STEP Support Programme

Hints and Partial Solutions for Assignment 17

Warm-up

1 You need to be quite careful with these proofs to ensure that you are not assuming something that should not be assumed.

For example, in general:

$$(x+y) \bmod a \neq x \bmod a + y \bmod a\,.$$

Take $x = 9$, $y = 13$ and $a = 5$ to see why it doesn't hold ($2 \neq 4 + 3$).

(i) We have

$$N_1 + N_2 = n_1 + am_1 + n_2 + am_2 = n_1 + n_2 + a(m_1 + m_2)$$

for some integers m_1 and m_2 and some integers n_1 and n_2 both in the interval 0 to $a - 1$, and therefore:

$$(N_1 + N_2) \bmod a = (n_1 + n_2) \bmod a$$

If $n_1 + n_2 < a$ then we have $(n_1 + n_2) \bmod a = n_1 + n_2$, but if $n_1 + n_2 > a$ then we have $n_1 + n_2 = a + (n_1 + n_2) \bmod a$.

Note that as both n_1 and n_2 are less than a, we must have $n_1 + n_2 < 2a$.

Similarly

$$N_1N_2 = (n_1 + am_1)(n_2 + am_2)$$

and

$$(n_1 + am_1)(n_2 + am_2) = n_1n_2 + a(m_1 + m_2 + am_1m_2)\,.$$

Then write $n_1n_2 = aq + (n_1n_2) \bmod a$ for some suitable q. This gives us:

$$N_1N_2 = a\left[q + m_1 + m_2 + am_1m_2\right] + (n_1n_2) \bmod a$$

and hence we have:

$$N_1N_2 \bmod a = n_1n_2 \bmod a\,.$$

(ii) We have:

$$\begin{aligned} 10a + b &= (10 \bmod 3) \times (a \bmod 3) + b \bmod 3 && [\text{mod } 3]\\ &\equiv 1 \times (a \bmod 3) + b \bmod 3 && [\text{mod } 3]\\ &\equiv (a + b) && [\text{mod } 3] \end{aligned}$$

If $10a + b$ is divisible by three then we need $10a + b = 0 \pmod 3$, which is true if and only if $a + b = 0 \pmod 3$ i.e. $a + b$ is divisible by three.

As we know $10 \bmod 3 = 1$, we have $10^n \equiv 1^n \bmod 3$. The n-digit number with digits $a_1 a_2 \cdots a_n$ is equal to $10^{n-1}a_1 + 10^{n-2}a_2 + \cdots + a_n$. We then have:

$$10^{n-1}a_1 + 10^{n-2}a_2 + \cdots + a_n = a_1 + a_2 + \cdots + a_n \pmod 3$$

and so any number is divisible by 3 if and only if the digit sum is divisible by 3.

(iii) We note that $10^n = (11 - 1)^n$ so

$$10^n \equiv (-1)^n \text{ (mod 11)}$$

and

$$10^3 a + 10^2 b + 10c + d \equiv -a + b - c + d \pmod{11}$$

and so a 4 digit number is divisible by 11 if and only if the negative of the first digit plus the second digit minus the third digit plus the last digit is divisible by 11 (this is a lot nicer written in symbols!). For example, 3916 is divisible by 11 as $-3+9-1+6 = 11$.

Similar calculations work for other sized numbers.

Preparation

2 **(i)** Easiest way to start here is to subtract the second equation from the first equation, and subtract the fourth equation from the third (it is a very good idea to label the equations!). You will then get two equations in just x and z, which are $2x + 2z = 1$ and $6x + 12z = 4$. These can be solved to give $x = \frac{1}{3}$ and $z = \frac{1}{6}$.

Substituting these values into the first and third equation (for example) gives the equations $w + y = \frac{1}{2}$ and $4w + 2y = \frac{3}{2}$. Solving these gives $w = \frac{1}{4}$ and $y = \frac{1}{4}$.

(ii) Note that we don't have to start with $i = 1$.

(a) $\frac{3}{2} + \frac{4}{3} + \frac{5}{4} = \frac{49}{12}$
(b) $4 + 1 + 0 + 1 + 4 = 10.$

(iii) To do the first part, you could use partial fractions on $\dfrac{1}{r(r+1)}$, or you could start on the RHS and use $\dfrac{1}{r} - \dfrac{1}{r+1} = \dfrac{(r+1) - r}{r(r+1)} = \dfrac{1}{r(r+1)}$.

Consider $\sum\limits_{r=1}^{n} \left(\dfrac{1}{r} - \dfrac{1}{r+1} \right)$. If you write the first few terms and the last one or two you will see that lots of bits cancel.

Answer: $\frac{n}{n+1}$

The check with $n = 3$ gives:

$$\begin{aligned} \frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \frac{1}{3 \times 4} &= \frac{1}{2} + \frac{1}{6} + \frac{1}{12} \\ &= \frac{9}{12} \\ &= \frac{3}{4} = \frac{n}{n+1} \qquad \text{as required} \end{aligned}$$

Note that $\sum\limits_{r=100}^{200} = \sum\limits_{r=1}^{200} - \sum\limits_{r=1}^{99}$, so the sum is equal to $\frac{200}{201} - \frac{99}{100} = \frac{101}{20100}$.

The numerator is equal to $200 \times 100 - 201 \times 99$. Re-writing the second term as $(200 + 1)(100 - 1)$ means that the numerator can be written as

$$\begin{aligned} &200 \times 100 - (200 + 1)(100 - 1) \\ =&\cancel{200 \times 100} - \left(\cancel{200 \times 100} - 200 + 100 - 1\right) \\ =&200 - 100 + 1 = 101 \end{aligned}$$

(iv) If two algebraic expressions are *equivalent* then they are equal in value for any value of x. Using the given values of x gives the simultaneous equations $a + \frac{1}{2}b = \frac{1}{2}$ and $\frac{1}{2}a + \frac{1}{3}b = \frac{1}{6}$ which you can solve to get $a = 1$ and $b = -1$.

Splitting the fraction up like this is known as "partial fractions". Another way of doing this is to equate numerators to get $1 \equiv a(x+2) + b(x+1)$ and then you can substitute $x = -1$ and $x = -2$ to find a and b.

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3 For the first part, using $n=0$ gives $s=1$. Then using $n=-1$ and $n=1$ gives the equations $1=-p+q-r+s$ and $2=p+q+r+s$ which means that we can cancel p and r easily to give $2q+2s=3 \implies q=\frac{1}{2}$. Using $n=1$ and $n=2$ gives $p+r=\frac{1}{2}$ and $8p+2r=3$ which can be solved to give $p=\frac{1}{3}$ and $r=\frac{1}{6}$.

Then use:

$$\sum_{r=-1}^{n} r^2 = \sum_{r=0}^{n} r^2 + (-1)^2$$

which gives:

$$\sum_{r=0}^{n} r^2 = \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n + 1 - 1\,.$$

You can then take out a factor of $\frac{1}{6}n$ to give $\frac{1}{6}n\left(2n^2+3n+1\right) = \frac{1}{6}n(2n+1)(n+1)$.

For the second part, the easiest thing is to take $n=-2,-1,0,1,2$ — note that we need 5 equations as there are 5 unknowns. Other values of n are possible but not as easy, For example, adding together the equations obtained by using $x=-2$ and $x=2$ will result in b and d cancelling.

Using $n=0$ gives $(-2)^3+(-1)^3+0^3=e \implies e=-9$.

Using $n=-1$ and $n=1$ gives the two equations $a-b+c-d+e=-9$ and $a+b+c+d+e=-8$. These can be added and subtracted to give:

$$2a+2c+2e=-17 \tag{1}$$
$$2b+2d=1 \tag{2}$$

Using $n=-2$ and $n=2$ gives $16a-8b+4c-2d+e=-8$ and $16a+8b+4c+2d+e=0$. These can be combined to give:

$$32a+8c+2e=-8 \tag{3}$$
$$16b+4d=8 \tag{4}$$

Solving the simultaneous equations given by (2) and (4) gives $b=\frac{1}{2}$ and $d=0$. Solving (1) and (3) (remembering that $e=-9$) gives $a=\frac{1}{4}$ and $c=\frac{1}{4}$.

We also need:

$$\sum_{r=-2}^{n} r^3 = \sum_{r=0}^{n} r^3 + (-1)^3 + (-2)^3$$

which gives:

$$\begin{aligned}\sum_{r=0}^{n} r^3 &= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2 - 9 + 9\\ &= \tfrac{1}{4}n^2(n^2+2n+1)\\ &= \tfrac{1}{4}n^2(n+1)^2\end{aligned}$$

Warm down

4 **(i)** If you use $3 \equiv -2 \pmod 5$ then you get:

$$3 \times 2^{2n} + 2 \times 3^{2n} \equiv -2 \times 2^{2n} + 2 \times (-2)^{2n} \pmod 5$$

and as $2n$ is even, $(-2)^{2n} = 2^{2n}$, hence $3 \times 2^{2n} + 2 \times 3^{2n} \equiv -2 \times 2^{2n} + 2 \times 2^{2n} \equiv 0 \pmod 5$.

(ii) We have $2 \equiv 5 \equiv -1 \pmod 3$ and $56 \equiv 2 \pmod 3$.
Hence $2^n + 5^n + 56 \equiv (-1)^n + (-1)^n + 2 \pmod 3$.
We also have $2^n + 5^n + 56 \equiv 2^n + (-2)^n + 0 \pmod 7$.

So if n is odd, $2^n + 5^n + 56 \equiv 0$ for both mod 3 and mod 7, so it is divisible by both 3 and 7. Since 3 and 7 have no common factors (as they are both prime) then $2^n + 5^n + 56$ must be divisible by 21.

And for the last part: $2^{3m} + 5^{3m} + 56 = 8^m + 125^m + 56 \equiv (-1)^m + (-1)^m + 2 \pmod 9$, which is equivalent to 0 if m is odd. Hence $2^n + 5^n + 56$ is divisible by 9 and 7 and so is divisible by 63.

In the general case for n odd, we have three cases, $n = 3m$ (where m must be odd - this is the case you were asked to consider) or $n = 3m + 1$ (here m must be even) or $n = 3m + 2$ (m must be odd). You then consider each case separately, so for example the second case gives:

$$\begin{aligned} 2^n + 5^n + 56 &= 2^{3m+1} + 5^{3m+1} + 56 && m \text{ even} \\ &\equiv 2 \times (2^3)^m + 5 \times (5^3)^m + 2 && [\text{mod } 9] \\ &\equiv 2 \times (-1)^m + 5 \times (-1)^m + 2 && [\text{mod } 9] \\ &\equiv 9 \equiv 0 && [\text{mod } 9] \end{aligned}$$

(iii) Lots of ways of doing this. For example, you could start by showing that:

$$2^{3n+1} + 3 \times 5^{2n+1} = 2 \times 8^n + 15 \times 25^n$$

and then consider this modulo 17 (using $25 \equiv 8\,[\text{mod } 17]$) to give:

$$\begin{aligned} 2^{3n+1} + 3 \times 5^{2n+1} &= 2 \times 8^n + 15 \times 25^n \\ &\equiv 2 \times 8^n + 15 \times 8^n && [\text{mod } 17] \\ &= 17 \times 8^n \equiv 0 && [\text{mod } 17] \end{aligned}$$