STEP Support Programme

A Level Support - Introduction to Calculus

General comments

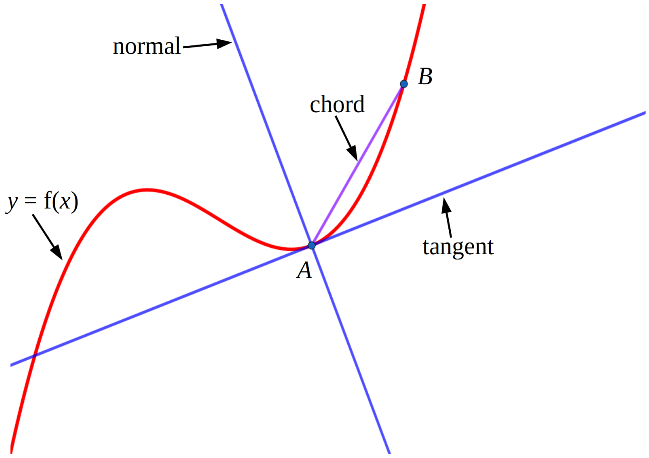
This is a short(ish) introduction to some of the A-level calculus topics.

There are some suggested NRICH and Underground Mathematics problems at various places in this document. You can find many more by visiting https://undergroundmathematics.org/ and https://nrich.maths.org/. You might find the search facilities useful to find problems and articles on specific topics.

Please send any corrections or comments to step@maths.org.

The Gradient of a Curve

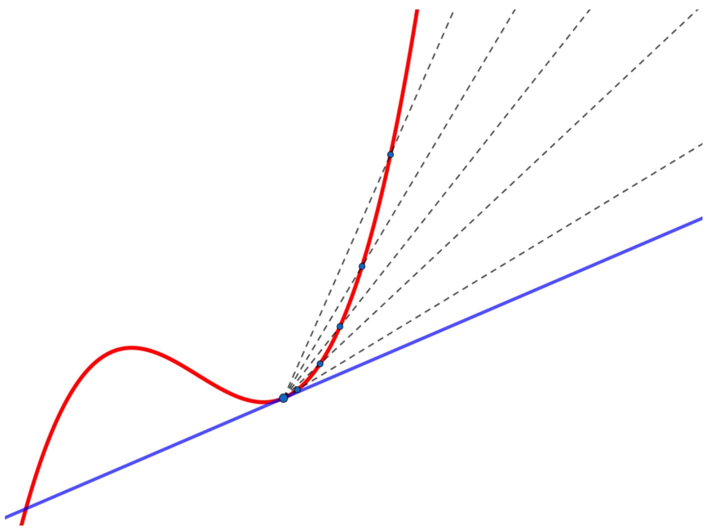
Some definitions



The above sketch shows a curve $y = \mathrm{f}(x)$. At point A on the curve the line that *touches* the curve at this point has been drawn. This means that the curve and the line meet at this point, and they are heading in the same direction — this line is called the *tangent* to the curve at the point A. The *gradient* of the curve at point A is equal to the gradient of the tangent at this point.

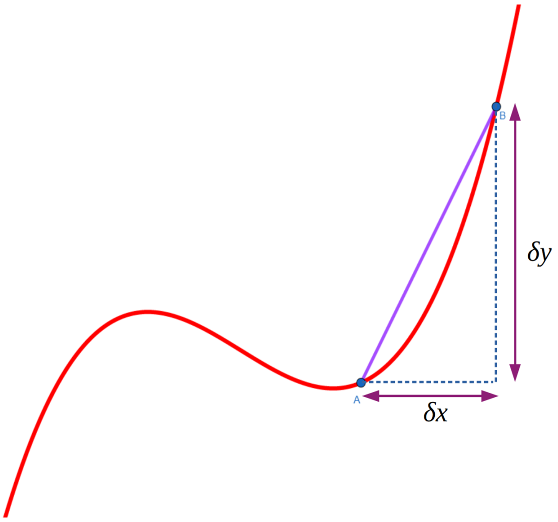
Another way of thinking of the point where a line touches a curve is that it is the limiting case between the line intersecting the curve twice near the point and not intersecting it at all. If you moved the tangent line in the above example a little bit it would either intersect the curve twice near A or it would not meet the curve near A at all. If you equate the equations of the curve and line then the point where they touch corresponds to a repeated root of the equation.

The line that passes through point A on the curve and is perpendicular to the tangent is called the *normal* to the curve at point A. A line joining two distinct points A and B on the curve is called a *chord*. As the point B gets closer to A then the gradient of the chord tends to the gradient of the tangent, as shown in the picture below:



You can explore how the tangent to the curve changes as the point A moves around the curve, and how the gradient of the chord tends to the gradient of the tangent as the end points of the chord get closer by using this GeoGebra diagram.

The gradient of a curve at a point can be found by considering what happens to the gradient of the chord as the end points get closer together. Consider two points on a curve $y = \mathrm{f}(x)$. The horizontal distance between then is δx and the vertical distance between them is δy.[1]



The gradient of the chord is given by $\dfrac{\delta y}{\delta x}$ and as δx gets smaller, δy gets smaller. The gradient of the curve at a point is given by:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

You cannot evaluate $\dfrac{\delta y}{\delta x}$ when $\delta x = 0$, as then the fraction would be undefined, but you can look at the behaviour of the fraction as δx gets smaller and smaller. This is called a *limit*, and the notation $\lim\limits_{\delta x \to 0}$ means "take the limit as δx tends to 0".

Note that $\dfrac{\delta y}{\delta x}$ is a fraction whereas $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is **not** a fraction.

Alternatively, we can consider the two points on the curve to be at $\big(x, \mathrm{f}(x)\big)$ and $\big(x+h, \mathrm{f}(x+h)\big)$. Then the gradient of the chord between them is $\dfrac{\mathrm{f}(x+h) - \mathrm{f}(x)}{(x+h) - x}$ and we have:

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h} \qquad (*)$$

There are two different notations used here for the derivative function. The notation $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is called Leibniz's notation, and the notation $\mathrm{f}'(x)$ was first used by Joseph Louis Lagrange in some of his unpublished works. Other notations include the *dot notation* used by Isaac Newton.

[1] The Greek letter δ is often used to represent a small change in a quantity, so δx is a small change in x. Note that δ is a prefix rather than a parameter in it's own right, so δx is a single parameter.
You cannot cancel the δs in $\dfrac{\delta y}{\delta x}$!

Example: Use $(*)$ to find the derivative of $y = x^3$.
We have:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\
&= \lim_{h\to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h} \\
&= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\
&= 3x^2
\end{aligned}$$

You might like to use this method to find the derivatives of $y = x^2$ and $y = x^n$. For the second one you can use the fact that $(x+h)^n = x^n + nx^{n-1}h + O(h^2)$. There is a note about "big O" notation at the bottom of this page.[2]

Example: Use $(*)$ to find the derivative of $y = x^{-2}$ where $x \neq 0$.
We have:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^{-2} - x^{-2}}{h} \\
&= \lim_{h\to 0} \left(\frac{1}{h(x+h)^2} - \frac{1}{hx^2} \right) \\
&= \lim_{h\to 0} \left(\frac{x^2 - (x+h)^2}{hx^2(x+h)^2} \right) \\
&= \lim_{h\to 0} \left(\frac{-2xh - h^2}{hx^2(x+h)^2} \right) \\
&= \lim_{h\to 0} \left(\frac{-2x - h}{x^2(x+h)^2} \right) \\
&= \frac{-2x}{x^4} \qquad \text{Since } x \neq 0 \text{ and } \lim_{h\to 0} (x+h)^2 = x^2 \\
&= -2x^{-3}
\end{aligned}$$

In general if $y = x^n$ then $\dfrac{dy}{dx} = nx^{n-1}$. This holds for all real values of n.

Differentiation is *linear*, which means we have:

- If a is a constant then $\dfrac{d}{dx}\big(af(x)\big) = a \times \dfrac{d}{dx}f(x)$
- $\dfrac{d}{dx}\big(f(x) + g(x)\big) = \dfrac{d}{dx}f(x) + \dfrac{d}{dx}g(x)$

This means we can differentiate linear combinations of terms such as:

$$\frac{d}{dx}\left(x^5 - 2x^3 + 3x^{-1} - x^{-3}\right) = 5x^4 - 6x^2 - 3x^{-2} + 3x^{-4}$$

Try applying some of these ideas to this Underground Mathematics problem.

[2] The notation $O(h^2)$ is saying "here there are terms in h^2 and even smaller bits (such as h^3 etc) and we can lump them all together". More formally we can say $f(h) = O(h^2)$ if $\dfrac{f(h)}{h^2}$ tends to a (non-zero) constant as h tends to zero. There is also a "little o" notation which is $f(h) = o(h^n)$ if $\lim_{h\to 0} \dfrac{f(h)}{h^n} = 0$. We could write $(x+h)^n = x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + o(h^2)$, i.e. we would need the h^2 term included before $o(h^2)$.

Stationary Points

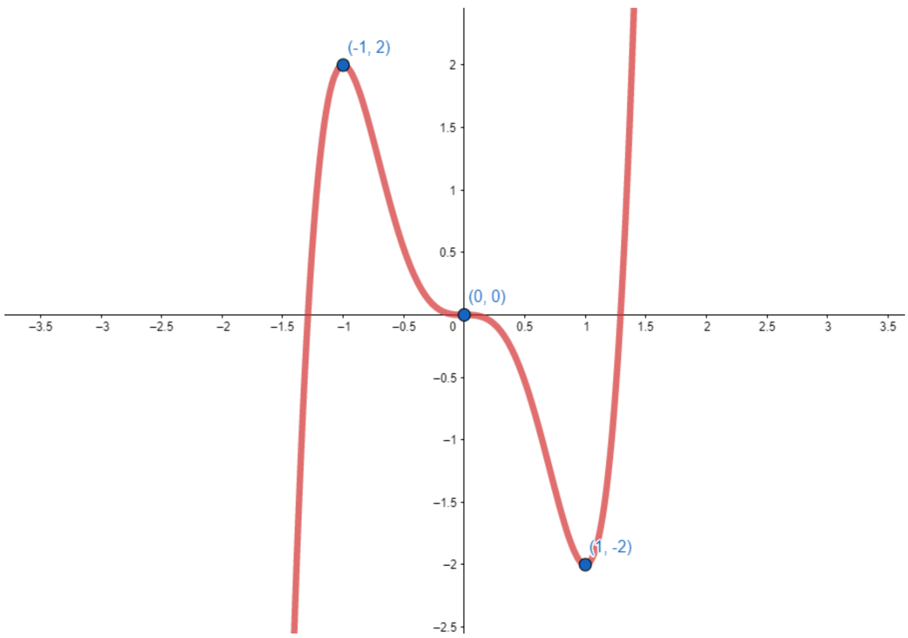
Stationary points on a curve are where the gradient is equal to 0, i.e. points where $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Example: find the stationary points of $y = 3x^5 - 5x^3$.
Differentiating gives:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 15x^4 - 15x^2 \\ 0 &= 15x^2(x^2 - 1) \\ 0 &= x^2(x+1)(x-1) \\ \implies x &= -1, x = 0, x = 1 \end{aligned}$$

Substituting these values for x into $y = 3x^5 - 5x^3$ gives the stationary points as being at $(-1, 2)$, $(0, 0)$ and $(1, -2)$.

Sketching the graph (which you could do using Desmos or GeoGebra) shows these three stationary points.



The point $(-1, 2)$ is a local maximum, the point $(0, 0)$ is a stationary point of inflection[3] and $(1, -2)$ is a local minimum. Local maxima and minima are sometimes called *turning points*.

This GeoGebra page shows how the gradient of the curve changes as you move along it. Note that the gradient of the curve is equal to zero at each of the stationary points, and for the point $(0, 0)$ there is a local maximum of the gradient. There are also two local minima of the gradient, these correspond to *non-stationary points of inflection* on the original curve.

There is more about points of inflection in this STEP Support Programme assignment.

[3] You can also get non-stationary points of inflection!

At local maxima, the gradient of the curve is changing from being positive to being negative, and at a local minimum the gradient is changing from being negative to being positive. This means that the gradient at a maximum is *decreasing*, and so "gradient of the gradient" is negative. The gradient at a minimum is *increasing* and the "gradient of the gradient" is positive.

The "gradient of the gradient" is the *second derivative*, $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$, and is found by differentiating the gradient of a curve twice. We can sometimes use the second derivative to help us determine the nature of stationary points.

- If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ then the gradient is decreasing and the curve has a **maximum** stationary point
- If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$ then the gradient is increasing and the curve has a **minimum** stationary point

If $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ then the stationary point could be any of the three types (maximum/minimum/point of inflection) and you need a different approach to classify the stationary point. A couple of ways of doing this are:

- Look at the sign of the gradient either side of the stationary point. If the gradient goes from positive to negative then it is a maximum, if it goes from negative to positive then it is a minimum and if it stays the same sign either side of the stationary point then it is a point of inflection.
- Look at how the y-coordinate changes either side of the stationary point by looking at points slightly to each side. For example, if $y = 3x^5 - 5x^3$ then considering the points with $x = 0.9, x = 1$ and $x = 1.1$ gives points with coordinates $(0.9, -1.87353)$, $(1, -2)$ and $(1.1, -1.82347)$. This shows that if you travel a little bit either side of $(1, -2)$ the y-coordinate increases slightly and so $(1, -2)$ is a minimum.

Try looking at the points on $y = 3x^5 - 5x^3$ with x-coordinates equal to -0.1, 0, and 0.1 and see if you can use these to show that $(0, 0)$ is a point of inflection.

You might like to try Underground Mathematics' Gradient Match and Floppy Hair.

Optimisation

You can (sometimes!) use the derivative to find maximum or minimum values. Care is needed as the maximum/minimum value of a function in a given range might not be at a local minimum/maximum. Sketching a graph can often be a good way of making sure your answer is sensible.

Example: A piece of paper measure 20 cm by 30 cm. Equal sizes squares are cut out of each corner and the sides turned up to make an open box. What is the maximum volume of this box?

Let the cut out squares have side length x. This means that the width of the box is $30 - 2x$ and then depth of the box is $20 - 2x$. The height of the box is x. The volume (V) of the box is given by:

$$V = x(30 - 2x)(20 - 2x)$$

Expanding and differentiating gives:

$$\begin{aligned} V &= 4x^3 - 100x^2 + 600x \\ \frac{\mathrm{d}V}{\mathrm{d}x} &= 12x^2 - 200x + 600 \end{aligned}$$

Equating the derivative to 0 and solving for x gives:

$$\begin{aligned} x &= \frac{200 \pm \sqrt{40000 - 4 \times 12 \times 600}}{24} \\ &= \frac{200 \pm \sqrt{11200}}{24} \\ &= \frac{25 \pm \sqrt{175}}{3} \\ &= \frac{25 \pm 5\sqrt{7}}{3} \end{aligned}$$

We seem to have two answers, but thinking back to the problem we must have $x \leqslant 10$ (as the original piece of paper had width 20cm) and if we take the positive root we get $x = \dfrac{25 + 5\sqrt{7}}{3} > \dfrac{25 + 5}{3} = 10$. Hence only the negative root makes sense. To make sure that it is a maximum we could consider the second derivative $\dfrac{\mathrm{d}^2V}{\mathrm{d}x^2} = 24x - 200$. When $x = \dfrac{25 - 5\sqrt{7}}{3}$ we have:

$$\begin{aligned} \frac{\mathrm{d}^2V}{\mathrm{d}x^2} &= 24 \times \left(\frac{25 - 5\sqrt{7}}{3}\right) - 200 \\ &= 8(25 - 5\sqrt{7}) - 200 \\ &= \cancel{200} - 40\sqrt{7} - \cancel{200} < 0 \end{aligned}$$

Since the second derivative is negative then the stationary point must be a local maximum.

If the second derivative was equal to zero then you would need to use another method to verify that the stationary point is a maximum. A couple of methods have been mentioned in the previous section, but you could also sketch a graph of the function. The volume $V = 4x^3 - 100x^2 + 600x$ is a cubic function with roots at $x = 0$, $x = 10$ and $x = 15$ (which are fairly obvious roots when you think about the problem!). These roots, and the fact that it is a "happy" cubic, are enough to sketch the curve and then you can see that the stationary point between $x = 0$ and $x = 10$ is a maximum.

Underground Mathematics' Maximising Volume activity involves a similar problem.

Tangents and Normals

You can use the gradient of a curve and equations of straight lines to find the equations of tangents (and normals - see the diagram on page 2) to the curve.

Example: A curve has equation $y = x(x+1)(x-1)$.

(a) Find the equation of the tangent at the point $(-1, 0)$. Find the coordinates of the point where this tangent meets the curve again.

(b) Find the equation of the normal at the point $(-1, 0)$. Show that this normal does not intersect the curve again.

It is often a good idea to sketch the curve first. This curve crosses the x axis at $x = -1, 0, 1$ and it is a "happy" cubic.

(a)

Expanding the equation for the curve gives $y = x^3 - x$ and differentiating this gives $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 1$. Therefore when $x = -1$ the gradient of the curve is equal to $3 \times (-1)^2 - 1 = 2$.
Using $y - y_1 = m(x - x_1)$ the equation of the tangent is $y - 0 = 2(x - -1) \implies y = 2x + 2$.

To find where the tangent meets the curve again we can equate y expressions and solve $2x + 2 = x^3 - x$. Rearranging gives:

$$\begin{aligned} x^3 - 3x - 2 &= 0 \\ (x+1)(x^2 - x - 2) &= 0 \\ (x+1)(x+1)(x-2) &= 0 \\ (x+1)^2(x-2) &= 0 \end{aligned}$$

A couple of points to note here. I could have used the factor theorem to show that $\mathrm{f}(-1) = 0$, but since the straight line is a tangent to the curve when $x = -1$ I already know that $x = -1$ must be a root and hence $(x+1)$ is a factor (which I can divide out by inspection). Note that I get a repeated root at $x = -1$, this is because at this point the line is a tangent[4]. Sometimes it is possible for a tangent to meet a curve again — a common misconception is that a tangent touches a curve once and does not meet it again at all.

The point where the tangent intersects with the curve again is when $x = 2$ and so $y = 2^3 - 2 = 6$. The coordinates are therefore $(2, 6)$.

(b)

The gradient of the normal to the curve when $x = -1$ will be $-\frac{1}{2}$ and so the equation is $y - 0 = -\frac{1}{2}(x+1) \implies y = -\frac{1}{2}x - \frac{1}{2}$. To find any other points of intersection solve $-\frac{1}{2}x - \frac{1}{2} = x^3 - x$.

$$\begin{aligned} x^3 - \frac{1}{2}x + \frac{1}{2} &= 0 \\ 2x^3 - x + 1 &= 0 \\ (x+1)(2x^2 - 2x + 1) &= 0 \end{aligned}$$

[4]Sometimes a helpful way of finding where tangents are is to search for repeated roots

The roots of the quadratic are $x = \dfrac{2 \pm \sqrt{4-8}}{4}$ and hence there are no more real roots and the normal does not meet the curve again.

This GeoGebra page shows the cubic with a tangent and normal at the point $(-1, 0)$. You can move the point along the curve and see when the tangent intersects the curve again and when it doesn't.

If you consider the equation of the tangent at a general point on the curve, $(t, t^3 - t)$ you can show that the tangent always intersects with the curve at one other point. You can also show that normal will intersect the curve again when $-\frac{1}{\sqrt{3}} < t < \frac{1}{\sqrt{3}}$, but this is a little trickier.

You might like to explore the Underground Mathematics resource Tangent or normal.

Reversing Differentiation

Consider the functions $y_1 = x^3 - x$, $y_2 = x^3 - x + 3$, $y_3 = x^3 - x - 47$. All three of these have the same gradient function: $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 1$. Reversing this, we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 1 \implies y = x^3 - x + c$$

where "c" is a constant that can be found if you know one point on the curve.

This Geogebra page shows a cubic of the form $y = x^3 - x + c$ and the gradient function (*derivative*) of that cubic. You can change c to see how the curve and gradient function are related.

In general we have:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x^n \implies y = \frac{1}{n+1}x^{n+1} + c \quad \text{for } n \neq -1$$

where c is a constant (it is often called the *constant of integration*).

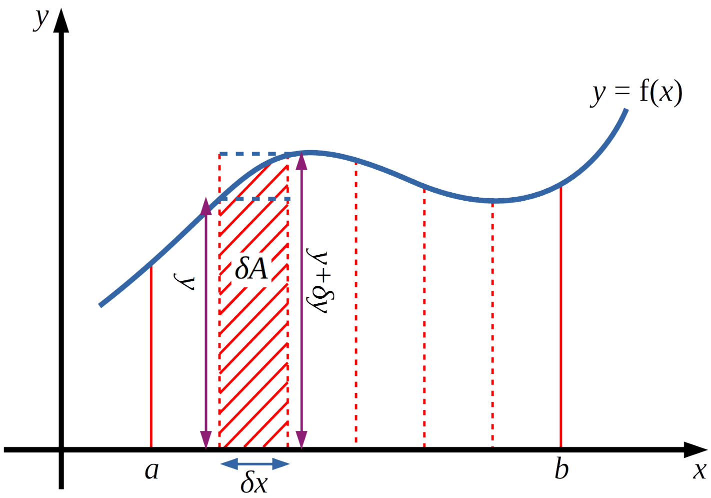
Note that we need to state that $n \neq -1$. There is a result that holds when $n = -1$, but it does not follow this rule. The case when $n = -1$ will be covered in another set of notes.

You might like to try this Underground Mathematics problem.

Reversing differentiation is sometimes called "Anti-differentiation".

Area Under a Curve

The sketch below shows a continuous curve $y = \mathrm{f}(x)$. Consider the area between the curve $y = \mathrm{f}(x)$ and the x axis, and the vertical lines $x = a$ and $x = b$. We can consider the area to be the sum of a lot of small[5] strips.



If we look at the shaded strip which has a width of δx and an area of δA and approximate it with a rectangle we have:

$$\delta A \approx y\,\delta x \tag{*}$$

Returning to $(*)$ we can re-write this to get:

$$\frac{\delta A}{\delta x} \approx y$$

Then, taking the limit as δx tends to 0 we have:

$$\begin{aligned} \lim_{\delta x \to 0} \frac{\delta A}{\delta x} &= y \\ \frac{\mathrm{d}A}{\mathrm{d}x} &= y \end{aligned}$$

This then becomes a “reverse differentiation” problem like those in the previous section.

The process of reversing differentiation is called *Integration*, and we can write:

$$\frac{\mathrm{d}A}{\mathrm{d}x} = y \quad \Longrightarrow \quad A = \int y\,\mathrm{d}x$$

The symbol $\int$ was introduced by Gottfried Wilhelm Leibniz and is based on the “long s” symbol as Leibniz thought of the integral as being an infinite sum of infinitely small terms.

[5]The strips in the diagram are not very small, in order to aid clarity!

Revisiting an example from the previous sections we have:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= 3x^2 - 1 \\ \implies \quad y &= \int 3x^2 - 1 \,\mathrm{d}x \\ \implies \quad y &= x^3 - x + c \end{aligned}$$

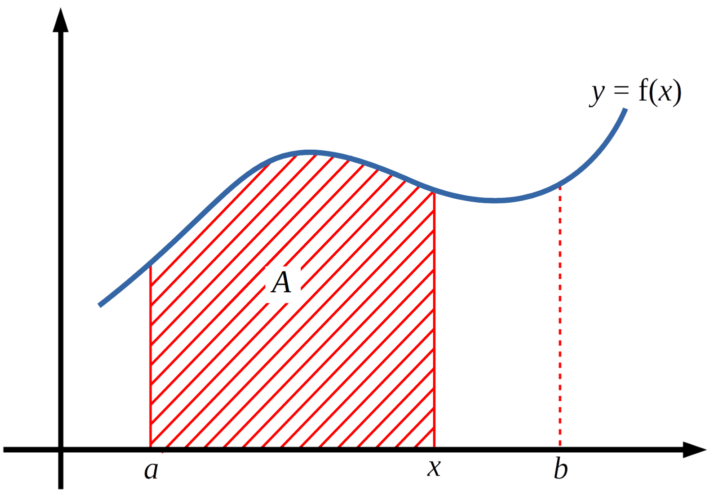
The notation "$\int 3x^2 - 1\,\mathrm{d}x$" means "the integral of $3x^2 - 1$ with respect to x". Sometimes the function you are trying to integrate is not written in terms of the variable you want to integrate with respect to — these sorts of situations will be discussed later in another set of notes.

To represent the area under a curve between $x = a$ and $x = b$ we can introduce *limits* on the integral to show where we start and stop measuring the area from. We can write:

$$A = \int_a^b y \,\mathrm{d}x$$

To evaluate the above integral we must think a little more about what it means.

Consider the area under the curve $y = \mathrm{f}(x)$, and let A be the area under the graph from a to x (where x can vary!). Let the integral of $\mathrm{f}(x)$ be $\mathrm{F}(x) + c$, so we have $A = \mathrm{F}(x) + c$.



When $x = a$ the area between a and x is equal to 0. Hence when $x = a$ we have $A = 0$. This gives:

$$\begin{aligned} 0 &= \mathrm{F}(a) + c \\ c &= -\mathrm{F}(a) \end{aligned}$$

Hence we have $A = \mathrm{F}(x) - \mathrm{F}(a)$, and in particular the area under the curve between a and b is $\mathrm{F}(b) - \mathrm{F}(a)$.

Note that this means that when we are considering an integral with limits (also called a *definite* integral) we do not need to include the constant of integration.

Example: Find the area enclosed by the curve $y = \sqrt{x}$, the x axis and the lines $x = 1$ and $x = 9$.

We need to find $\displaystyle\int_1^9 x^{\frac{1}{2}}\,\mathrm{d}x$:

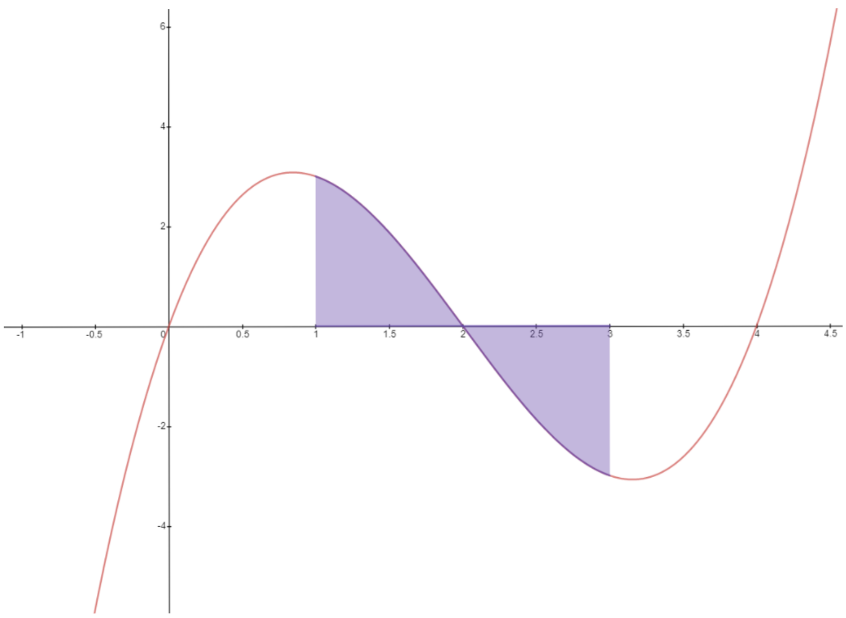
$$\begin{aligned}\int_1^9 x^{\frac{1}{2}}\,\mathrm{d}x &= \left[\tfrac{2}{3}x^{\frac{3}{2}}\right]_1^9\\ &= \tfrac{2}{3}\times 27 - \tfrac{2}{3}\times 1\\ &= 17\tfrac{1}{3}\end{aligned}$$

Example: Find the area enclosed by the curve $y = x(x-2)(x-4)$, the x axis and the lines $x = 1$ and $x = 3$.

We have:

$$\begin{aligned}\int_1^3 x(x-2)(x-4)\,\mathrm{d}x &= \int_1^3 x^3 - 6x^2 + 8x\,\mathrm{d}x\\ &= \left[\tfrac{1}{4}x^4 - 2x^3 + 4x^2\right]_1^3\\ &= \left(\frac{81}{4} - 2\times 27 + 4\times 9\right) - \left(\frac{1}{4} - 2 + 4\right)\\ &= 2\tfrac{1}{4} - 2\tfrac{1}{4} = 0\end{aligned}$$

It seems a bit surprising to have zero area, and a quick sketch suggests that the area isn't equal to 0.



If instead we look at the area between $x = 2$ and $x = 3$ we get:

$$\begin{aligned}\int_2^3 x^3 - 6x^2 + 8x \, \mathrm{d}x &= \left[\tfrac{1}{4}x^4 - 2x^3 + 4x^2\right]_2^3 \\ &= \left(\frac{81}{4} - 2 \times 27 + 4 \times 9\right) - \left(\frac{1}{4} \times 16 - 2 \times 8 + 4 \times 4\right) \\ &= 2\tfrac{1}{4} - 4 \\ &= -1\tfrac{3}{4}\end{aligned}$$

When using integration to find an area which lies *below* the x axis then this is identified with a negative sign. This means that if a curve passes below the x axis then you need to consider the regions above and and below the x axis separately[6]. In this case the area is given by:

$$\begin{aligned}\int_1^2 x(x-2)(x-4) \, \mathrm{d}x + \left| \int_2^3 x(x-2)(x-4) \, \mathrm{d}x \right| &= 4 - \left(\tfrac{1}{4} - 2 + 4\right) + \left|-1\tfrac{3}{4}\right| \\ &= 1\tfrac{3}{4} + 1\tfrac{3}{4} = 3\tfrac{1}{2}\end{aligned}$$

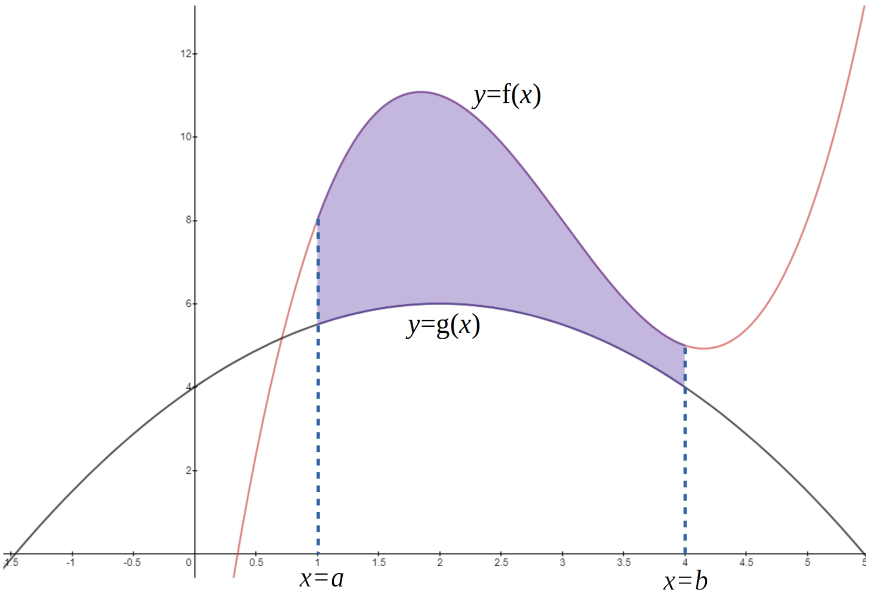
You might not have met modulus "$|x|$" signs before — these find the "size" or *magnitude* of a number, and when dealing with real numbers they remove any negative signs. Another term for "the modulus of x" is "the absolute value of x", and some calculators have an "ABS" function.

In this section we have shown that the area *enclosed* by the curve $y = \mathrm{f}(x)$, the x axis and the lines $x = a$ and $x = b$ is given by $\displaystyle\int_a^b y \, \mathrm{d}x$. If instead we wanted to find the area enclosed between $y = \mathrm{f}(x)$, the y axis and the horizontal lines $y = c$ and $y = d$ then we would to evaluate the integral $\displaystyle\int_{y=c}^{y=d} x \, \mathrm{d}y$. You would need to rearrange $y = \mathrm{f}(x)$ to get x in terms of y in order to do this (i.e. find g such that $x = \mathrm{g}(y)$).

[6] Yet another reason why sketching a graph is often a good idea!

Area Between Two Curves

Consider the area between the curves $y = \mathrm{f}(x)$, $y = \mathrm{g}(x)$ and the lines $x = a$ and $x = b$.



This area is given by $\displaystyle\int_a^b \big[\mathrm{f}(x) - \mathrm{g}(x)\big]\,\mathrm{d}x$. This formula holds as long as $\mathrm{f}(x) \geqslant \mathrm{g}(x)$ for all values of x in the integration range, even if the curves fall below the x axis. However if the curves cross at any point then the integral needs to be split up into the different regions for $\mathrm{f}(x) \geqslant \mathrm{g}(x)$ and $\mathrm{g}(x) \geqslant \mathrm{f}(x)$.

You might like to try Area between a parabola and a line and Area between a parabola and a tangent both from Underground Maths.

Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus says that for a *continuous* function $f(x)$[7]:

$$\frac{\mathrm{dF}}{\mathrm{d}x} = \mathrm{f}(x) \iff \mathrm{F}(x) - \mathrm{F}(a) = \int_a^x \mathrm{f}(t)\,\mathrm{d}t\,,$$

or, if you prefer,

$$\frac{\mathrm{dF}}{\mathrm{d}x} = \mathrm{f}(x) \iff \mathrm{F}(x) = \int \mathrm{f}(x)\,\mathrm{d}x + C\,,$$

where C is a constant. If you are not sure of this, try substituting $\mathrm{f}(x) = x^3$ and $\mathrm{F}(x) = \frac{1}{4}x^4$ and verify that it holds in this case.

Note the use of a different letter for the variable in the first integral. Here t is called a 'dummy variable': it doesn't matter what letter you use because it will disappear when you do the integral and evaluate the result at $t = x$ and $t = a$. The only letters you should not use as a dummy variable here are x and a, because they have been used as limits.

This can be used to prove some properties of definite integrals:

Property 1: $\displaystyle\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c \mathrm{f}(x)\,\mathrm{d}x = \int_a^c \mathrm{f}(x)\,\mathrm{d}x$

$$\begin{aligned}\int_a^b \mathrm{f}(x)\,\mathrm{d}x + \int_b^c \mathrm{f}(x)\,\mathrm{d}x &= \big[\mathrm{F}(b) - \mathrm{F}(a)\big] + \big[\mathrm{F}(c) - \mathrm{F}(b)\big]\\ &= \mathrm{F}(c) - \mathrm{F}(a)\\ &= \int_a^c \mathrm{f}(x)\,\mathrm{d}x\end{aligned}$$

Property 2: $\displaystyle\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$

$$\begin{aligned}\int_a^b \mathrm{f}(x)\,\mathrm{d}x &= \mathrm{F}(b) - \mathrm{F}(a)\\ &= -\big[\mathrm{F}(a) - \mathrm{F}(b)\big]\\ &= -\int_b^a \mathrm{f}(x)\,\mathrm{d}x\end{aligned}$$

You might like to try these Underground Maths activities:

- Meaningful Areas (you don't need to actually evaluate any integrals!)
- Integrating, then differentiating
- Additional integrals
- Simultaneous integral equations

[7] The function only needs to be continuous over the interval in which you are integrating, and there are some non-continuous functions for which this holds.

Improper and Infinite Integrals

Consider the integrals:

1. $$\int_1^s \frac{1}{x^2}\,\mathrm{d}x \text{ where } s > 1$$
2. $$\int_r^1 \frac{1}{x^2}\,\mathrm{d}x \text{ where } 0 < r < 1$$
3. $$\int_1^s \frac{1}{\sqrt{x}}\,\mathrm{d}x \text{ where } s > 1$$
4. $$\int_r^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x \text{ where } 0 < r < 1$$

For the first two we have $\displaystyle\int x^{-2}\,\mathrm{d}x = \left[-x^{-1}\right](+c)$ and for the second two we have $\displaystyle\int x^{-\frac{1}{2}}\,\mathrm{d}x = \left[2x^{-\frac{1}{2}}\right](+c)$. This gives the four definite integrals:

1. $$\int_1^s \frac{1}{x^2}\,\mathrm{d}x = -\frac{1}{s} - (-1) = 1 - \frac{1}{s}$$
2. $$\int_r^1 \frac{1}{x^2}\,\mathrm{d}x = -1 - \left(-\frac{1}{r}\right) = \frac{1}{r} - 1$$
3. $$\int_1^s \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2\sqrt{s} - 2$$
4. $$\int_r^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2 - 2\sqrt{r}$$

Now consider what happens as $s \to \infty$ and $r \to 0$. For the first two integrals we have:

$$\lim_{s\to\infty} \int_1^s \frac{1}{x^2}\,\mathrm{d}x = \lim_{s\to\infty}\left[1 - \frac{1}{s}\right] = 1$$

and

$$\lim_{r\to 0} \int_r^1 \frac{1}{x^2}\,\mathrm{d}x = \lim_{r\to 0}\left[\frac{1}{r} - 1\right] = ??$$

For this second integral we can see that as $r \to 0$, the value gets larger and larger, and when $r = 0$ it is undefined. This Geogebra file shows the areas under $y = \dfrac{1}{x^2}$ with sliders for r and s, and you can use this to help you visualise the limits as $s \to \infty$ and $r \to 0$. If you right click on the sliders you can change the settings so that you can investigate certain ranges of r and/or s in more detail.

For the next two integrals we have:

$$\lim_{s\to\infty}\int_1^s \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{s\to\infty}\left[2\sqrt{s}-2\right] = ??$$

and

$$\lim_{r\to 0}\int_r^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{r\to 0}\left[2-2\sqrt{r}\right] = 2$$

In this case it is the first integral which is undefined and the second interval which converges to a limit. This Geogebra file illustrates the areas in these cases. Like in the previous one, you can right click on the sliders and change the settings.

An integral of the form $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1$ is called an *infinite* integral.

Often people will write something along the lines of $\displaystyle\int_1^\infty \frac{1}{x^2}\,\mathrm{d}x = 1 - \frac{1}{\infty} = 1 - 0 = 1$ which makes some mathematicians wince, but is basically shorthand for letting $s \to \infty$. Probably best not to write this during an interview to study maths at university!

An integral of the form $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = 2$ is called an *improper* integral, as at a value in the range of integration the integrand is undefined. You do need to be careful and check for undefined values within the range of integration, otherwise you might end up with nonsensical results like $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,\mathrm{d}x = 0$.