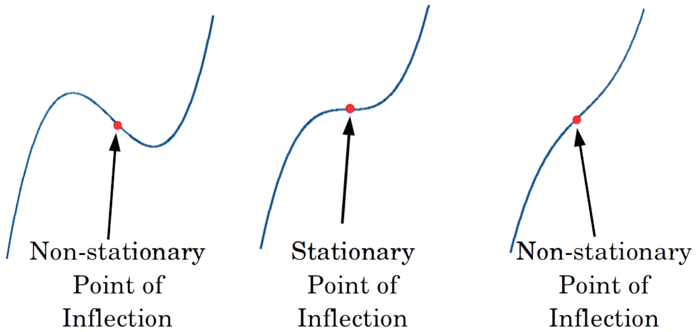
STEP Support Programme

Hints and Partial Solutions for Assignment 13

Warm-up

1 If you zoom in on the graph at a point of inflection, it looks like a straight line. For a *stationary* point of inflection the straight line is horizontal and for a *non-stationary* point of inflection the straight line is not horizontal.

A cubic graph can have a maximum and minimum (with a non-stationary point of inflection in between), or just a stationary point of inflection, or just a non-stationary point of inflection. The picture below shows the basic three cubic graph shapes (the graphs are $y = x^3 - x$, $y = x^3$ and $y = x^3 + x$).

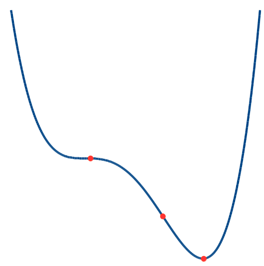


(i) We have $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 12$ and so the graph is concave when $-1 < x < 1$.

(ii) The second derivative is given by $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x - 4$. The point is therefore $\left(\frac{2}{3}, -\frac{70}{27}\right)$.

(iii) We have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 6x^2$ which is equal to 0 when $x = 0$ or $x = \frac{3}{2}$. We also have $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 12x$ which is equal to 0 when $x = 0$ or $x = 1$.

There is a stationary point of inflection at $(0, 0)$, a non-stationary point of inflection at $(1, -1)$ and a minimum when $x = \frac{3}{2}$. As it is a quartic with a positive x^4 term, it will be large and positive at both "ends"— which justifies the nature of $(0, 0)$ (i.e. that it is a point of inflection rather than a minimum or maximum).



(iv) By differentiating twice, you can show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12(x-1)^2 = 0$ when $x = 1$. You can then look at the sign of the gradient either side of $x = 1$, **or** use the fact that the graph is a translation of $y = x^4$ to show that when $x = 1$ the graph has a minimum (and not a point of inflection).

If you know the chain rule, you can use this to differentiate $y = (x-1)^4$ twice, but if not you can expand $(x-1)^4$ and differentiate this.

Preparation

2 You may be familiar with the idea of using intersecting graphs to see how many roots an equation has. For example, if you wanted to see how many roots $y = x^3 - 3x + 4$ has you could draw $y = -2$ onto your graph of $y = x^3 - 3x + 2$. For this question, we were asking you to use a slightly different technique, which is translating the whole graph up and down.

(i) **(a)** We have $y = (x-1)\left(x^2 + x - 2\right) = (x-1)(x-1)(x+2)$ and hence there are roots at $x = -2$ and $x = 1$.

(b) $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 3$ which is equal to 0 when $x = \pm 1$. Since $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x$ this is not equal to 0 when $x = \pm 1$ and so we know that these are turning points. The turning points are at $(-1, 4)$ and $(1, 0)$.

(c) The y intercept is at $(0, 2)$ (and this is also a point of inflection).

There are two distinct roots of $x^3 - 3x + 2 = 0$.

Use Desmos to **check** your graph (but make you have tried to sketch it yourself first!).

(ii) When sketching these, probably the first thing to do is work out where the stationary points and y intercept go to. The transformations are:

(a) down 2, so there are 3 roots,
(b) up 2, so there is 1 root and
(c) down 6, so there is 1 root.

When the equation has two distinct roots then we need one of the turning points to be on the x-axis, so we need $k = 2$ or $k = -2$. For three distinct roots we need the turning points either side of the x-axis which means $-2 < k < 2$.

You can use Desmos to plot $y = x^3 - 3x + k$ with a "slider" which you can vary to see what happens to the graph as k varies.

(iii) **(a)** We have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 12x^3 + 12x^2 - 12x - 12 = 12\left(x^3 + x^2 - x - 1\right) = (x-1)(x+1)^2$. The stationary points are at $(1, -6)$ and $(-1, 10)$.

(b) $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 36x^2 + 24x - 12 = 12\left(3x^2 + 2x - 1\right)$. This is equal to zero when $x = -1$ or $x = \frac{1}{3}$.

(c) This is a quartic with a positive x^4 term, so will tend to $+\infty$ as x tends to $\pm\infty$. From the previous two parts we can deduce that there is a minimum at $(1, -6)$ (use the second derivative when $x = 1$ to confirm this) and that there is a non-stationary point of inflection when $x = \frac{1}{3}$. The shape of the graph as $x \to -\infty$ tells us the the point $(1, -10)$ is a stationary point of inflection. It looks very much like the graph in 1**(iii)**. There are no other points where the gradient is zero, or points of inflection, so make sure that you do not put any extra ones in.

(d) For the equation to have just one root we need the graph to move up by 6 (so that the minimum lies on the x axis), hence we need $k = 5 + 6 = 11$.

The STEP question (2012 STEP I Q2)

3 **(i)** We have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 12x$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 12x^2 - 12$. The y-intercept is at $(0, 9)$ and there are turning points at $(\pm\sqrt{3}, 0)$ and $(0, 9)$. There are non-stationary points of inflection at $(1, 4)$ and $(-1, 4)$. It is shaped like a "W" (but with smooth bumps — a Mexican hat).

There are 2 distinct roots of $y = x^4 - 6x^2 + 9$, which are the two minimum turning points. Note that the y-intercept of $y = x^4 - 6x^2 + b$ is $(0, b)$ and that this graph is a vertical translation of the first graph.

(a) Here we need the two minimums to be above the x-axis, so we need $b > 9$

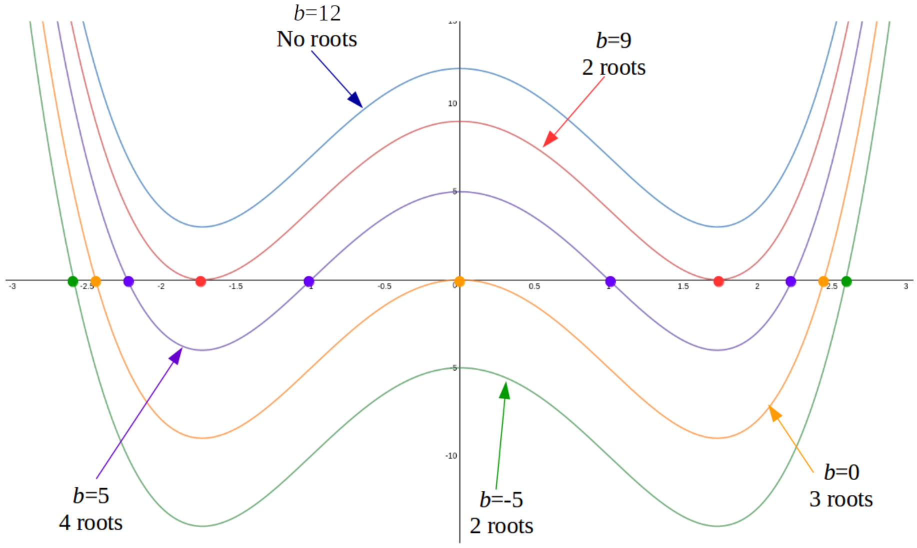
(b) We can only have 0, 2, 3 or 4 roots so this is not possible (no values of b)

(c) Either we have the two minimum points on the x-axis or the maximum point is below the x-axis, so $b = 9$ or $b < 0$

(d) This is the case when the maximum point lies on the x-axis, so $b = 0$

(e) Here we have the minimum points below the x-axis, but the maximum point is above the x-axis which gives $0 < b < 9$

Below is a sketch showing what happens for some different values of b.

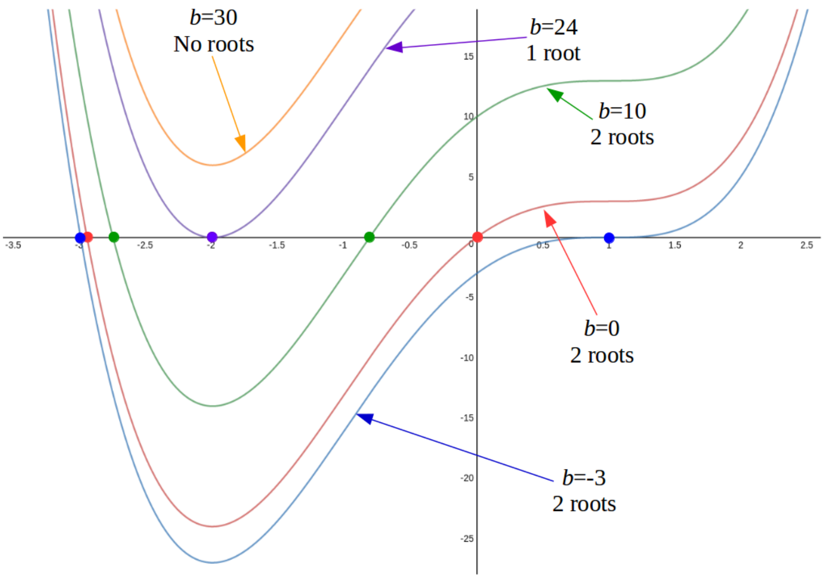


(ii) Setting $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ gives $12x^2 - 12 = 0 \implies x = \pm 1$ and you can substitute these in $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 12x + a = 0$ to find the corresponding values of a ($a = \pm 8$).

For the next part, start by taking $a = 8$ and sketching $y = x^4 - 6x^2 + 8x$ (i.e. taking $b = 0$). We have $\frac{dy}{dx} = 4\left(x^3 - 3x + 2\right)$ and $\frac{d^2y}{dx^2} = 12\left(x^2 - 1\right)$ This graph passes through the origin, has a minimum turning point at $(-2, -24)$, a non-stationary point of inflection at $(-1, -13)$ and a stationary point of inflection at $(1, 3)$.

The sketch should show that when $b = 0$ there are two roots of the equation. Considering vertical translations of this graph leads to the answers:

- no roots when $b > 24$
- one root when $b = 24$
- 2 roots when $b < 24$



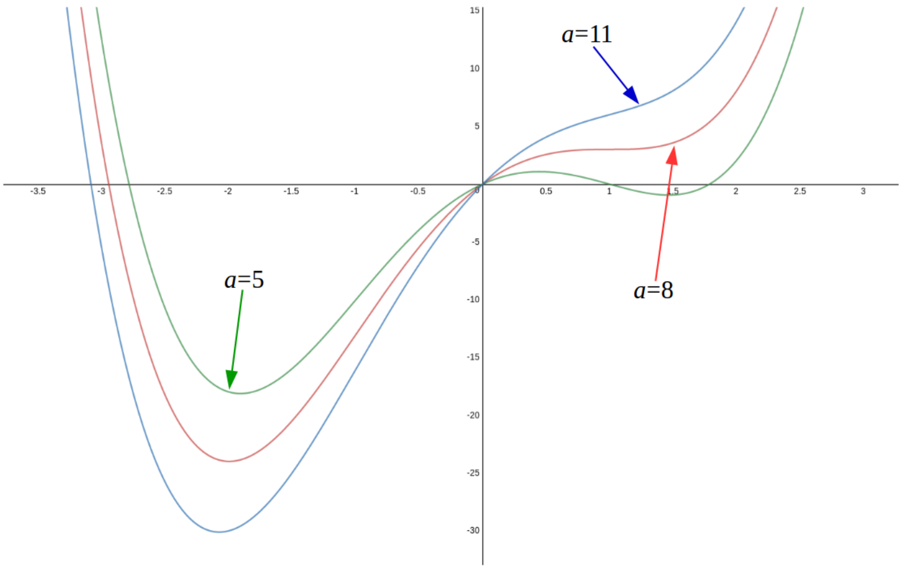
For $a = -8$, you can show that if the first graph is $f(x)$ then the second one is $f(-x)$, so is a reflection of the first graph in the y-axis. The answer is therefore the same as when $a = 8$.

(iii) You can show that there are two non-stationary points of inflection, when $x = \pm 1$. It is trickier to know how many stationary points there are (whether inflection or turning points). One approach is to start by drawing the graph of $\frac{dy}{dx}$ in the case when $a = 8$, and should be able to show that there are two roots (i.e. two values of x for which $\frac{dy}{dx} = 0$) and these correspond to two stationary points of the original graph. You can then consider what $\frac{dy}{dx}$ looks like for $a > 8$ and hence show that in this case you have only one root, so the original graph has only one stationary point.

We want to sketch $f(x) = \dfrac{dy}{dx} = 4x^3 - 12x + a$. The points where we have $f'(x) = 0$ satisfy $12x^2 - 12 = 0$ so are $(1, a-8)$ and $(-1, a+8)$. By considering $f''(x) = 24x$ you can show that $(1, a-8)$ is a minimum point and $(-1, a+8)$ is a maximum point. As a varies, the cubic translates vertically up and down. If $a > 8$ then both the minimum and maximum points lies above the x-axis and so there is only one root of $f(x) = 0$.

Hence there is only one solution to $\dfrac{dy}{dx} = 0$ and only one stationary point of $y = x^4 - 6x^2 + ax$. The x-coordinate of this stationary point is less than -1.

The sketch below shows the graph for three values of a (although the question only asked you to sketch the graph when $a > 8$).



Warm down

4 The relationship can be written as $n+5p+10q = p+5q+10n$, or equivalently $4p+5q = 9n$.

(a) It is obviously true when $n = p = q$. So in the case when $n = 6$ we have a solution $n = p = q = 6$.

(b) We have $4p + 5q = 54$ (since $9n = 54$). This is a straight line with negative gradient. Since p and q have to be positive integers, the solutions lie on a finite line segment.

(c) If $(6, p, q)$ is a solution, then we have $4p + 5q = 54$. This is one equation for two unknowns, so we may expect that the solution is not unique — there are many solutions. It is only the fact that p and q are integers which prevents there being an infinite number of solutions.

Now let $p' = p + 5k$ and $q' = q - 4k$. We have:

$$\begin{aligned} 4p' + 5q' &= 4(p + 5k) + 5(q - 4k) \\ &= 4p + \cancel{20k} + 5q - \cancel{20k} \\ &= 4p + 5q = 54 \end{aligned}$$

Hence $4p' + 5q'$ is also equal to 54, i.e. p' and q' also satisfy our equation: and we have found a way of generating new solutions from old!

Graphically, each solution lies on the line $4p+5q = 54$ and the coordinates of (p, q) are both integers. To find another solution we have to move along the line until we find another point on the line where both coordinates are integers. Given that the gradient of the line is $-\frac{4}{5}$ to get to another solution we have to move an integer multiple of the vector $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$.

(d) We know that $p = 6, q = 6$ is a solution, and we can find other solutions by using $p + 5k, q - 4k$. Remember that p and q have to be positive integers. The solutions are $(p = 6, q = 6)$, $(p = 1, q = 10)$ (taking $k = -1$) and $(p = 11, q = 2)$.

If, instead of $n = 6$, we are given that $p = 6$ then the relationship between n and q is $9n - 5q = 24$. This line has positive gradient, which means that we can find infinitely many solutions. These can be written in the form $(n = 6 + 5k, p = 6, q = 6 + 9k)$.