STEP Support Programme

Hints and Partial Solutions for Assignment 15

Warm-up

1 **(i)** It is best to start simplifying from the bottom. You should end up with x.

$$\begin{aligned}1-\frac{1}{1-\frac{1}{1-\frac{1}{x}}} &= 1-\frac{1}{1-\frac{1}{\frac{x-1}{x}}}\\ &= 1-\frac{1}{1-\frac{x}{x-1}}\\ &= 1-\frac{1}{\frac{(x-1)-x}{x-1}}\\ &= 1-\frac{x-1}{-1}\\ &= 1+(x-1)=x\end{aligned}$$

(ii) The simplest way to show that $x=15$ is a root is to set $x=15$ in the equation and check that this gives 0. Factorising out some common factors helps with the arithmetic:

$$\begin{aligned}&15^4-18\times 15^3+35\times 15^2+180\times 15-450\\ =&15\left(15^3-18\times 15^2+35\times 15+180-30\right)\\ =&15^2\left(15^2-18\times 15+35+12-2\right)\\ =&15^2\left(225-270+35+12-2\right)\\ =&15^2(272-272)=0\end{aligned}$$

Having established that 15 is a root of the equation, you can write

$$x^4-18x^3+35x^2+180x-450\equiv(x-15)(x^3-3x^2-10x+30)$$

more or less by inspection, without doing the full long division.

Alternatively, instead of substituting $x=15$ into the equation, you could do the long division (dividing the polynomial by $(x-15)$) carefully, remembering to show that the last subtraction does indeed gives 0 (so that $x-15$ is indeed a factor); and you should state "therefore $x=15$ is a root" at the end — this is what you were asked to show, so you should indicate clearly that you realise that you have actually shown it and not just stopped.

Looking at the cubic $x^3-3x^2-10x+30$, you can see that $x=3$ is a root, and so we have $x^3-3x^2-10x+30=(x-3)(x^2-10)$. The roots are therefore $x=15$, $x=3$ and $x=\pm\sqrt{10}$.

(iii) For part **(a)** you should get $1 \times 2 \times 3 \times 4 = 24$.

For the other two parts write out the first few terms, then write $\cdots$ once you have seen the pattern, then write the last one or two terms. You should see that lots of things cancel.

Answers: **(b)** $\dfrac{1}{n+1}$ and **(c)** $\dfrac{\mathrm{g}(n)}{\mathrm{g}(0)}$.

Preparation

2 **(i)** **(a)** This is a periodic sequence with period 2 (values are $1, -2, 1, -2, \ldots$).

(b) This sequence appears to tend towards a limit. To find this limit using the suggested technique gives us:

$$
\begin{aligned}
& l = 6 - \frac{4}{l} \\
\Rightarrow\ & l^2 - 6l + 4 = 0 \\
\Rightarrow\ & l = 3 \pm \sqrt{5}
\end{aligned}
$$

There appear to be two limits! You have to do a bit of work to find out which of the two limits is the relevant one (it depends on the value of u_1). For both of the above values of l, if $u_1 = l$ then $u_2 = l$ and the whole sequence is constant, but if the value of u_1 is not equal to one of the two limits then the sequence will always tend to $3+\sqrt{5}$: this is a *stable* limit and $3-\sqrt{5}$ is an *unstable* limit. Even if you use a starting point very close to $3-\sqrt{5}$ (such as 0.763932) the sequence will still converge to the other limit. You can use a spreadsheet to see this happening.[1] Note that for this sequence some starting values are problematic, for example if we had $u_1 = \frac{3}{4}$ then the third term would be 0 which presents some difficulties in finding the fourth term.

(c) This sequence converges to the limit $2-\sqrt{2}$.

(d) This sequence diverges. The initial value is greater than the *unstable* limit of $2+\sqrt{2}$ and the sequence gets further away from this. If the initial value was just a little less than $2+\sqrt{2}$ then the sequence would converge to the *stable* limit of $2-\sqrt{2}$.

(ii) This is a periodic sequence with period 6 (the values are $4, 1, -3, -4, -1, 3, \ldots$).

(iii) **(a)** $u_2 = 4 - b$ and $u_3 = 16 - 9b + b^2$

(b) Here you need to solve $16 - 9b + b^2 = 2 \implies b^2 - 9b - 14 = 0$. The solutions are $b = 2$ and $b = 7$.

(c) When $b = 2$ the sequence is constant (all the terms are the same) but when $b = 7$ the sequence is periodic with period 2.

(iv) This is a way to estimate the square root of 2, and we were expecting you to use a calculator or spreadsheet here. Try comparing u_5 with what your calculator gives you for $\sqrt{2}$.

Note that solving $l = \dfrac{1}{2}\left(\dfrac{2}{l} + l\right)$ gives 2 possible values of l, but as $u_1 > 0$, and $u_{k+1} > 0$ if $u_k > 0$ then the whole sequence is positive and it will tend to $\sqrt{2}$ (rather than $-\sqrt{2}$).

[1] The reason why one limit is unstable is connected to the gradient of $y = 6 - \frac{4}{x}$ at the limit. Try researching "Cobweb diagrams" for examples of why this happens.

This iterative formula was derived by rearranging $x^2 = 2$ as follows:

$$\begin{aligned} x^2 &= 2 \\ \Rightarrow \quad 2x^2 &= 2 + x^2 \\ \Rightarrow \quad 2x &= \frac{2}{x} + x \\ \Rightarrow \quad x &= \frac{1}{2}\left(\frac{2}{x} + x\right) . \end{aligned}$$

This suggests using the sequence

$$u_{n+1} = \frac{1}{2}\left(\frac{2}{u_n} + u_n\right)$$

will give successively better approximations for $\sqrt{2}$. There are other ways to rearrange the equation $x^2 = 2$ to get $x = \ldots$, some will give a sequence which converges to $\sqrt{2}$ but others will converge to $-\sqrt{2}$ and some might not converge at all.

(v) Solving $l = \frac{1}{3}\left(\frac{a}{l^2} + 2l\right)$ will give the limit as $\sqrt[3]{a}$ and so if you set $a = 7$ you will be able to find an approximation for $\sqrt[3]{7}$. Six iterations should be enough to find $\sqrt[3]{7}$ correct to 8 decimal places.

The STEP question (2006 STEP II Q1)

3 **(i)** In each case we start with $u_1 = 2$.

(a) Here we can put $u_2 = u_1 = 2$ and solve for k to get $k = 20$.

(b) Here we need to find an expression for u_3 in terms of $u_1 = 2$.

$$\begin{aligned} u_2 &= k - \tfrac{36}{2} = k - 18 \\ u_3 &= k - \frac{36}{k-18} \\ &= \frac{k(k-18)-36}{k-18} \end{aligned}$$

Then setting $u_3 = u_1 = 2$ gives:

$$\begin{aligned} 2 &= \frac{k(k-18)-36}{k-18} \\ 2(k-18) &= k^2 - 18k - 36 \\ 2k - 36 &= k^2 - 18k - 36 \\ k^2 - 20k &= 0 \end{aligned}$$

We already know that $k = 20$ gives the constant sequence, so the value that gives a sequence with period 2 is $k = 0$.

We are taking the definition of *period* m to mean that the sequence repeats itself after m terms, but does not repeat itself after any number of terms smaller than m.

(c) Here we need $u_5 = u_1 = 2$, so start by finding an expression for u_5. We have:

$$\begin{aligned} u_3 &= \frac{k^2 - 18k - 36}{k - 18} \\ u_4 &= k - \frac{36(k-18)}{k^2 - 18k - 36} \\ &= \frac{k^3 - 18k^2 - 36k - 36k + 18 \times 36}{k^2 - 18k - 36} \\ u_5 &= k - \frac{36(k^2 - 18k - 36)}{k^3 - 18k^2 - 72k + 18 \times 36} \\ &= \frac{k^4 - 18k^3 - 72k^2 + 18 \times 36k - 36(k^2 - 18k - 36)}{k^3 - 18k^2 - 72k + 18 \times 36} \\ &= \frac{k^4 - 18k^3 - 108k^2 + 36 \times 36k + 36^2}{k^3 - 18k^2 - 72k + 18 \times 36} \end{aligned}$$

Setting $u_5 = 2$ gives:

$$\begin{aligned} 2(k^3 - 18k^2 - 72k + \cancel{18 \times 36}) &= k^4 - 18k^3 - 108k^2 + 36 \times 36k + \cancel{36^2} \\ 2k^3 - 36k^2 - 4 \times 36k &= k^4 - 18k^3 - 108k^2 + 36 \times 36k \\ 0 &= k^4 - 20k^3 - 72k^2 + 40 \times 36k \\ 0 &= k^4 - 20k^3 - 72k^2 + 1440k \end{aligned}$$

Notice that I did not evaluate 18×36 etc. If I can avoid long multiplication, I do so (whilst being prepared to do it when necessary).

We already know that two of the roots must be $k = 0$ and $k = 20$ (from the previous work), so we can factorise the quartic. This gives:

$$\begin{aligned} k^4 - 20k^3 - 72k^2 + 1440k &= k\left(k^3 - 20k^2 - 72k + 1440\right) \\ &= k(k-20)\left(k^2 - 72\right) \end{aligned}$$

Remembering that we only want the values of k for which the sequence has period 4 (and not period 2 or constant) we have $k = \pm\sqrt{72} = \pm 6\sqrt{2}$.

(ii) If $u_n \geqslant 2$ we have $u_{n+1} = 37 - \frac{36}{u_n} \geqslant 37 - \frac{36}{2} = 19$ and of course $19 \geqslant 2$. Therefore, as $u_1 \geqslant 2$, we have $u_n \geqslant 2$ for all n.

To find the limit, solve $l = 37 - \dfrac{36}{l}$. This gives:

$$\begin{aligned} l^2 &= 37l - 36 \\ l^2 - 37l + 36 &= 0 \\ (l-36)(l-1) &= 0 \end{aligned}$$

There are two solutions, but only one is greater than 2. Hence $l = 36$.

You might like to test this with a spreadsheet.

Warm down

4 **(i)** Creating a dump 100 miles from Cairo will solve this one. On the first journey leave 100 miles worth of fuel at the dump 100 miles from Cairo and then return to Cairo and refill. By the time you get back to the dump, you have 200 miles worth of fuel left, add in the 100 miles worth from the dump and you can now make the 300 miles to your destination.

The distances travelled are:

100 miles to the dump
100 miles back to Cairo
100 miles to the dump
300 miles to the destination

So a total of 600 miles is travelled.

(ii) If you can set up a dump 400 miles from the destination with enough petrol in it, you can use part **(i)**. You need to create a dump 60 miles from Cairo twice, leaving 180 miles worth of fuel each time. When you reach the dump for the third time you will have travelled 5 lots of 60 miles, have 240 miles worth of fuel in the car and a dump of 360 miles worth. Put 60 miles worth into the car and then the situation is just like part **(i)**.

The distances travelled are:

60 miles to dump 1 (leave 180 miles worth of fuel)
60 miles back to Cairo
60 miles to dump 1 (leave 180 miles worth of fuel)
60 miles back to Cairo
60 miles to dump 1 (put 60 miles worth of fuel into the car)
100 miles to dump 2 (leave 100 miles worth of fuel)
100 miles back to dump 1 (put 300 miles worth of fuel into the car)
100 miles to dump 2 (put 100 miles worth of fuel into the car)
300 miles to the destination

So a total of 900 miles is travelled.

To make a longer trip, you have to set up more frequent dumps. If the range of the car with a full tank is R, then the best strategy turns out to be having dumps at distances R, $R + R/3$, $R + R/3 + R/5$, $R + R/3 + R/5 + R/7$, etc, from the destination. Since the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

(related to the so-called *harmonic* series $\sum \frac{1}{n}$) does not converge (it gets bigger without limit the more terms you add), there is no limit to the width of the dessert you can cross like this.