STEP Support Programme

A Level Support - Coordinate Geometry

General comments

This is a short(ish) introduction to some of the A-level coordinate geometry topics.

There are some suggested NRICH and Underground Mathematics problems at various places in this document. You can find many more by visiting https://undergroundmathematics.org/ and https://nrich.maths.org/. You might find the search facilities useful to find problems and articles on specific topics.

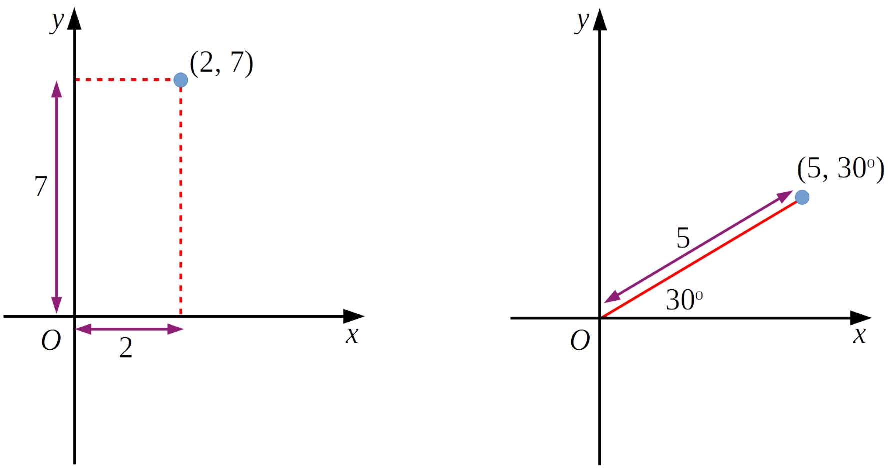
Please send any corrections or comments to step@maths.org.

Coordinate systems

There are various ways that you can define the position of a point in a plane.

- **Cartesian coordinates:** These are an ordered[1] set of numbers — such as $(2, 7)$ — which describe the left-right and down-up position of a point from a fixed point (the *origin*). These are named after René Descartes. A general pair of Cartesian coordinates can be written as (x, y).
- **Polar coordinates:** These are an ordered set of numbers — such as $(5, 30°)$ — which describe the distance "as the bird flies" of a point from a fixed point (the origin) and the angle that the direct path to the point makes with the positive horizontal axis. A general pair of polar coordinates can be written as (r, θ).

The diagram below illustrates the difference between these two systems.



Using Pythagoras' theorem and standard trigonometry ratios we have:

$$r = x^2 + y^2$$
$$\tan\theta = \frac{y}{x}$$

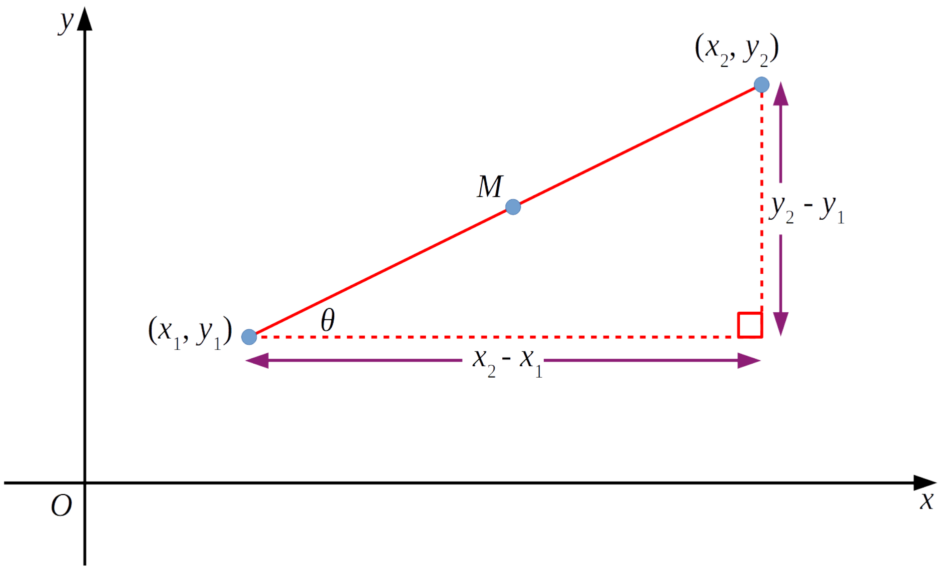
Cartesian coordinates of the form (x, y, z) can be used to describe the position of points in 3D space, as can Cylindrical Coordinates and Spherical Coordinates.

In this document we will be using Cartesian Coordinates.

[1] This just means that the order is important, i.e. $(2, 7)$ is different to $(7, 2)$.

Straight line coordinate geometry

The diagram below shows two points with Cartesian coordinates (x_1, y_1) and (x_2, y_2) in a plane.



The horizontal distance between the two points is given by $x_2 - x_1$ and the vertical distance is $y_2 - y_1$. By Pythagoras' theorem the distance, d, between the two points satisfies

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$$

The point M is the midpoint of the line-segment connecting (x_1, y_1) and (x_2, y_2). The x component of the coordinates of M is given by $x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$. A similar result holds for the y coordinate, and so the position of M is given by:

$$\left(\tfrac{1}{2}[x_1 + x_2], \tfrac{1}{2}[y_1 + y_2]\right)$$

The gradient of the line connecting the two points is given by:

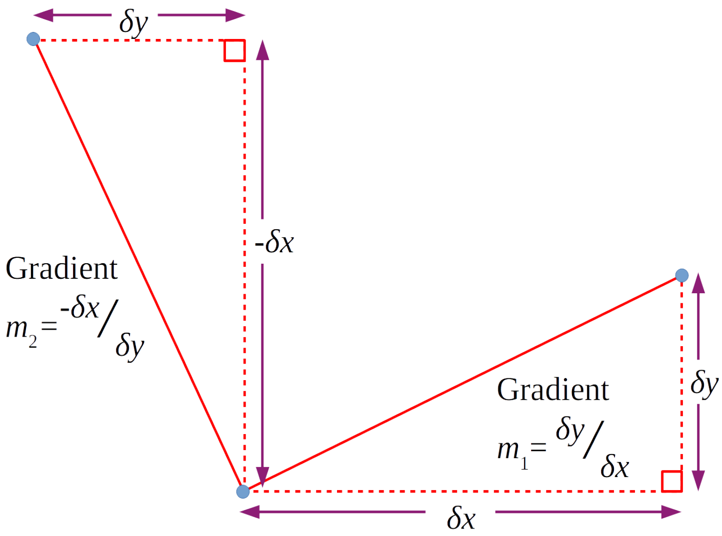
$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Alternatively, if the angle the line makes with the horizontal is θ, then the gradient is given by $m = \tan\theta$.

With all of the above formulae, care must be taken especially when the coordinates involve negative numbers.

Perpendicular lines

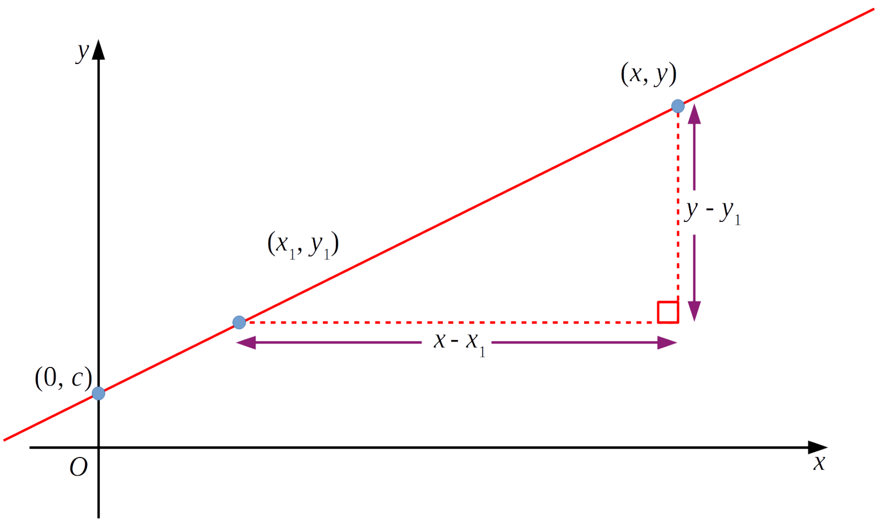
If we take a line (or line segment) with gradient $m_1 = \dfrac{\delta y}{\delta x}$ and rotate it by $90°$, then we get a diagram that looks a bit like this:



I have chosen to rotate the line segment anti-clockwise, but rotating it clockwise is equally valid.

The gradient of the rotated segment is given by $m_2 = \dfrac{-\delta x}{\delta y} = -\dfrac{1}{m_1}$, i.e. the gradient of the perpendicular line segment is the negative reciprocal of the gradient of the original line segment. Sometimes this condition is written as $m_1 m_2 = -1$.

The equation of a straight line



Take a straight line with a *fixed* point (x_1, x_2) and a *variable* point (x, y). The gradient between the two points satisfies:

$$\frac{y - y_1}{x - x_1} = m$$
$$\implies y - y_1 = m(x - x_1)$$

The equation of a straight line with gradient m passing through a point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

If we know that the line passes through $(0, c)$ (i.e. the y intercept is c), then the equation of the straight line becomes $y - c = m(x - 0) \implies y = mx + c$, which you may have seen before.

If we don't know the gradient of the line, but know two points that it passes through, then you can use these two points to find the gradient, m, and then use either one of the two given fixed points in $y - y_1 = m(x - x_1)$.[2]

Example: A triangle has vertices at $A(0, 4)$, $B(4, 2)$ and $C(1, -1)$.

(i) Find the perpendicular bisectors of AB and BC.

(ii) Find the point of intersection of these two lines.

(iii) Find the perpendicular bisector of AC and hence show that the three perpendicular bisectors meet at the same point.

The first thing you should do is draw a sketch of the three points, roughly in the correct places (but they do not have to be accurately plotted). You can then use your sketch to make sure that your answers are sensible.

(i)
The perpendicular bisector of $A(0, 4)$ and $B(4, 2)$ passes through the midpoint of AB and is perpendicular to the line segment AB. The midpoint of AB is $(2, 3)$ and the gradient of AB is $\frac{2-4}{4-0} = -\frac{1}{2}$. Therefore the gradient of the perpendicular bisector of AB is 2 and the equation is:

$$y - 3 = 2(x - 2) \implies y = 2x - 1$$

The midpoint of $B(4, 2)$ and $C(1, -1)$ is $(2.5, 0.5)$ and the gradient of line segment BC is $\frac{2--1}{4-1} = 1$. Therefore the gradient of the perpendicular bisector of BC is -1 and it has equation:

$$y - 0.5 = -1(x - 2.5) \implies y = -x + 3$$

(ii)
When these two lines meet we have:

$$\begin{aligned} 2x - 1 &= -x + 3 \\ 3x &= 4 \\ x &= \tfrac{4}{3} \\ y &= \tfrac{5}{3} \end{aligned}$$

(iii)
The midpoint of AC is $(0.5, 1.5)$ and the gradient of the line segment is $\frac{-1-4}{1-0} = -5$. Therefore the gradient of the perpendicular bisector is $\frac{1}{5}$ and the equation of the line is:

$$y - 1.5 = \frac{1}{5}(x - 0.5) \implies y = \frac{1}{5}x + \frac{7}{5}$$

[2] There is a standard equation for this situation, $\dfrac{y - y_1}{x - x_1} = \dfrac{y_2 - y_1}{x_2 - x_1}$, but personally I find this too much to try and remember, so I use $y - y_1 = m(x - x_1)$ instead.

To check to see if the three lines pass through the same point it is sufficient to check that the point of intersection of two of them is a point on the third line. So we can substitute the point $\left(\frac{4}{3}, \frac{5}{3}\right)$ into the equation for the bisector of AC to get:

$$\begin{aligned} y &= \frac{1}{5}x + \frac{7}{5} \\ \frac{5}{3} &= \frac{1}{5} \times \frac{4}{3} + \frac{7}{5} \quad (?) \\ \frac{5}{3} &= \frac{4}{15} + \frac{21}{15} \quad (?) \\ \frac{5}{3} &= \frac{25}{15} \quad \checkmark \end{aligned}$$

and so the point $\left(\frac{4}{3}, \frac{5}{3}\right)$ **does** lie on the third perpendicular bisector and so all three perpendicular bisectors meet at the same point.

Note the use of a tick (✓) to indicate that we have shown what was required.

The three perpendicular bisectors of the sides of a triangle **always** meet at a single point, and this point is known as the *circumcentre* of the triangle. You can draw a circle passing through the three vertices of the triangle with centre at the circumcentre — this is called the *circumcircle* of the triangle.

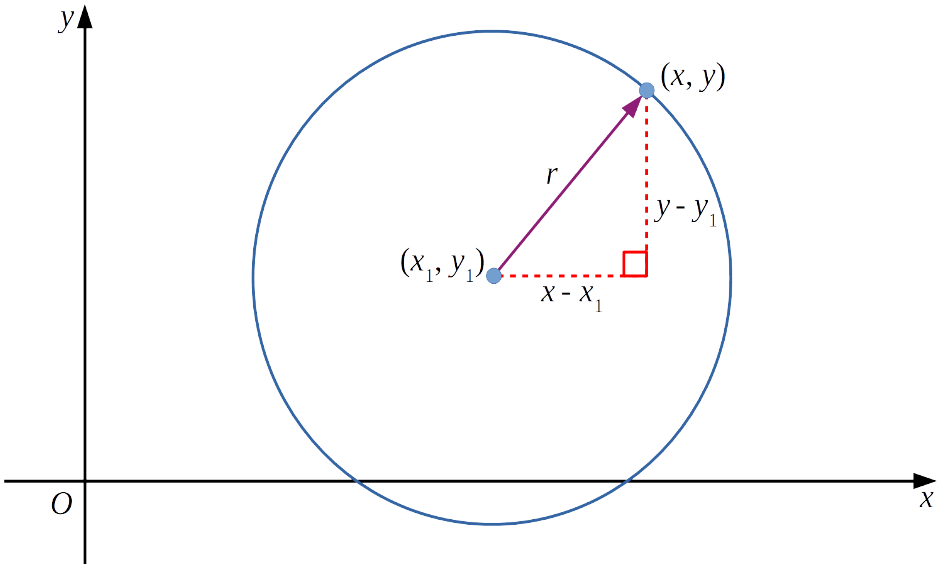
This Geogebra page shows the three points A, B and C, the perpendicular bisectors, the circumcentre and the circumcircle. You can move the vertices of the triangle and the circumcentre and circumcircle will move to match. If you like, you can *hide* the perpendicular bisectors etc, move the vertices of the triangle and then calculate the equations of the perpendicular bisectors and location of the circumcentre. You can then *show* the hidden lines and check your calculations.

You might like to look at some of these Underground Mathematics activities:

- Lots of Lines
- Straight Line Pairs
- Tilted Rectangle
- Simultaneous Inequalities
- No solutions, or infinitely many?

The equation of a circle

Consider a circle with centre at (x_1, y_1) with radius r. This means that every point on the circle is a distance of r from the centre. Consider a general point (x, y) on the circle.



Using Pythagoras' theorem we have:

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$

which is the equation of a circle with radius r and centre (x_1, y_1).

You can investigate equations of circles with the Underground Mathematics activity Teddy bear.

Example: Find the equation of the tangent to the circle with equation $x^2 + y^2 + 2x - 4y - 20 = 0$ at the point $(-4, 6)$.

Completing the square on the x and y terms gives:

$$\begin{aligned} x^2 + 2x + y^2 - 4y &= 20 \\ [x+1]^2 - 1 + [y-2]^2 - 4 &= 20 \\ (x+1)^2 + (y-2)^2 &= 25 \end{aligned}$$

therefore the centre of the circle is at $(-1, 2)$ (and the radius of the circle is 5).

The gradient between $(-4, 6)$ and $(-1, 2)$ is $\dfrac{2-6}{-1--4} = -\dfrac{4}{3}$. Therefore the gradient of the tangent to the circle must be $\dfrac{3}{4}$ as they are perpendicular.[3]

[3] This comes from one of the Circle Theorems — the tangent to a circle is perpendicular to the radius drawn at the point of contact.

The equation of the tangent is therefore:

$$\begin{aligned} y - 6 &= \frac{3}{4}(x - -4) \\ 4y - 24 &= 3x + 12 \\ 4y &= 3x + 36 \end{aligned}$$

This Desmos diagram shows the circle and tangent for this question. Drawing a sketch is always a good idea to check if your answer is sensible, for example is the gradient of the tangent is positive on your sketch then you should probably have a positive gradient in the equation of your tangent.

You might like to try these Underground Mathematics activities:

- Touching circles
- Pairs of circles
- Implicit circles