STEP Support Programme

A Level Support - Algebra

General comments

This is a short(ish) introduction to some of the A-level algebra topics.

There are some suggested NRICH and Underground Mathematics problems at various places in this document. You can find many more by visiting https://undergroundmathematics.org/ and https://nrich.maths.org/. You might find the search facilities useful to find problems and articles on specific topics.

Please send any corrections or comments to step@maths.org.

Some definitions

An *equation* is a mathematical statement showing that two expressions have equal value. An example would be $2x + 4 = 10$, which implies that $x = 3$. An equation is true for only certain values of x[1].

An *identity* is something that is true for all values of x, for example $2x + 4 \equiv 2(x + 2)$. Sometimes the equivalence sign $\equiv$ is used in an identity, but often the equal sign is used instead.
If you try and "solve" the identity $2x + 4 \equiv 2(x + 2)$ you will end up with the true, but unhelpful, statement $0 = 0$.

- Expression, e.g. $2x + 4$
- Equation, e.g. $2x + 4 = 10$
- Identity, e.g. $2x + 4 \equiv 2(x + 2)$
- Formula, e.g. $P = 2x + 4$
- Function, e.g. $\mathrm{f}(x) = 2x + 4$

It is hard to write down precise definitions for these terms. For example, $y = 2x + 1$ and $y = 3 - x$ might be considered as two formulae or they could be a pair of simultaneous equations. Trying to distinguish between *parameters*, *variables* and *unknowns* is even more difficult.

Expanding brackets

You may be asked to multiply together two brackets with a not-inconsiderable number of terms in each. In these cases I would use a table (or *grid*) to help me organise the working out and make sure that I don't miss any terms.

Example: Multiply out and simplify $(x + 2y - 5)(2x - y - 3)$

	x	$+2y$	-5
$2x$	$2x^2$	$4xy$	$-10x$
$-y$	$-xy$	$-2y^2$	$5y$
-3	$-3x$	$-6y$	15

and so we have $(x + 2y - 5)(2x - y - 3) = 2x^2 + 3xy - 2y^2 - 13x - y + 15$.

You can also use tables/grids to help you factorise expressions, see the Polynomials section for an example.

[1] Other parameters are available.

Surds

A surd is a root which is irrational, e.g. $\sqrt{5}$, $\sqrt[3]{10}$ etc. Roots which can be turned into a rational number such as $\sqrt[4]{16}$ and $\sqrt{\frac{9}{25}}$ are not surds.

The main result you need when manipulating surds is that $\sqrt[n]{a \times b} = \sqrt[n]{a} \times \sqrt[n]{b}$.[2] Most of the time you will be working with square roots.

Example: Simplify $\sqrt{48} - \sqrt{75} + \sqrt{300}$.

We have:
$$\begin{aligned}\sqrt{48} - \sqrt{75} + \sqrt{300} &= \sqrt{16 \times 3} - \sqrt{25 \times 3} + \sqrt{100 \times 3}\\ &= \sqrt{16} \times \sqrt{3} - \sqrt{25} \times \sqrt{3} + \sqrt{100} \times \sqrt{3}\\ &= 4\sqrt{3} - 5\sqrt{3} + 10\sqrt{3}\\ &= 9\sqrt{3}\end{aligned}$$

Example: Find the square roots of $11 - 6\sqrt{2}$.

You know that if you square something of the form $(a + b\sqrt{2})$ then you will get something of the same form. Start by equating $11 - 6\sqrt{2}$ to $(a + b\sqrt{2})^2$ (where a and b are rational):
$$\begin{aligned}11 - 6\sqrt{2} &= (a + b\sqrt{2})^2\\ 11 - 6\sqrt{2} &= a^2 + 2ab\sqrt{2} + 2b^2\end{aligned}$$

Then equating the rational and irrational parts gives:
$$\begin{aligned}11 &= a^2 + 2b^2\\ -6 &= 2ab\end{aligned}$$

Then substituting $a = -\frac{3}{b}$ into the first equation gives:
$$\begin{aligned}\left(-\frac{3}{b}\right)^2 + 2b^2 &= 11\\ \frac{9}{b^2} + 2b^2 &= 11\\ 2b^4 - 11b^2 + 9 &= 0\\ (2b^2 - 9)(b^2 - 1) &= 0\\ b &= \pm 1\end{aligned}$$

Note that we stated that b was rational and so we ignore the $2b^2 = 9$ solution. When $b = \pm 1$ we have $a = \mp 3$ and so the square roots of $11 - 6\sqrt{2}$ are $\pm(3 - \sqrt{2})$.

If instead we take $2b^2 = 9 \implies b = \pm\frac{3\sqrt{2}}{2}$ then this gives $a = \mp\sqrt{2}$ and so $a + b\sqrt{2}$ becomes $\mp(\sqrt{2} - 3)$, i.e. the same square roots that we found by using $b^2 = 1$.

The same method is used to find square roots of complex numbers — but for some reason finding square roots of expressions of the form $a + b\sqrt{q}$ is less likely to be covered at school.

[2]Note that, in general, $\sqrt[n]{a + b} \neq \sqrt[n]{a} + \sqrt[n]{b}$.

Example: Simplify $\dfrac{1}{3-\sqrt{2}}+\dfrac{1}{3+\sqrt{2}}$.

Rationalising the denominators gives:

$$\begin{aligned}\frac{1}{3-\sqrt{2}}+\frac{1}{3+\sqrt{2}} &= \frac{3+\sqrt{2}}{(3+\sqrt{2})(3-\sqrt{2})}+\frac{3-\sqrt{2}}{(3-\sqrt{2})(3+\sqrt{2})}\\ &= \frac{3+\sqrt{2}}{9+3\sqrt{2}-3\sqrt{2}-2}+\frac{3-\sqrt{2}}{9+3\sqrt{2}-3\sqrt{2}-2}\\ &= \frac{3+\sqrt{2}}{7}+\frac{3-\sqrt{2}}{7}\\ &= \frac{6}{7}\end{aligned}$$

Try the NRICH problems The root of the problem and Ab Surd Ity. There are also lots of problems involving surds on the Underground Mathematics website, including Nested Surds.

Quadratic equations and functions

The "standard" quadratic equation is $ax^2+bx+c=0$. Some of the ways of solving it include:

- Factorisation: rewriting the quadratic equation so that it is the product of two brackets and then setting each of them equal to zero in turn, e.g.

$$\begin{aligned}2x^2+x-3&=0\\ (2x+3)(x-1)&=0\\ \implies x&=-\tfrac{3}{2} \text{ or } x=1\end{aligned}$$

- Completing the square: forcing the equation into the form $(x+p)^2=q$. This can be a very useful technique if trying to show that something is always positive, or for finding the maximum or minimum of something.

$$\begin{aligned}2x^2-7x+3&=0\\ 2\left[x^2-\tfrac{7}{2}x\right]&=-3\\ 2\left[\left(x-\tfrac{7}{4}\right)^2-\left(\tfrac{7}{4}\right)^2\right]&=-3\\ 2\left(x-\tfrac{7}{4}\right)^2-2\times\tfrac{49}{16}&=-3\\ 2\left(x-\tfrac{7}{4}\right)^2&=\tfrac{25}{8}\\ \left(x-\tfrac{7}{4}\right)^2&=\tfrac{25}{16}\\ x-\tfrac{7}{4}&=\pm\tfrac{5}{4}\\ x&=\tfrac{7}{4}\pm\tfrac{5}{4}\end{aligned}$$

- The Quadratic Formula[3]: This is the result of completing the square on the general quadratic equation $ax^2 + bx + c = 0$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

You might like to use this NRICH Proof Sorter activity to see how the quadratic formula is derived.

Care is needed when some of the coefficients have negative signs, and also if you are looking at a general equation such as $x^2 + ax = b$. The bit under the the square root $(b^2 - 4ac)$ is called the *discriminant* and the sign of this tells us the number of real roots of the equation. If $b^2 - 4ac > 0$ there are two real roots, if $b^2 - 4ac = 0$ then there is one real (repeated) root and if $b^2 - 4ac < 0$ then there are no real roots.

You may like to try NRICH's Power Quady and Underground Mathematics' Discriminating.

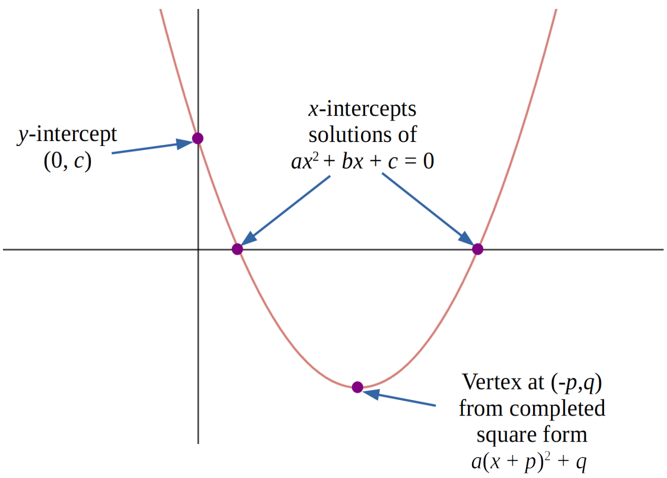
All quadratic functions have the same basic shape (a *parabola*), and if the coefficient of x^2 is positive then we have a "happy parabola", and if it is negative then we have a "sad parabola". The key points of the graph can be found via the various ways of expressing the quadratic.

The y **intercept** can be read off from the expanded form $y = ax^2 + bx + c$; the y intercept is $(0, c)$.

The x **intercepts** (if there are any) are given by the solutions to the quadratic equation $ax^2 + bx + c = 0$. If there is just one solution then the curve only meets the x axis at one point, and the x axis is a tangent to the curve at this point (the curve "touches" the x axis). In this situation the quadratic will be a perfect square, i.e. we can write it in the form $(Ax + B)^2$. Here the vertex of the curve will lie on the x axis.

Completing the square will show where the vertex of the curve is. By writing the quadratic in the form $a(x + p)^2 + q$, and by using the fact that $(x + p)^2$ cannot be negative, we have:

- If $a > 0$ then $a(x + p)^2 + q \geqslant q$ and so parabola has a minimum at $(-p, q)$
- If $a < 0$ then $a(x + p)^2 + q \leqslant q$ and so parabola has a maximum at $(-p, q)$



[3] This can be sung to the tune of "pop goes the weasel".

Quadratic functions are symmetrical about the vertex. To show this consider two points $x_1 = -p-\epsilon$ and $x_2 = -p + \epsilon$ (i.e. two points with x values equally spaced around $x = -p$).

Using the completed square form $y = a(x + p)^2 + q$ we have:

$$\begin{aligned} y_1 &= a[(-p-\epsilon)+p]^2 + q = a\epsilon^2 + q \\ y_2 &= a[(-p+\epsilon)+p]^2 + q = a\epsilon^2 + q \end{aligned}$$

and so we have $y_1 = y_2$ and hence the curve is symmetrical about the line $x = -p$.

The Greek letter ϵ (*epsilon*) is often used to stand for a small quantity.

You can investigate quadratic curves and their properties with this Underground Mathematics activity.

Inequalities

Inequalities can be *strict*, $<$ or $>$ or they can be *non-strict*, $\leqslant$ or $\geqslant$. When comparing two numbers strict inequalities make more sense (you would write $3 < 5$ rather than $3 \leqslant 5$).

Adding or subtracting the same thing to both sides leaves the inequality the same (in a similar way to equations), but more care is needed if you want to multiply or divide both sides.

For example, we know that $-2 < 3$ but if we multiply both sides by -4 we get $8 > -12$, i.e. the inequality has changed sign. If you multiply or divide both sides by a negative value then the inequality "flips". It is best to try to avoid multiplying both sides by something that might, or might not, be negative such as $x - 1$.

Example: find the range of values of x such that $2x - 3 \leqslant 2 + x < 5 - x$.

When you have a set of inequalities like this, the usual method is to split them into two separate inequalities and solve independently.

$$\begin{aligned} 2x - 3 &\leqslant 2 + x \\ x - 3 &\leqslant 2 \\ x &\leqslant 5 \end{aligned}$$

and

$$\begin{aligned} 2 + x &< 5 - x \\ 2 + 2x &< 5 \\ x &< \frac{3}{2} \end{aligned}$$

This means that we need both $x \leqslant 5$ **and** $x < \frac{3}{2}$ to be true hence we need to have $x < \frac{3}{2}$.

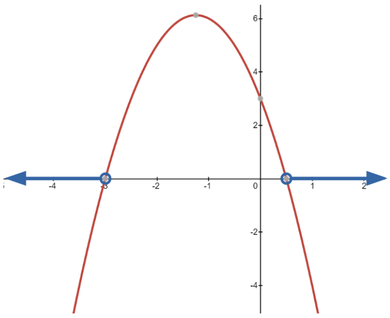
Another thing to be careful about is squaring both sides of an inequality. If both sides are positive then squaring both sides will leave the inequality sign unchanged, but if both sides are negative the sign will flip. If one side is positive and one negative then the sign might flip, or it might not depending on the magnitude of each side.

To solve a quadratic inequality it is often best to draw a sketch.

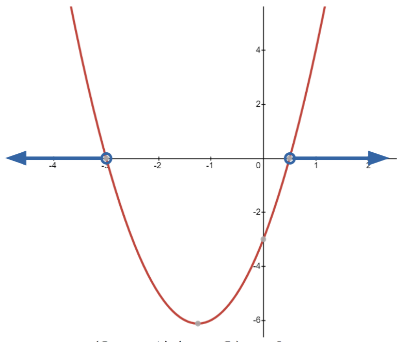
Example: solve $3 - 5x - 2x^2 < 0$.

The critical points here are where the curve meets the x axis, i.e. where it changes from being negative to positive and vice-versa. These can be found by factorising to get $(1 - 2x)(x + 3) < 0$ or, if you first rearrange the inequality to $2x^2 + 5x - 3 > 0$ you get $(2x - 1)(x + 3) > 0$.

Your sketch should look like one of the following:



$(1 - 2x)(x + 3) < 0$
$x < -3$ or $x > {}^1/_2$



$(2x - 1)(x + 3) > 0$
$x < -3$ or $x > {}^1/_2$

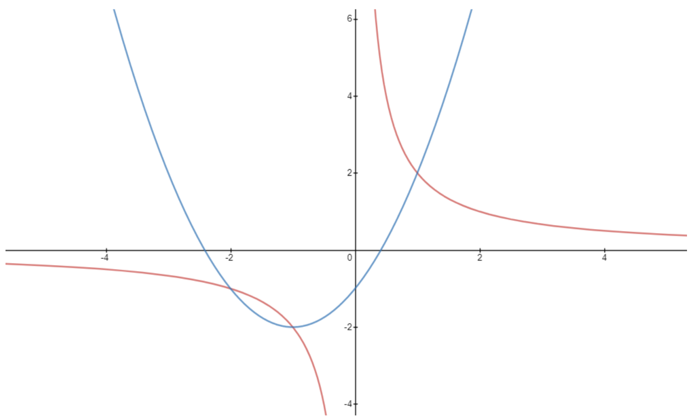
You can practice solving quadratic inequalities with these Underground Mathematics activities: Inequalities for some occasions and When are these quadratic inequalities true together?.

Example: Solve $x^2 + 2x - 1 < \dfrac{2}{x}$

In this case it is very tempting to multiply throughout by x to get $x^3 + 2x^2 - x - 2 < 0$, however this would only be true when x is positive so you would in fact need to solve $x^3 + 2x^2 - x - 2 < 0$ for $x > 0$ and $x^3 + 2x^2 - x - 2 > 0$ for $x < 0$.

A perhaps safer method is to sketch a graph showing the two sides of the original inequality, i.e. sketch $y = \frac{2}{x}$ and $y = x^2 + 2x - 1$. The curve $y = \frac{2}{x}$ is a basic reciprocal graph and the other curve is a parabola. By completing the square we have $x^2 + 2x - 1 = (x + 1)^2 - 2$ and so the vertex of the parabola is at $(-1, -2)$ and it will intersect the y axis at $(0, -1)$.[4]

Your sketch should look something like:



[4] You can also find where it intersects the x axis but that isn't actually useful here!

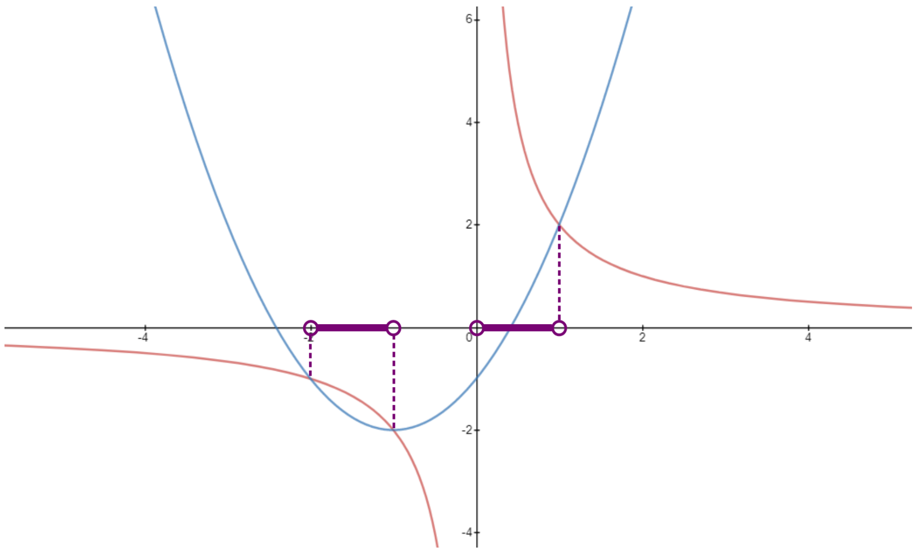
Note that I have drawn the curves intersecting at three distinct points. To see if that is true we need to solve $x^2 + 2x - 1 = \dfrac{2}{x}$.

$$\begin{aligned} x^2 + 2x - 1 &= \frac{2}{x} \\ x^3 + 2x^2 - x - 2 &= 0 \\ (x-1)(x^2+3x+2) &= 0 \\ (x-1)(x+1)(x+2) &= 0 \end{aligned}$$

Therefore the curves intersect at $x = -2, x = -1$ and $x = 1$.

To solve the cubic I have used the *Factor Theorem* to find one root and then used *Polynomial Division* to write the cubic as a product of a linear and quadratic factor. See the Polynomials section for more details on these.

The final stage is to identify in which regions the curve $y = x^2 + 2x - 1$ lies *below* $y = \dfrac{2}{x}$. The graph below shows how I could use my sketch graph to indicate these regions:



Therefore the solution to $x^2 + 2x - 1 < \dfrac{2}{x}$ is $-2 < x < -1$ **or** $0 < x < 1$.[5]

An alternative method is to solve $x^2 + 2x - 1 = \dfrac{2}{x}$ and use this to split the range of x into regions ($x < -2$, $-2 < x < -1$ etc.). You can then pick a value of x in each region ($x = -3$, $x = -1.2 \cdots$) and then use these to see if the inequality is true or not in each region.

You can use Desmos to help make sure that your solutions are correct. This page shows the two curves and the regions where $x^2 + 2x - 1 < \dfrac{2}{x}$ is true, whereas this one shows what happens if you solve $x^3 + 2x^2 - x - 2 < 0$ instead. By comparing the two you can see that they agree on the region for positive x, but are the opposite of each other for negative x.

This Underground Mathematics activity explores the relationships between graphs and inequalities.

[5]Note the use of "or" here, "and" implies that both of the inequalities are true at the same time which is impossible. This is similar to the solutions of $x^2 = 9$ being $x = 3$ or $x = -3$.

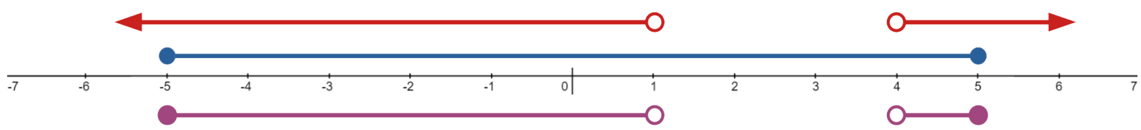
Example: find the range of x for which $5x - 7 < x^2 - 3 \leqslant 22$.

Start by splitting the inequality up into two parts:

$$\begin{aligned} 5x - 7 &< x^2 - 3 \\ 0 &< x^2 - 5x + 4 \\ 0 &< (x-1)(x-4) \\ x < 1 \quad &\text{or} \quad x > 4 \end{aligned}$$

$$\begin{aligned} x^2 - 3 &\leqslant 22 \\ x^2 - 25 &\leqslant 0 \\ (x-5)(x+5) &\leqslant 0 \\ -5 \leqslant x &\leqslant 5 \end{aligned}$$

We need to find the range of x for which these are both true. A quick diagram can help:



In the above diagram $-5 \leqslant x \leqslant 5$ is shown in blue and $x < 1$ or $x > 4$ is shown in red (note the empty and solid circles to show the difference between the strict and non-strict inequalities). The region where both of the inequalities are true is shown in purple. Hence the range of x is:

$$-5 \leqslant x < 1 \quad \text{or} \quad 4 < x \leqslant 5$$

This Underground Mathematics activity explores an inequality involving square roots.

Algebraic fraction manipulation

The same rules of manipulation apply for both numerical and algebraic fractions such as:

- Multiplying the numerator and denominator by the same fraction leaves the fraction unchanged *this can be very useful!*
- You can multiply two fractions together by multiplying the numerators and denominators
- To divide by a fraction, multiply by the reciprocal of the divisor
- To add (or subtract) two algebraic fractions you need to find a common denominator. Your life will be easiest if you find the lowest possible one.

Multiplying the top and bottom by the same thing can often make your life easier.

Example: Which is bigger, $\frac{\frac{2}{3}}{5}$ or $\frac{2}{\frac{3}{5}}$?[6]

Multiply the top and bottom of the first fraction by 3, and the second fraction by 5 to get:

$$\frac{\frac{2}{3}\times 3}{5\times 3}=\frac{2}{15}$$
$$\frac{2\times 5}{\frac{3}{5}\times 5}=\frac{10}{3}$$

and from this it is easy to see that the second fraction is the larger one.

Example:
Simplify $\frac{1}{1+\frac{1}{1+\frac{1}{x}}}$. Start by multiplying top and bottom of the "innermost" fraction by x.

$$\begin{aligned}\frac{1}{1+\frac{1}{1+\frac{1}{x}}}&=\frac{1}{1+\frac{x}{x(1+\frac{1}{x})}}\\&=\frac{1}{1+\frac{x}{x+1}}\\&=\frac{x+1}{x+1+x}\\&=\frac{x+1}{2x+1}\end{aligned}$$

This is an example of a *continued fraction.* You can explore these further with the NRICH Comparing Continued Fractions and the Underground Mathematics Staircase Sequences and Staircase Sequences Revisited problems.

Example: Write $\frac{1}{x+1}-\frac{x}{x^2+1}+\frac{1}{x^2-1}$ as a single fraction.

The first thing to notice is that $x^2-1=(x+1)(x-1)$ and so the lowest common denominator is $(x^2+1)(x-1)(x+1)$. This means we have:

$$\begin{aligned}&\frac{1}{x+1}-\frac{x}{x^2+1}+\frac{1}{x^2-1}\\=&\frac{(x^2+1)(x-1)}{(x^2+1)(x-1)(x+1)}-\frac{x(x-1)(x+1)}{(x^2+1)(x-1)(x+1)}+\frac{x^2+1}{(x^2+1)(x-1)(x+1)}\\=&\frac{(x^3-x^2+x-1)-x(x^2-1)+(x^2+1)}{(x^2+1)(x-1)(x+1)}\\&=\frac{\cancel{x^3}-\cancel{x^2}+x-\cancel{1}-\cancel{x^3}+x+\cancel{x^2}+\cancel{1}}{(x^2+1)(x-1)(x+1)}\\&=\frac{2x}{(x^2+1)(x-1)(x+1)}\end{aligned}$$

You might like to have a try at Underground Mathematics' Can we simplify these algebraic fractions? and Frightening function problems.

[6] LaTeX makes it clear which fraction is 'inside" the other fraction. Make sure that your handwriting also makes it clear.

Polynomials

A *polynomial* is a function which is a collection of terms of the form $a_i x^i$, where i is a non-negative integer, i.e. each term is a power of x multiplied by a constant.

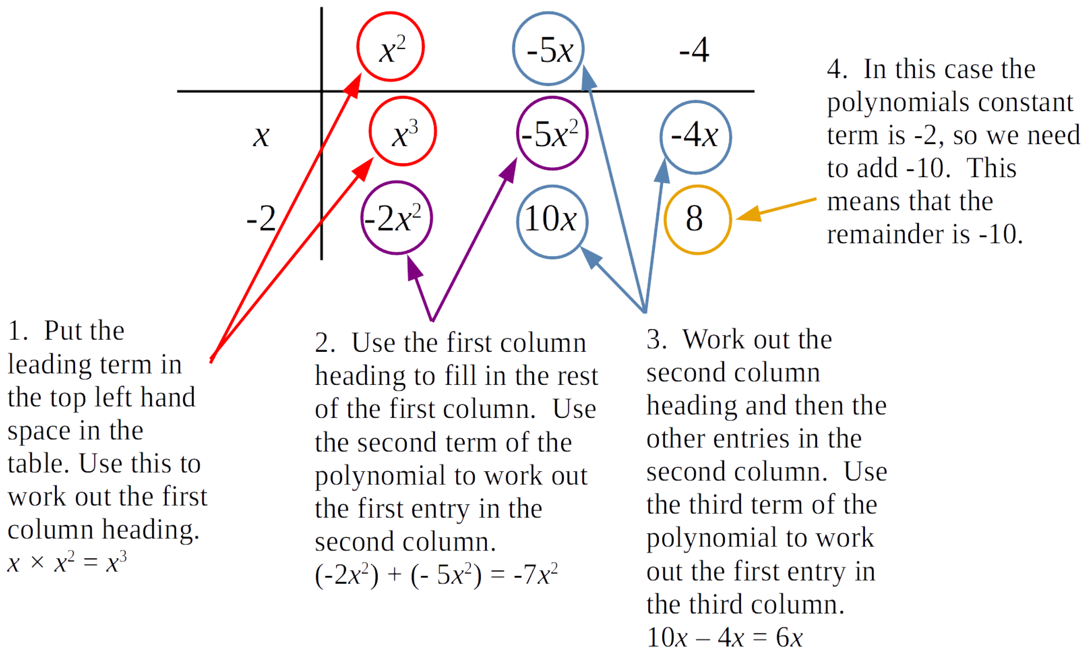
A general polynomial looks like:

$$\mathrm{f}(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0 = \sum_{i=0}^{n} a_i x^i$$

The highest power of x which appears in a polynomial is the *degree* or *order* of the polynomial - so a cubic is a polynomial of order 3 etc.

You might need to divide a polynomial by another polynomial. There are various ways to do this including *Polynomial long division* and *Synthetic division* (both of these can be googled - some people really like synthetic division, I personally can never remember the steps). I tend to use use a method where I use a combination of inspection and using a table “backwards”.

Example: Divide $x^3 - 7x^2 + 6x - 2$ by $x - 2$.



This means that when $x^3 - 7x^2 + 6x - 2$ is divided by $x - 2$ then the *quotient* is $x^2 - 5x - 4$ and the *remainder* is -10. Equivalently we can write $x^3 - 7x^2 + 6x - 2 = (x^2 - 5x - 4)(x - 2) - 10$. The same idea can be used when dividing by quadratics, or higher order polynomials. Whichever method you use, it is usually a good idea to multiply out your brackets to check your answer.

This NRICH problem involves factorising a polynomial.

If we take a polynomial $\mathrm{f}(x)$ then we can write:

$$\mathrm{f}(x) = (ax+b)\mathrm{q}(x) + r$$

Where $\mathrm{q}(x)$ is the quotient (and will have an order one less than that of $\mathrm{f}(x)$) and r is the remainder — since we are dividing by a linear function r will be a constant.

Substituting $x = -\frac{b}{a}$ gives:

$$\begin{aligned}
\mathrm{f}\left(-\tfrac{b}{a}\right) &= \left(-\tfrac{b}{a} \times a + b\right)\mathrm{q}(x) + r \\
\mathrm{f}\left(-\tfrac{b}{a}\right) &= 0 \times \mathrm{q}(x) + r \\
\mathrm{f}\left(-\tfrac{b}{a}\right) &= r
\end{aligned}$$

This gives us two useful theorems about polynomials:

- **The Remainder Theorem** when $\mathrm{f}(x)$ is divided by $ax+b$ the remainder is $\mathrm{f}\left(-\frac{b}{a}\right)$
- **The Factor Theorem** $(ax+b)$ is a factor of $\mathrm{f}(x)$ if and only if $\mathrm{f}\left(-\frac{b}{a}\right) = 0$

To prove the Factor Theorem you should really be a little more careful (as it is an if and only if). A slightly more robust proof would be:

If $(ax+b)$ is a factor of $\mathrm{f}(x)$ then we have $\mathrm{f}(x) = (ax+b)\mathrm{q}(x)$ and substituting $x = -\frac{b}{a}$ gives $\mathrm{f}\left(-\frac{b}{a}\right) = 0$.

Dividing $\mathrm{f}(x)$ by $(ax+b)$ gives $\mathrm{f}(x) = (ax+b)\mathrm{q}(x) + r$. Substituting $x = -\frac{b}{a}$ gives $\mathrm{f}\left(-\frac{b}{a}\right) = r$ and so if $\mathrm{f}\left(-\frac{b}{a}\right) = 0$ this implies that $r = 0$ and so $(ax+b)$ is a factor of $\mathrm{f}(x)$.

A potentially useful consequence of the Remainder and Factor theorems is that if $\mathrm{f}(a) = p$ then $(x-a)$ is a factor of $\mathrm{f}(x) - p$.

To solve an equation of the form $\mathrm{f}(x) = 0$ where $\mathrm{f}(x)$ is a polynomial you can use the factor theorem to find one solution and then use polynomial division to extract a factor, and repeat!

These Underground Mathematics problems use the factor theorem and/or the remainder theorem:

- Can we find a?
- $(x^2+1)(x+1) = (a^2+1)(a+1)$
- Dividing by $x-k$
- Divisible by x^2-1?

Two useful results are:

- $a^3 - b^3 \equiv (a-b)(a^2+ab+b^2)$
- $a^3 + b^3 \equiv (a+b)(a^2-ab+b^2)$

These facts are useful when solving the Underground Mathematics Ab-surder! problem.

Roots of polynomials

If we consider a cubic polynomial with roots α, β and γ then it can be written as $\mathrm{f}(x) = a(x-\alpha)(x-\beta)(x-\gamma)$. Expanding the brackets and simplifying gives:

$$a(x-\alpha)(x-\beta)(x-\gamma) = ax^3 - a(\alpha+\beta+\gamma)x^2 + a(\alpha\beta+\beta\gamma+\gamma\alpha)x - a\alpha\beta\gamma$$

If we compare this expansion to $\mathrm{f}(x) = ax^3 + bx^2 + cx + d$ we can see that:

$$\begin{aligned} \alpha+\beta+\gamma = \sum \alpha &= -\frac{b}{a} \\ \alpha\beta+\beta\gamma+\gamma\alpha = \sum \alpha\beta &= \frac{c}{a} \\ \alpha\beta\gamma &= -\frac{d}{a} \end{aligned}$$

These relationships are sometimes called *Vieta's Formulae* after François Viète.

For a more general polynomial of degree n with roots $r_1, r_2, \cdots, r_n$ (where some, or all, of these roots might be complex, or repeated) we have:

$$a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = a_n(x-r_1)(x-r_2)\cdots(x-r_n)$$

For this general polynomial we have:

$$\begin{aligned} \sum r_i &= -\frac{a_{n-1}}{a_n} \\ \sum_{i<j} r_i r_j &= \frac{a_{n-1}}{a_n} \\ \sum_{i<j<k} r_i r_j r_k &= -\frac{a_{n-2}}{a_n} \\ \cdots r_1 r_2 \cdots r_{n-1} r_n &= (-1)^n \frac{a_0}{a_n} \end{aligned}$$

The restrictions $i<j$ and $i<j<k$ are used so that products are not double counted, so r_1r_2 is counted but not r_2r_1.

This Underground Mathematics problem involves roots of a cubic.

Rational functions

A function of the form $\mathrm{f}(x) = \dfrac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ where both $\mathrm{P}(x)$ and $\mathrm{Q}(x)$ are polynomials is called a rational function. It is a *proper* rational function if the degree of the numerator is less than the degree of the denominator.

If the rational function is not *proper* then you can use polynomial division to re-write the function in the form $\mathrm{f}(x) = \mathrm{P}_1(x) + \dfrac{\mathrm{P}_2(x)}{\mathrm{Q}(x)}$. The example below shows one possible way of doing this.

Example:

$$\begin{aligned}
\frac{x^3}{(x+1)(x-2)} &= \frac{x^3}{x^2-x-2} \\
&= \frac{x(x^2-x-2)+x^2+2x}{x^2-x-2} \\
&= x + \frac{x^2+2x}{x^2-x-2} \\
&= x + \frac{1(x^2-x-2)+3x+2}{x^2-x-2} \\
&= x + 1 + \frac{3x+2}{x^2-x-2} \\
&= x + 1 + \frac{3x+2}{(x+1)(x-2)}
\end{aligned}$$

It is a good idea to check your answer by rewriting $x + 1 + \dfrac{3x+2}{(x+1)(x-2)}$ as a single fraction.

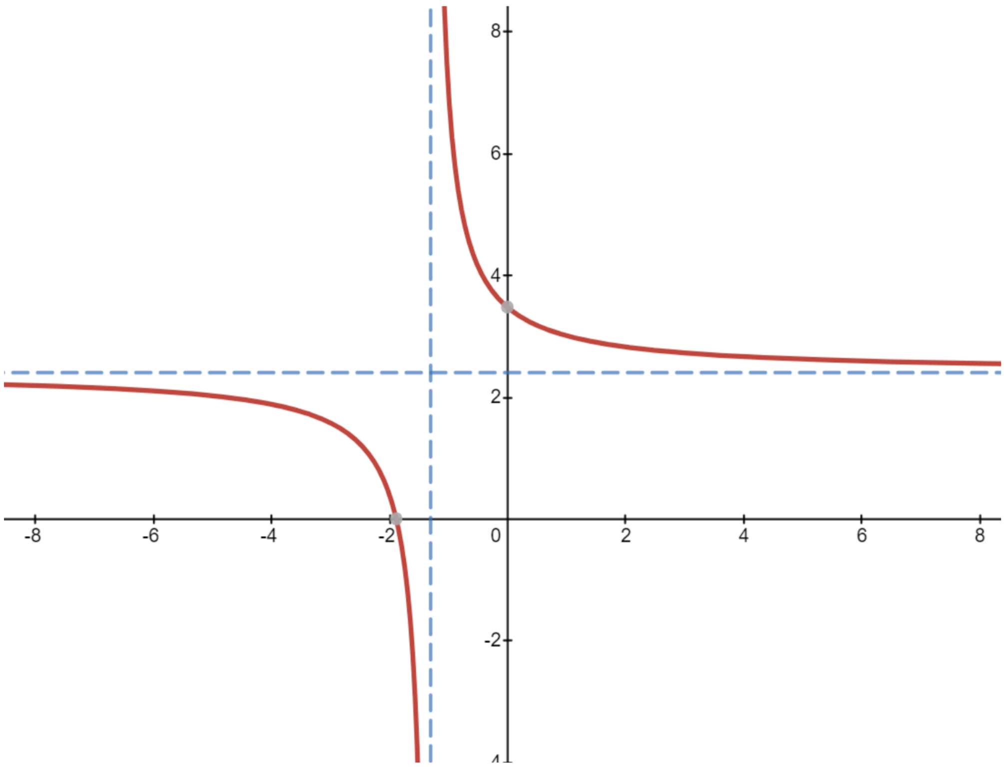
Sketching rational functions

Consider a rational function of the form $\mathrm{f}(x) = a + \dfrac{b}{x+c}$, where $b > 0$.

As x gets very large and positive $(x \to +\infty)$, $\dfrac{b}{x+c}$ gets very small. Since we have taken b to be positive this term is also positive. This means that as $x \to +\infty$, $\mathrm{f}(x) \to a_+$; which means that $\mathrm{f}(x)$ tend to a "from above". Similarly, as $x \to -\infty$, $\dfrac{b}{x+c} \to 0$ and this term is negative. Hence we have $\mathrm{f}(x) \to a_-$ as $x \to \infty$, i.e. $\mathrm{f}(x)$ tend to a "from below". There is a horizontal asymptote at $y = a$.

As x gets close to $-c$ the $\dfrac{b}{x+c}$ term gets very large. If $x < -c$ then the term is negative and if $x > -c$ then this term is positive. We can write; as $x \to -c_-$, $\mathrm{f}(x) \to -\infty$ and as $x \to -c_+$, $\mathrm{f}(x) \to +\infty$. There is a vertical asymptote at $x = -c$.

This gives us all the information we need to sketch the graph, which should look something like this:



You might like to use Desmos with some sliders to see what the affects of varying a, b and c are. See this page for an example.

An alternative way of sketching this graph is to consider it as a sequence of translations of the graph $\mathrm{f}(x) = \dfrac{1}{x}$.

This Underground Mathematics problem uses a sketch of a rational function to help solve an inequality. You might also like to try this Underground Mathematics card sort activity.

Partial fractions

It can sometimes be helpful to split a rational function up into two (or more) separate fractions.

Example: Express $\dfrac{5x-1}{(x+3)(x-5)}$ in partial fractions.

$$\begin{aligned}\frac{5x-1}{(x+3)(x-5)} &= \frac{A}{x+3} + \frac{B}{x-5} \\ &= \frac{A(x-5)}{(x+3)(x-5)} + \frac{B(x+3)}{(x-5)(x+3)} \\ &= \frac{(A+B)x + (3B-5A)}{(x+3)(x-5)}\end{aligned}$$

Equating coefficients gives $A + B = 5$ and $3B - 5A = -1$. Substituting for B gives:

$$\begin{aligned} 3B - 5A &= -1 \\ 3(5 - A) - 5A &= -1 \\ 15 - 3A - 5A &= -1 \\ 8A &= 16 \end{aligned}$$

This gives $A = 2$ and so $B = 3$. This gives $\dfrac{5x-1}{(x+3)(x-5)} = \dfrac{2}{x+3} + \dfrac{3}{x-5}$.

The standard forms for partial fractions are:

$$\begin{aligned} \frac{\mathrm{f}(x)}{(x-\alpha)(x-\beta)(x-\gamma)} &= \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{x-\gamma} \\ \frac{\mathrm{f}(x)}{(x-\alpha)(x-\beta)^2} &= \frac{A}{x-\alpha} + \frac{B}{x-\beta} + \frac{C}{(x-\beta)^2} \\ \frac{\mathrm{f}(x)}{(x-\alpha)(x^2+\beta)} &= \frac{A}{x-\alpha} + \frac{Bx+c}{x^2+\beta} \qquad \text{where } \beta > 0 \end{aligned}$$

In all of these cases $\mathrm{f}(x)$ has degree less than or equal to 2. If you have an *improper* fraction then you can use polynomial division to reduce the numerator to have degree less than the denominator.

This Underground Mathematics problem uses partial fractions to evaluate a sum. If you have met $\mathrm{i} = \sqrt{-1}$ then you might like to investigate this NRICH problem.

Logarithms

A *Logarithm* tells you the value power or index that you need to apply to a base number to get a certain value. This will probably be clearer with an example: you need to multiply five "2's" together to get 32, so the logarithm of base 2 of 32 is 5, or:

$$2^5 = 32 \implies \log_2(32) = 5$$

Logarithms do not have to have integer values, for example $9^{\frac{1}{2}} = 3 \implies \log_9(3) = \frac{1}{2}$. In general we have:

$$b^x = y \implies x = \log_b(y)$$

You can investigate logarithms to base 10 with this NRICH interactivity.

Similarly to the laws of indices, there are laws of logarithms which are very closely linked.

$$\begin{aligned} \log_b 1 &= 0 \\ \log_b b &= 1 \\ \log_b(xy) &= \log_b(x) + \log_b(y) \\ \log_b\left(\frac{x}{y}\right) &= \log_b x - \log_b y \\ \log_b x^a &= a \log_b x \end{aligned}$$

For the first two, note that $b^0 = 1 \implies \log_b 1 = 0$ and $b^1 = b \implies \log_b b = 1$.

For the third one, let $\log_b x = s$ and let $\log_b y = t$. This means that we have $b^s = x$ and $b^t = y$.

$$\begin{aligned}
xy &= b^s \times b^t \\
xy &= b^{s+t} \\
\log_b(xy) &= \log_b\left(b^{s+t}\right) \\
\log_b(xy) &= s + t \\
\log_b(xy) &= \log_b x + \log_b y
\end{aligned}$$

The derivation of the forth result runs along the same lines, starting with $\dfrac{x}{y} = b^s \div b^t$. The last result can be derived by repeated application of the third result.

Change of base With modern calculators which can find a logarithm to any base this is perhaps a slightly less important result than it once was. However it is still worth knowing (especially as TMUA/MAT/STEP do not allow the use of calculators). We have:

$$\log_b x = \frac{\log_c x}{\log_c b}$$

With a slightly more old-fashioned calculator this was used to calculate logs using logarithms to be base 10 (which is the key marked "log" or "lg"). Using the change of base formula we have:

$$\log_b x = \frac{\log x}{\log b} \quad \text{where } \log = \log_{10}$$

To show the change of base formula start by writing $\log_b x = y$:

$$\begin{aligned}
\log_b x &= y \\
x &= b^y \\
\log_c x &= \log_c\left(b^y\right) \\
\log_c x &= y \log_c b \\
\frac{\log_c x}{\log_c b} &= y
\end{aligned}$$

If we take the change of base formula and substitute $c = x$ we get:

$$\begin{aligned}
\log_b x &= \frac{\log_c x}{\log_c b} \\
\log_b x &= \frac{\log_x x}{\log_x b} \\
\log_b x &= \frac{1}{\log_x b}
\end{aligned}$$

Logarithms can be used to solve equations where the unknown is a power.

Example: Solve $5^x = 8$:

$$\begin{aligned} 5^x &= 8 \\ \log(5^x) &= \log 8 \\ x \log 5 &= \log 8 \\ x &= \frac{\log 8}{\log 5} = 1.292 \text{ (to 3 d.p.)} \end{aligned}$$

For more on logarithms check out these Underground Mathematics resources:

- Logarithm Lineup
- Changing Bases
- How far did the earth move?
- 1950s Calculators

Binomial expansions

A *binomial expression* is the sum of two terms, such as $a + 2b$, $x - y$, etc.

Consider $(x + y)^n$ where n is a positive integer. If we want to find the coefficient of $x^r y^{n-r}$ in the expansion of $(x + y)^n = (x + y)(x + y) \cdots (x + y)$ then we need to find the number of ways of *choosing* r of the n brackets to contribute an x in the product (and then the rest of the brackets will be contributing a y).

There are n choices for which bracket we select first, then $n - 1$ choices for the second one, etc. So the number of ways of selecting r brackets *where order matters* is:

$$n(n-1)(n-2)(n-r+1) = \frac{n!}{(n-r)!}$$

However the order doesn't matter (it doesn't matter if you multiply the x's in the 3^{rd}, 7^{th} and then 4^{th} brackets or if you do multiply them in the order $4^{\text{th}}, 3^{\text{rd}}, 7^{\text{th}}$). The total number of ways of choosing the x brackets is therefore:

$$\frac{n!}{(n-r)!r!} = {}^nC_r = \binom{n}{r}$$

The binomial expansion of $(x + y)^n$ is:

$$(x+y)^n = y^n + \binom{n}{1} xy^{n-1} + \binom{n}{2} x^2y^{n-2} + \binom{n}{3} x^3y^{n-3} + \cdots + \binom{n}{n-1} x^{n-1}y + x^n$$

A few points to note:

- $\binom{n}{r} = \binom{n}{n-r}$ as the number of ways of choosing r of the brackets to "include" is the sams as the number of ways of choosing r of the brackets to exclude.
- $\binom{n}{0} = \binom{n}{n} = 1$ as there is only one way to choose all of the brackets, or none of the brackets. In order for this to be consistent with the definition of $\binom{n}{r} = \dfrac{n!}{(n-r)!r!}$ we need $0! = 1$.
- $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n-1} + \binom{n}{n} = (1+1)^n = 2^n$. This result comes from taking $x = y = 1$ in the above expansion of $(x+y)^n$.[7]

This NRICH article discusses binomial coefficients and how they are connected to "choices".

Here are some Underground Mathematics problems involving binomial coefficients and expansions:

- Tennis
- Coefficient of x^2
- Coefficient of x^3y^5
- Binomial Coefficient Relationships

[7] This explains why the sums of the rows of Pascal's Triangle are equal to powers of 2.