STEP Support Programme

Hints and Partial Solutions for Assignment 12

Warm-up

1 **(i)** You don't need a particularly fancy argument here. $n(n+1)$ is the product of two consecutive integers, so one of them is divisible by 2. Hence $n(n+1)$ is divisible by 2.

$n(n+1)(n+2)$ is the product of three consecutive integers, so one of them is a multiple of 3. Hence $n(n+1)(n+2)$ is divisible by 3.

(ii) $n^3 - n = (n-1)n(n+1)$, and so is the product of three consecutive integers. Hence one of the three is divisible by 3, and at least one of them is divisible by 2. Hence the product of the three is divisible by 6.

(iii) $n^5 - n^3 = n^3(n-1)(n+1)$. From part **(ii)** it is divisible by three. Then if n is even, n^3 is divisible by 8, and if n is odd, $n-1$ and $n+1$ are consecutive even numbers — which means one of them is a multiple of 4 and the other is a multiple of 2 (and therefore the product is a multiple of 8).

More formally we could let $n = 2k+1$, then we have $(2k+1)^3 \times 2k \times (2k+2) = (2k+1)^3 \times 4k(k+1)$. Since we know that $k(k+1)$ is even we have another factor of 2 and are done.

(iv) $2^{2n} - 1 = (2^n - 1)(2^n + 1)$. Note that you cannot say "this is the product of two consecutive odd numbers, so one of them must be divisible by 3" — consider 5 and 7!

Instead consider $(2^n - 1)2^n(2^n + 1)$ which is the product of three consecutive integers, so must be divisible by 3. But 2^n is **not** divisible by 3, so one of $2^n - 1$ and $2^n + 1$ must be divisible by 3.

(v) There are various ways to proceed, but perhaps the simplest is to note that $n - 1 = 3k$, so $n = 3k+1$. Substitute and expand $(3k+1)^3 - 1$ which gives $27k^3 + 27k^2 + 9k = 9(3k^3 + 3k^2 + k)$.

Preparation

2 **(i)** **(a)**

$$\frac{1}{(x-1)(x+2)} - \frac{1}{(x+1)(x-2)} = \frac{(x+1)-(x-1)}{(x-1)(x+1)(x+2)}$$
$$= \frac{2}{(x-1)(x+1)(x+2)}$$

(b) Note you can cancel out some ms in the second batch of fractions as a first step.

$$\frac{m}{m+2} \times \frac{m-1}{m+1} + \frac{\cancel{m}}{m+2} \times \frac{2}{m+1} \times \frac{m-1}{\cancel{m}}$$
$$= \frac{m(m-1)+2(m-1)}{(m+1)(m+2)}$$
$$= \frac{m^2+m-1}{(m+1)(m+2)}$$
$$= \frac{(m+2)(m-1)}{(m+1)(m+2)}$$
$$= \frac{m-1}{m+1}$$

(ii) Remember that the probabilities for the second sweet depend upon what happened for the first sweet (they are **conditional**). We either take 2 mint imperials or 2 lemon sherbets.

Method 1:

$$\frac{9}{15} \times \frac{8}{14} + \frac{6}{15} \times \frac{5}{14} = \frac{24}{70} + \frac{10}{70}$$
$$= \frac{34}{70}$$
$$= \frac{17}{35}$$

Method 2:

$$\frac{{}^9C_2 + {}^6C_2}{{}^{15}C_2} = \frac{36+15}{105}$$
$$= \frac{51}{105}$$
$$= \frac{17}{35}$$

It makes no difference to the probability if you take both sweets in one go, or one after the other. If doing one after the other then a tree diagram is useful (Method 1), if taking them simultaneously then arrangements are probably the way to go (Method 2).

(iii) Again, the probabilities change depending on what happened with the first sweet. If the first was an apple sour, and you remove it, you are then left with $a-1$ apple sours and a total of $a+b-1$ sweets.

Answers: **(a)** $\dfrac{a}{a+b}$ and **(b)** $\dfrac{b}{a+b} \times \dfrac{b-1}{a+b-1} \times \dfrac{b-2}{a+b-2}$.

(iv) **(a)** The easiest approach here is to find the probability that they have all forgotten and subtract from 1. Answer: $1 - \left(\frac{3}{4}\right)^3 = \frac{37}{64}$.

(b) Answer: $\frac{1}{4}$.

(c) The order is not particularly important, the important bit is that one particular child has not got goggles — we could have said "Philip" has forgotten his goggles whilst the other two have them. Answer: $\frac{3}{4} \times (\frac{1}{4})^2 = \frac{3}{64}$.
If asked to find the probability that (any) one child forgets goggles whilst the other two have them this probability would be multiplied by 3, as the child who forgets the goggles could be child 1, 2 or 3.

The STEP question

3 **(i)** As long as you have a £1 coin person first, then you can give change to the one £2 coin person whenever they arrive. The answer is therefore $\dfrac{m}{m+1}$.

(ii) Here the options that work are (omitting the pound signs):

- 1, 1, anything (as then you have two £1 coins for change)
- 1, 2, 1, anything

This gives $P = \dfrac{m}{m+2} \times \dfrac{m-1}{m+1} + \dfrac{m}{m+2} \times \dfrac{2}{m+1} \times \dfrac{m-1}{m}$ which gives the required result (see part **2 (i)(b)** for the working out).

You can consider the options that **don't** work instead (which will be the sequences starting "2," and "1, 2, 2,").

(iii) Here the options that work are:

- 1, 1, 1, anything, which has probability $\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{m-2}{m+1}$
- 1, 1, 2, 1, anything, which has probability $\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{m-2}{m}$
- 1, 1, 2, 2, 1, anything, which has probability $\frac{m}{m+3} \times \frac{m-1}{m+2} \times \frac{3}{m+1} \times \frac{2}{m} \times \frac{m-2}{m-1}$
- 1, 2, 1, 1, anything, which has probability $\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{m-2}{m}$
- 1, 2, 1, 2, 1, anything, which has probability $\frac{m}{m+3} \times \frac{3}{m+2} \times \frac{m-1}{m+1} \times \frac{2}{m} \times \frac{m-2}{m-1}$

You need to be sure you have covered all the possible options, and explain in your proof why this is so. Working systematically helps.

The fractions look a bit of a mess, but there are some common factors in the numerators and denominators that you can factorise out as a first step. This leaves the probability as:

$$\frac{m(m-1)(m-2)+3(m-1)(m-2)+6(m-2)+3(m-1)(m-2)+6(m-2)}{(m+3)(m+2)(m+1)}$$

$$=\frac{m^3-3m^2+2m+6m^2-18m+12+12m-24}{(m+3)(m+2)(m+1)}$$

$$=\frac{m^3+3m^2-4m-12}{(m+3)(m+2)(m+1)}$$

$$=\frac{(m+2)\left(m^2+m-6\right)}{(m+3)(m+2)(m+1)}$$

Looking at the given answer suggest that we would like $(m+2)$ to be a factor.

$$=\frac{(m+2)(m+3)(m-2)}{(m+3)(m+2)(m+1)}$$

$$=\frac{m-2}{m+1}$$

This was a question from the Probability and Statistics section of a STEP paper. Note that it doesn't actually need anything other than GCSE probability and algebraic fraction manipulation, but does require you to read the question carefully, work logically and have fluent algebraic manipulation skills.

Warm down

4 This is a bit of a "Marmite" problem, with some people really liking it and some not so much. However it is a useful example of being systematic and ensuring that you have considered all the cases.

As a starting point, write 2450 as a product of prime factors, and then carefully and logically write down all possible sets of three ages for the bell ringers.

$2450 = 2 \times 5^2 \times 7^2$, and so the possibilities[1] are:

Age 1	Age 2	Age 3	Sum ages	Imam's age
1	1	$2 \times 5^2 \times 7^2$	2452	1226
1	2	$5^2 \times 7^2$	1228	614
1	5	$2 \times 5 \times 7^2$	496	248
1	7	$2 \times 5^2 \times 7$	358	179
1	2×5	5×7^2	256	128
1	2×7	$5^2 \times 7$	190	95
1	5×5	2×7^2	124	62
1	5×7	$2 \times 5 \times 7$	106	53
1	7×7	2×5^2	100	50
2	5	5×7^2	252	126
2	7	$5^2 \times 7$	184	92
2	5×5	7×7	76	38
2	5×7	5×7	72	36
5	5	2×7^2	108	54
5	7	$2 \times 5 \times 7$	82	41
5	2×5	7^2	64	**32**
5	2×7	5×7	54	27
7	7	2×5^2	64	**32**
7	2×5	5×7	52	26
7	2×7	5×5	46	23

Don't forget that the Imam knows how old they are — so if the Imam is 41 then they are looking for the ages which sum to 82. The fact that the Imam cannot immediately answer the question is important, and will enable you to work out how old the Imam is — i.e. the Imam is 32 as this is the only value that is not unique.

You then need to use the fact that knowing the Rabbi is older than anyone else in the room enables the Imam to solve the problem.

The two possible sets of ages are $(5, 10, 49)$ and $(7, 7, 50)$. If the Rabbi was 60 (say) then this does not eliminate either possibility. The Rabbi needs to be 50 so that the second case is eliminated and the bell ringers are aged 5, 10 and 49[2].

[1] Note that some of these "possibilities" are rather implausible.

[2] 5 is perhaps a little young for full-sized church bells — maybe they ring hand bells instead?