STEP Support Programme

Hints and Partial Solutions for Assignment 14

Warm-up

1 **(i)** To show the first result, the easiest approach is to expand the brackets. For the second request you can replace b with $-b$ (or if you are uncomfortable with this you can let $b=-c$). Expand your answer for a^3+b^3 to check your factorisation was correct.

(ii) **(a)** $x^3+5^3=(x+5)(x^2-5x+25)$.

(b) You may get slightly different answers depending on whether your first step is

$$(x^3)^2-(y^3)^2=(x^3+y^3)(x^3-y^3)$$

(difference of two squares) or

$$(x^2)^3-(y^2)^3=(x^2-y^2)(x^4+x^2y^2+y^4)$$

(using the result of part **(i)**).

The expression can be written as $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$, which is easier to spot if you take the first approach above.

(iii) The sum of the geometric progression is $\dfrac{1-t^5}{1-t}$ (or equivalent). This means that we have $(1-t)\left(1+t+t^2+t^3+t^4\right)=1-t^5$.

To deduce the factorisation for a^5-b^5, use the substitution $t=\frac{b}{a}$ and then multiply by b^5.

For a^5+b^5 you can replace b with $-b$ to get $a^5+b^5=(a+b)\left(a^4-a^3b+a^2b^2-ab^3+b^4\right)$.

(iv) Multiplying out the brackets gives $\left(x^2+y^2\right)^2-2x^2y^2=x^4+y^4$.

Then, taking $y=1$, you get

$$x^4+1=(x^2+1-\sqrt{2}\,x)(x^2+1+\sqrt{2}\,x).$$

Setting this equal to zero gives two quadratic equations which you can solve (using the quadratic formula) to find the four fourth roots of -1. Clearly, they will be complex (not real), since $x^4\geqslant 0$ if x is real. They can be written as $x=\dfrac{\pm\sqrt{2}(1\pm i)}{2}$.

Preparation

2 **(i)** We have:

$$\frac{1}{3+\sqrt{5}} + \frac{1}{3-\sqrt{5}} = \frac{3-\sqrt{5}}{9-5} + \frac{3+\sqrt{5}}{9-5} = \frac{6}{4} = \frac{3}{2}$$

(ii) This is a GP of the form $\sum r^k$, so for convergence you need to show that $|r| < 1$. which you can do by noting (or proving) that $\sqrt{3} < 2$. The sum to infinity is:

$$\begin{aligned}\frac{1}{1-\frac{1+\sqrt{3}}{3}} &= \frac{3}{3-(1+\sqrt{3})}\\ &= \frac{3}{2-\sqrt{3}}\\ &= \frac{3(2+\sqrt{3})}{1}\\ &= 3(2+\sqrt{3})\end{aligned}$$

The last bit says "write down", which suggests that no real work is required. You can do this by just replacing $\sqrt{3}$ with $-\sqrt{3}$ to get the sum to infinity as $3(2-\sqrt{3})$. A quick sanity check tells us that each term in the last series is smaller than the corresponding term in the first series, so you would expect the sum to be smaller.

(iii) By factorising as suggested you get $(x-y)(x+y) = z$ i.e. $z(x+y) = z$. This means that either $z = 0$ or $x + y = 1$. **It is important that you consider the case $z = 0$ separately and don't just divide by z.**

If we have $z = 0$ then we have $x = y$ and $xy = -2$ which implies that $x^2 = -2$ so there are no real solutions in this case. In fact, the question does not ask for *real* solutions, so you should give these solutions as well ($x = \pm i\sqrt{2}$, $y = \pm i\sqrt{2}$, $z = 0$).

If $z \neq 0$, then we have $x + y = 1$ and so we have $x(1-x) = -2 \implies x^2 - x + 2 = 0$. The solutions in this case are $x = 2, y = -1, z = 3$ and $x = -1, y = 2, z = -3$.

The STEP question (2010 STEP II Q3)

3 **(i)** For $n = 0$, $F_0 = 0$ so $a + b = 0$. Thus $b = -a$ and $F_n = a(\lambda^n - \mu^n)$. The other equations ($n = 1, 2, 3$) are:

$$a(\lambda - \mu) = 1$$
$$a(\lambda^2 - \mu^2) = 1$$
$$a(\lambda^3 - \mu^3) = 2.$$

You can then use the result from question **1(i)** to write $\lambda^3 - \mu^3 = (\lambda - \mu)\left(\lambda^2 + \lambda\mu + \mu^2\right)$. Hence, using $a(\lambda - \mu) = 1$ in $a(\lambda^3 - \mu^3) = 2$ gives $\lambda^2 + \lambda\mu + \mu^2 = 2$. In a similar way you can use the first two equations to get $\lambda + \mu = 1$.

When solving the equations, it is helpful to notice that as $a > 0$ (given in the "stem") and $a(\lambda - \mu) = 1$ then $\lambda > \mu$.

We have:

$$\begin{aligned} \lambda^2 + \lambda\mu + \mu^2 &= 2 \\ \lambda^2 + \lambda(1-\lambda) + (1-\lambda)^2 &= 2 \\ \lambda^2 - \lambda - 1 &= 0 \\ \lambda &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

And since $\lambda > \mu$ we have $\lambda = \frac{1+\sqrt{5}}{2}$. Substituting back into $\lambda + \mu = 1$, $a(\lambda - \mu) = 1$ and $b = -a$ will give the other values.

The final answers are:

$$a = \frac{1}{\sqrt{5}}, \quad b = -\frac{1}{\sqrt{5}}, \quad \lambda = \frac{1+\sqrt{5}}{2}, \quad \mu = \frac{1-\sqrt{5}}{2}\,.$$

(ii) If you noticed that the sequence is the Fibonacci sequence, then you could easily calculate that $F_6 = 8$ using the implicit formula $F_{n+2} = F_{n+1} + F_n$. However you have to show this by using $F_n = a(\lambda^n - \mu^n)$, and as the answer is known (or easily found by other means) you must be careful to fully justify your answer. It is helpful to factorise the brackets, i.e. write it as:

$$F_6 = \frac{1}{\sqrt{5}} \times \frac{1}{2^6}\left[(1+\sqrt{5})^6 - (1-\sqrt{5})^6\right].$$

You could use $x^6 - y^6$ to help simplify $(1+\sqrt{5})^6 - (1-\sqrt{5})^6$, but is is probably simpler to use a binomial expansion. Noting that lots of terms cancel we have:

$$\begin{aligned} F_6 &= \frac{1}{\sqrt{5}} \times \frac{1}{2^6}\left[12\sqrt{5} + 40\left(\sqrt{5}\right)^3 + 12\left(\sqrt{5}\right)^5\right] \\ &= \frac{1}{2^6}\left[4 \times 3 + 4 \times 10 \times 5 + 4 \times 3 \times 25\right] \\ &= \frac{1}{2^4}\left[3 + 10 \times 5 + 3 \times 25\right] \\ &= \frac{128}{2^4} = \frac{2^7}{2^4} = 2^3 = 8 \end{aligned}$$

(iii) Note that the sum can be written as $\displaystyle\sum_{n=0}^{\infty} \frac{a(\lambda^n - \mu^n)}{2^{n+1}}$ or $\displaystyle\frac{a}{2}\sum_{n=0}^{\infty}\left(\frac{\lambda}{2}\right)^n - \frac{a}{2}\sum_{n=0}^{\infty}\left(\frac{\mu}{2}\right)^n$. It is probably easier to find the infinite sums in terms of a, λ and μ before substituting for them.

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}} &= \frac{a}{2}\left[\frac{1}{1-\frac{\lambda}{2}} - \frac{1}{1-\frac{\mu}{2}}\right] \\
&= \frac{a}{2}\left[\frac{4}{4-2\lambda} - \frac{4}{4-2\mu}\right] \\
&= \frac{a}{2}\left[\frac{4}{4-(1+\sqrt{5})} - \frac{4}{4-(1-\sqrt{5})}\right] \\
&= 2a\left[\frac{1}{3-\sqrt{5}} - \frac{1}{3+\sqrt{5}}\right] \\
&= 2a\left[\frac{3+\sqrt{5}}{4} - \frac{3-\sqrt{5}}{4}\right] \\
&= 2a\left[\frac{2\sqrt{5}}{4}\right] \\
&= \sqrt{5}\,a \\
&= 1
\end{aligned}$$

Warm down

4 If we start on the first day with:

Group 1 : ABC
Group 2 : DEF
Group 3 : GHI

then on subsequent days A, B and C must be in separate groups, say A in group 1, B in group 2 and C in group 3. D must then go with A, B and C in turn etc.

One solution is:

Day 1	Day 2	Day 3	Day 4
ABC	ADG	AFH	AEI
DEF	BEH	BDI	BFG
GHI	CFI	CEG	CDH