STEP Support Programme

Hints and Partial Solutions for Assignment 16

Warm-up

1 **(i)** **(a)** The range of f is $f(y) \geqslant 1$.
(b) Here we replace y with x^2 so $f(x^2) = 1 + x^4$.

(ii) Let $y = x^2$ which gives $f(y) = \sqrt{1+y^2}$. As $y = x^2$ (and x is real) we must have $y \geqslant 0$.

(iii) Let $y = \sqrt{x}$ which gives $f(y) = \sqrt{1+y^8}$. We have $y \geqslant 0$ again as the square root is defined to be positive.

(iv) If we have $f(x) = a^x$ then $f(x+y) = a^{x+y}$. You also have $f(y) = a^y$ and hence $f(x)f(y) = a^x a^y = a^{x+y} = f(x+y)$ and hence $f(x) = a^x$ is a possible solution. To find the value of a use the condition $f(1) = 2$ in $f(x) = a^x$ to give $a^1 = 2$. and so the solution is $f(x) = 2^x$

(v) This function appears to work in the opposite way from the one in part (v), so think about using the inverse to a^x, i.e. $\log_a(x)$. You then need to show that your function does indeed satisfy the given relationship. To find a use $f(2) = 1$ to give $\log_a(2) = 1$ i.e. $a = 2$.

(vi) A function of the form $f(x) = Ax$ will work since $A(x+y) = Ax + Ay$. We have $f(2) = 1$ so $2A = 1$ i.e. $A = \frac{1}{2}$.

Preparation

2 **(i)** There are lots of ways you can do these (e.g. $\sin 105° = \sin(60° + 45°)$). There is nothing wrong with using your calculator (or a website such as Wolfram Alpha) to check your solution approximately, but only **after** working it exactly out yourself.

Answers: $\sin(105°) = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and $\cos(75°) = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(ii) You should end up with $\sin 3A = 3\sin A - 4\sin^3 A$. You might find it easier to manipulate if you use the substitution $s = \sin A$.

$$\begin{aligned}
\sin(3A) &= \sin(2A + A)\\
&= \sin 2A \cos A + \cos 2A \sin A\\
&= 2\sin A \cos A \cos A + (1 - 2\sin^2 A)\sin A\\
&= 2s(1-s^2) + s - 2s^3\\
&= 3s - 4s^3\\
&= 3\sin A - 4\sin^3 A
\end{aligned}$$

The equation in $s = \sin A$ is:

$$\begin{aligned}
3s - 4s^3 - s &= 2(1-2s^2)\\
4s^3 - 4s^2 - 2s + 2 &= 0\\
2s^3 - 2s^2 - s + 1 &= 0\\
(s-1)(2s^2-1) &= 0
\end{aligned}$$

which has solutions $s = 1$ and $s = \pm\frac{1}{\sqrt{2}}$. These give the values of A as $A = \frac{1}{2}\pi, \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$.

(iii) We suggested that to show that $1+\sqrt{2}$ is a root you can substitute it and and show that does indeed give 0. When do the substitution, you see that each term has a common factor of $1+\sqrt{2}$ which you can factorise out. This gives:

$$(1+\sqrt{2})\left[2(1+\sqrt{2})^2 - (2\sqrt{2}+6)(1+\sqrt{2}) + (4\sqrt{2}+5) - 1\right]$$

which makes the surd manipulation a little easier. This simplifies to:

$$\begin{aligned}
&(1+\sqrt{2})\left[2(3+2\sqrt{2}) - (8\sqrt{2}+10) + (4\sqrt{2}+5) - 1\right]\\
=&(1+\sqrt{2})\left[6 + 4\sqrt{2} - 8\sqrt{2} - 10 + 4\sqrt{2} + 5 - 1\right]\\
=&(1+\sqrt{2})\left[16 - 16 + 8\sqrt{2} - 8\sqrt{2}\right] = 0\,.
\end{aligned}$$

You can also show that $x = 1+\sqrt{2}$ is a root by showing that $x - 1 - \sqrt{2}$ is a factor by algebraic division (which, since you are going to have to find the other roots, is not a totally bad idea). However, if you are going to show that $x = 1+\sqrt{2}$ is a root by this method then you need to show the long division very carefully and show that the last subtraction does give zero explicitly. You also need to state your final conclusion such as "hence $1+\sqrt{2}$ is a root".

We know that $x-1-\sqrt{2}$ is a factor, so we have:

$$2x^3-(2\sqrt{2}+6)x^2+(4\sqrt{2}+5)x-(\sqrt{2}+1)=(x-1-\sqrt{2})(Ax^2+Bx+C)$$

Equating coefficients gives $(x-1-\sqrt{2})(2x^2-4x+1)=0$, and so the other two roots are $x=1\pm\frac{1}{2}\sqrt{2}$.

(iv) The first equation has a root of $x=1$ so we can factorise to get $(x-1)(2x^2-3x-9)=0 \implies (x-1)(x-3)(2x+3)=0$ and so the roots are $x=1,3,-\frac{3}{2}$.

Substituting $x=ky$ gives $2k^3y^3-5k^2y^2-6ky+9=0$. The given equation in y has constant term 1, which suggests that we want to divide the above equations throughout by 9, and the coefficient of y is -2 which means we need $6k/9=2$ and hence $k=3$. Using the substitution $k=3$ gives:

$$\begin{aligned} 2x^3-5x^2-6x+9&=0 && \text{substitute } x=3y \\ 2\times 27y^3-5\times 9y^2-6\times 3y+9&=0 && \text{divide by 9} \\ 6y^3-5y^2-2y+1&=0 \end{aligned}$$

Therefore the roots of the second equation are given by $y=\frac{1}{3}x=\frac{1}{3},1,-\frac{1}{2}$.

The STEP question (2015 STEP I Q2)

3 **(i)** You can use the identities given in question **2 (i)** e.g.

$$\begin{aligned}\cos 15^\circ &= \cos(45^\circ - 30^\circ)\\ &= \cos 45^\circ \cos 30^\circ + \sin 30^\circ \sin 45^\circ\\ &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{2} \times \frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}\end{aligned}$$

There are many ways of finding $\sin 15^\circ$. You could start with $\sin(45^\circ - 30^\circ)$ (which is probably the easiest approach). You could use your value of $\cos 15^\circ$ in $\cos^2 15^\circ + \sin^2 15^\circ = 1$ to find $\sin 15^\circ$. You could use $\cos 2A = 2\cos^2 A - 1$, but in this case you will need to simplify $\sqrt{\sqrt{3}+2}$ which is a little off putting. Whichever you use you should obtain $\sin 15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}}$.

(ii) Using the same idea as in question **2 (ii)**, write $\cos 3\alpha$ in terms of $\cos\alpha$ to get the equation $4x^3 - 3x - 4\cos^3\alpha + 3\cos\alpha = 0$. You should then be able to demonstrate that $x = \cos\alpha$ is a root of this (substitute $x = \cos\alpha$ and show that this makes the LHS equal to 0).

You now know that $(x - \cos\alpha)$ is a factor, so the equation can be written as:

$$4x^3 - 3x - 4\cos^3\alpha + 3\cos\alpha = (x - \cos\alpha)(Ax^2 + Bx + C) = 0$$

Equating coefficients gives $A = 4$, $-\cos\alpha \times A + B = 0 \implies B = 4\cos\alpha$ and $-C \times \cos\alpha = -4\cos^3\alpha + 3\cos\alpha \implies C = 4\cos^2\alpha - 3$. It is a good idea to check that the coefficient of x is also correct.

This gives us the equation $(x - \cos\alpha)(4x^2 + 4\cos\alpha x + 4\cos^2\alpha - 3) = 0$.

Using the quadratic formula gives the other two roots as:

$$\begin{aligned}x &= \frac{-4\cos\alpha \pm \sqrt{16\cos^2\alpha - 4\times 4(4\cos^2\alpha - 3)}}{8}\\ &= -\frac{1}{2}\cos\alpha \pm \frac{\sqrt{\cos^2\alpha - (4\cos^2\alpha - 3)}}{2}\\ &= -\frac{1}{2}\cos\alpha \pm \frac{\sqrt{3 - 3\cos^2\alpha}}{2}\\ &= -\frac{1}{2}\cos\alpha \pm \frac{\sqrt{3}}{2}\sin\alpha.\end{aligned}$$

(iii) If you set $y = 2x$ the equation will become:

$$\begin{aligned} y^3 - 3y - \sqrt{2} &= 0 \\ 8x^3 - 6x - \sqrt{2} &= 0 \\ 4x^3 - 3x - \frac{\sqrt{2}}{2} &= 0 \end{aligned}$$

which has the same form as the equation in part **(ii)** with $\cos 3\alpha = \frac{1}{\sqrt{2}}$. This means that $\alpha = 15°$ and you can use the work from part **(ii)** to find the values of x which are $\cos 15°, -\frac{1}{2}\cos 15° \pm \frac{\sqrt{3}}{2}\sin 15°$.

Using the values of $\cos 15°$ and $\sin 15°$ found in part **(i)** gives the final answers as $y = -\sqrt{2},\ \dfrac{1 \pm \sqrt{3}}{\sqrt{2}}$.

Warm down

4 It will probably be helpful to mark in all the right angles and equal sides you are given before starting.

(i) Use ASA to show that the triangles are congruent.

(ii) Here use SAS to show that $\triangle GCD$ is congruent to $\triangle GBD$. Remember that D is the midpoint of BC.

(iii) We have just shown $GC = CB$, and from part **(i)** we know $GE = GF$. It follows by Pythagoras' theorem that $EC = FB$.

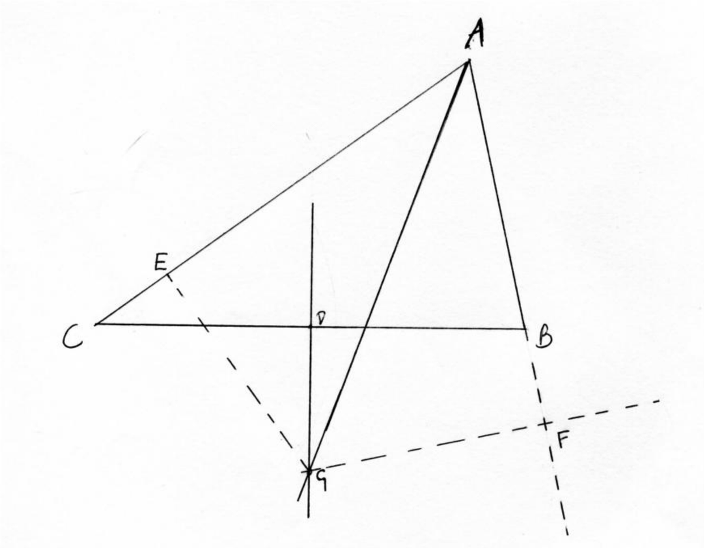
Hence as $AE = AF$ and $EC = FB$ we have $AC = AB$ and the triangle is isosceles.

(iv) It cannot be the case that **all** triangles are isosceles, so something is wrong with the "proof". There does not seem to be any wrong steps in the argument, so there must be something wrong with the diagram, or we made an incorrect assumption somewhere.

If you draw a non-isosceles triangle and carefully construct the points and lines as shown you should find that the point G lies outside of the triangle[1]. This is not a problem in itself, because the proof still works.

However, you should also find that exactly one of E and F lies outside the triangle, i.e. on either AC produced or AB produced. In the picture below, we have $AE = AF$ and $EC = FB$ **but** $AC = AE + EC$ and $AB = AF - FB$, so the proof now fails.

Please note that, although I have rubbed out the construction lines for clarity, they were there!



[1]You can prove that this must be the case. Assume that $AB < AC$ and let the angle bisector at A intersect the line BC at X. You can use the Sine Rule to show that $CX > XB$ and hence that the angle bisector at A intersects the perpendicular bisector to BC outside the triangle.