&= \frac{16a}{3\pi}
\end{aligned}$$