STEP Support Programme

Hints and Partial Solutions for Assignment 11

Warm-up

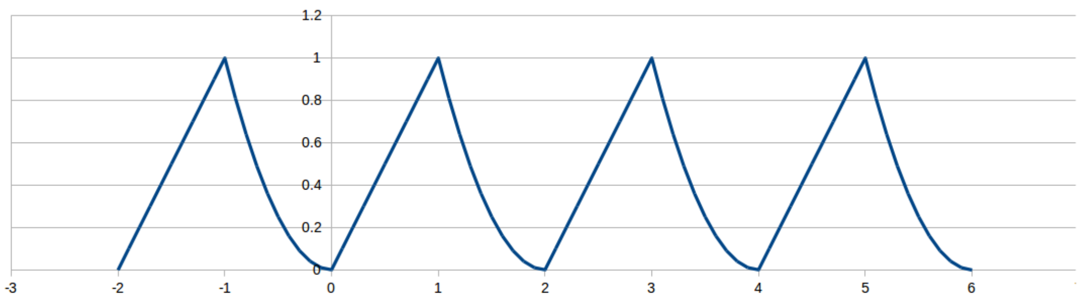
1 **(i)** $T(n)$, or T_n as it is sometimes written, is the nth *triangular number*, so called because you need $T(n)$ billiard (or snooker or pool) balls to make an equilateral triangular shape of side n balls[1]. You can see from the given recurrence formula that $T(n)$ is the sum of the first n integers.

Nothing to do with the question, but if you arrange the balls in a right-angled triangle, you can prove by means of a picture that

$$T(n) + T(n-1) = n^2.$$

You can also fit two lots of $T(n)$ together to make a rectangle, with dimensions n by $n+1$, and use this to find a formula for $T(n)$. You could use this to show that $T(14) = 105$.

(ii) This function is periodic, with period 2. It looks a bit like saw teeth, as shown below.



This graph is continuous (you can draw it without taking the pencil off the paper), but the gradient of the graph is not continuous.

[1] Or an isosceles right-angled triangle.

Preparation

2 **(i)** Writing

$$4^x = (2^2)^x = 2^{2x} = (2^x)^2 = y^2,$$

gives $y^2 - 7y - 8 = 0 \implies (y-8)(y+1) = 0$ and, apparently, two solutions for y. But only one solution is relevant, because $y = 2^x$ has to be positive. Hence we have $y = 8 = 2^3$ and so $x = 3$ (substitute $x = 3$ back into the original equation to make sure it works!).

(ii) You need to be very careful if you use squaring when solving an equation. It often means that extra "solutions" are created.

As a very simple example, consider the equation:

$$x + 1 = 2$$

which has just the one solution, $x = 1$.

However, if our first step towards solving the question was to square both sides (admittedly a pretty daft thing to do), we would get:

$$\begin{aligned} (x+1)^2 &= 4 \\ \implies x^2 + 2x + 1 &= 4 \\ \implies x^2 + 2x - 3 &= 0 \\ \implies (x+3)(x-1) &= 0 \\ \implies x = -3 \quad \text{or } x &= 1\,, \end{aligned}$$

so this way we seem to get two solutions. Of course, only one of them satisfies the original equation $x + 1 = 2$.

If you square both sides of an equation, you must make sure that you check your solutions in the **original** equation to make sure that they are indeed solutions.

To solve $\sqrt{3x-5} - \sqrt{x+6} = 1$, you could you square both sides (or take $\sqrt{x+6}$ to the other side and then square both sides). You will get a cross term (note that $(\sqrt{A} - \sqrt{B})^2 \neq A - B$ or even $A + B$) so there is still a square root in the squared equation. However, there is only one square root term, so you can isolate this on one side of the equation and square again. This should result in the equation $x^2 - 13x + 30 = 0$ which has two solutions, of which **only one** is a solution of the original equation!

One possible solution

$$\begin{aligned}
\sqrt{3x-5} &= 1+\sqrt{x+6} \\
3x-5 &= 1+(x+6)+2\sqrt{x+6} \\
2x-12 &= 2\sqrt{x+6} \\
x-6 &= \sqrt{x+6} \\
x^2-12x+36 &= x+6 \\
x^2-13x+30 &= 0 \\
(x-10)(x-3) &= 0
\end{aligned}$$

This gives us $x = 10$ or $x = 3$, but if we substitute these back into the original equation we find that only $x = 10$ satisfies it.

The STEP question (2013 STEP I Q1)

3 Substitutions work quite nicely for all three equations.

In the first case you should end up with $y = \dfrac{-3 \pm \sqrt{11}}{2}$. Then you should note that as $y = \sqrt{x}$, which is positive (or zero) by definition, only the positive solution for y makes sense, and $x = \left(\dfrac{-3+\sqrt{11}}{2}\right)^2 = \cdots$

There is one root for part **(ii) (a)**, start by substituting $y = \sqrt{x+2}$ and you should obtain $y^2 + 10y - 24 = 0$. Using the fact that $y \geqslant 0$ will give you one possible value of y which you can then use to find the value of x. Substituting this back into the original equation should be reassuring.

There are two roots for part **(ii) (b)**, start by substituting $y = \sqrt{2x^2 - 8x - 3}$ (or you can do two substitutions, $y = 2x^2 - 8x - 3$ and $z = \sqrt{y}$). You should end up with $x = 2 \pm \sqrt{10}$. You should then check whether these two solutions actually solve the original equation.

Squaring can be used as an alternative to substitution in order to eliminate the square roots, but you end up with an unpleasant quartic in part **(ii)(b)**. If you take this approach, you must check your solutions satisfy the original equation as "false solutions" will be created by squaring.

Solution

(i) Using the substitution $y = \sqrt{x}$ gives:

$$\begin{aligned} y^2 + 3y - \tfrac{1}{2} &= 0 \\ y &= \frac{-3 \pm \sqrt{9+2}}{2} \\ y &= \frac{-3 \pm \sqrt{11}}{2} \end{aligned}$$

Since $y > 0$, we need to take the positive square root, and as $x = y^2$ we have:

$$\begin{aligned} x &= \left(\frac{-3+\sqrt{11}}{2}\right)^2 \\ &= \tfrac{1}{4}\left(9 + 11 - 6\sqrt{11}\right) \\ &= \tfrac{1}{2}\left(10 - 3\sqrt{11}\right) \end{aligned}$$

(ii) **(a)** Using $y = \sqrt{x+2}$, and $y^2 = x + 2$, gives:

$$\begin{aligned} x + 10\sqrt{x+2} - 22 &= 0 \\ \left(y^2 - 2\right) + 10y - 22 &= 0 \\ y^2 + 10y - 24 &= 0 \\ (y+12)(y-2) &= 0 \end{aligned}$$

Hence we have $y = -12$ or $y = 2$, but since $y = \sqrt{x+2}$ we must have $y > 0$ and so $y = 2$. This means that $x = y^2 - 2 = 2$.

(b) Here use $y = \sqrt{2x^2 - 8x - 3}$. This means that $y^2 = 2x^2 - 8x - 3$ and so we have $x^2 - 4x = \frac{1}{2}(y^2 + 3)$. Substituting gives:

$$\begin{aligned}
x^2 - 4x + \sqrt{2x^2 - 8x - 3} - 9 &= 0 \\
\tfrac{1}{2}(y^2 + 3) + y - 9 &= 0 \\
y^2 + 2y - 15 &= 0 \\
(y+5)(y-3) &= 0
\end{aligned}$$

As before, $y > 0$ and so $y = 3$. This gives:

$$\begin{aligned}
\sqrt{2x^2 - 8x - 3} &= 3 \\
2x^2 - 8x - 3 &= 9 \\
x^2 - 4x - 6 &= 0 \\
x &= \frac{4 \pm \sqrt{16 + 24}}{2} \\
x &= 2 \pm \sqrt{10}
\end{aligned}$$

Then substituting these both back into the equation shows that they are both solutions of the original equation. It may help to note that if $x = 2 \pm \sqrt{10}$ then $x^2 = 14 \pm 4\sqrt{10}$.

Warm down

4 **(i)** Noting that $2^{m+1} + 2^m = 2^m(2+1)$, and similarly for the other side then we have

$$\begin{aligned} 2 \times 2^m + 2^m &= 3^2 \times 3^n - 3^n \\ (2+1) \times 2^m &= (9-1) \times 3^n \\ 3 \times 2^m &= 8 \times 3^n \end{aligned}$$

Since each number can be written as a *unique* product of primes, we can deduce from this what m and n are. Since the prime factorisation is unique, the powers of 3 and 2 must match on both sides, so $m = 3$ and $n = 1$.

If you prefer, you can divide throughout by 3 and 8 to get:

$$2^{m-3} = 3^{n-1}$$

and then as one side is a power of 2 and the other is a power of 3, the indices on each side must be equal to 0. Hence we have $m = 3$ and $n = 1$.

(ii) One way to approach this is to use two substitutions, such as $3^x = a$ and $5^x = b$. This gives:

$$\begin{aligned} 3^{2x} - 34 \times 15^{x-1} + 5^{2x} &= 0 \\ a^2 - 34 \times \frac{3^x \times 5^x}{15} + b^2 &= 0 \\ a^2 - \tfrac{34}{15}ab + b^2 &= 0 \\ 15a^2 - 34ab + 15b^2 &= 0 \\ (3a - 5b)(5a - 3b) &= 0 \end{aligned}$$

There are now two cases to consider. If $3a = 5b$ we have:

$$\begin{aligned} 3a &= 5b \\ 3 \times 3^x &= 5 \times 5^x \\ 3^{x+1} &= 5^{x+1} \\ x &= -1 \end{aligned}$$

And if $5a = 3b$ we have:

$$\begin{aligned} 5a &= 3b \\ 5 \times 3^x &= 3 \times 5^x \\ 3^{x-1} &= 5^{x-1} \\ x &= 1 \end{aligned}$$

Hence there are two solutions, $x = \pm 1$. It is not a bad idea to check that these both work in the original equation.