STEP Support Programme

Hints and Partial Solutions for Assignment 18

Warm-up

1 You can use Desmos to check your graphs (and please do so for parts **(i)** and **(ii)**). However, if your graph is wrong it is **vitally** important that you work out why **before** correcting it.

(i) Note that $y = \dfrac{1}{x-1}$ can be considered as a translation of $y = \dfrac{1}{x}$.

(ii) If we write $\dfrac{x}{x-1} = 1 + \dfrac{1}{x-1}$ we could draw this as a translation of part **(i)**.

(iii) Here we can use long division, i.e. do $x^2 \div (x-1)$ (you may find it easier to write x^2 as $x^2 + 0x + 0$), or write $\dfrac{x^2}{x-1} \equiv ax + b + \dfrac{c}{x-1}$ (by comparison with part **(ii)**), to get $x^2 \equiv (ax+b)(x-1) + c$. This gives $\dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$ and so we can see that as $x \to \infty$, $y \approx x + 1$, so there is an *oblique* asymptote at $y = x + 1$.

A different approach would be to sketch $y = \dfrac{(x+1)^2}{x}$, i.e. $y = x + 2 + \dfrac{1}{x}$, and then translate this to get $y = \dfrac{x^2}{x-1}$.

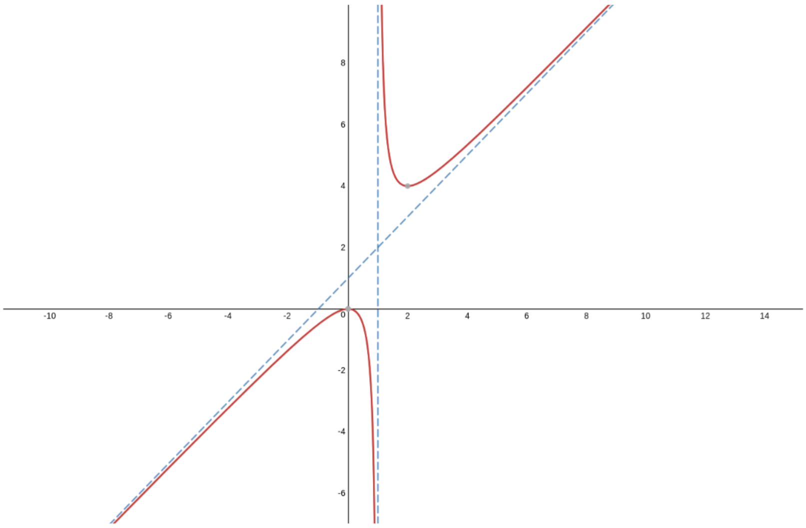
You can show that the graph cannot cross the oblique asymptote by trying to solve $\dfrac{x^2}{x-1} = x + 1$. This gives:

$$\begin{aligned} \frac{x^2}{x-1} &= x + 1 \\ x^2 &= x^2 - 1 \end{aligned}$$

Hence there are no solutions and the curve does not cross the asymptote.

Graphs **can** cross oblique or horizontal asymptotes in some cases; do not regard them as impassible laser beams. In this particular case there are no solutions to $\dfrac{x^2}{x-1} = x+1$ so the curve cannot cross the asymptote.

The derivative is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1 - \dfrac{1}{(x-1)^2}$ which gives turning points at $(0,0)$ and $(2,4)$. The graph is shown on the next page, with asymptotes in blue dotted lines.

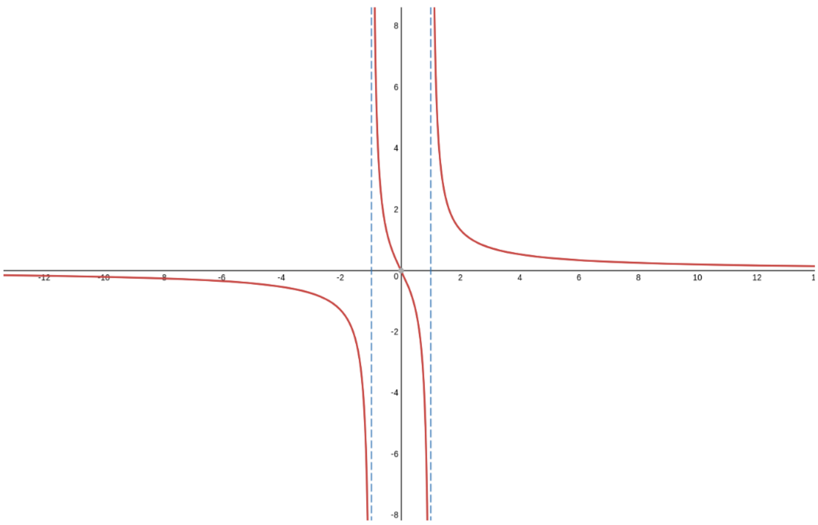


(iv) There are two vertical asymptotes at $x = 1$ and $x = -1$. Also as $x \to \pm\infty$, $y \to 0$, and for added brownie points you can show whether y tends to 0 from above the x-axis or below the x-axis (i.e. whether $y \to 0_+$ or $y \to 0_-$).

$y \to 0_+$ means that y tends to 0 whilst being positive (so tends to 0 "from above").

The derivative is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\dfrac{1}{(x-1)^2} - \dfrac{1}{(x+1)^2}$ and so the gradient is always negative. This means that there are no turning points and the graph always heads "downwards".

You can also show that the graph is of an *odd* function, i.e. $\mathrm{f}(-x) = -\mathrm{f}(x)$, and so it will have rotation symmetry about the origin. Another point of possible interest is that there is a (non-stationary) point of inflection at the origin (this is where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$).



Preparation

2 **(i)** You can either use the quadratic formula or spot an "obvious" root and factorise to obtain x in terms of a. If you spot that $x = 1$ is a root you can factorise to get $[x-1]\,[(a+2)x+a]$ to give the second root as $x = -\frac{a}{a+2}$.

(ii) If $-7x - 7 < 0$ then $x > -1$.

Then consider $3x^3 - x^2 - 7x - 7 < 0$. If $x = -\frac{7}{9}$, then the first two terms are both negative (why?) and since $x > -1$ you know that $-7x - 7 < 0$ and hence the whole thing is negative.

(iii) If $a/b > 0$, then either both a and b are positive, or they are both negative.

For the second part, if $x > 0$ then to have $y > 0$ we must have $x - 1 > 0$ and hence $x > 1$.

If $y > x$ then we have $\dfrac{x}{x-1} > x$. Since $x > 0$ we can divide by x to get $\dfrac{1}{x-1} > 1$. We also know that $x > 1$, so $x - 1 > 0$ and hence we can multiply by $x - 1$ to get $1 > x - 1$ and so $x < 2$.

The STEP question (2014 STEP I Q3)

3 It may be quickest to do the whole question using the relationship:

$$\tfrac{1}{3}\left(b^3-a^3\right)=\left(\tfrac{1}{2}\left(b^2-a^2\right)\right)^2$$

which comes directly from evaluating the integrals given in the stem. This becomes:[1]

$$\begin{aligned}\tfrac{1}{3}\cancel{(b-a)}\left(b^2+ab+a^2\right)&=\tfrac{1}{4}\cancel{(b-a)}(b+a)\left(b^2-a^2\right)\quad\text{since } b\neq a\\ 4\left(b^2+ab+a^2\right)&=3\left(b^3-a^2b+ab^2-a^3\right)\end{aligned}\qquad (*)$$

Or is you prefer you can leave the RHS as $3(b+a)^2(b-a)$.

When you divide by $(b-a)$ you do need to state why it is possible to do this, i.e. that since $b>a$ then $b-a>0$.

(i) Substitute $a=0$ into $(*)$ gives $4b^2=3b^3$. Since $b>0$ we have $b=\frac{4}{3}$.

(ii) Substituting $a=1$ into $(*)$ gives $4(b^2+b+1)=3(b^3+b^2-b-1)$ which simplifies to $3b^3-b^2-7b-7=0$.

The turning points of $y=3x^3-x^2-7x-7$ are where $y'=9x^2-2x-7=0$ i.e. we have $(x-1)(9x+7)=0$ and $x=1$ or $x=-\frac{7}{9}$. For both of these x values the y value is negative and so both turning points lie below the x-axis. This means that there is only one root of the cubic. Evaluating for $x=2$ and $x=3$ will show that the root lies between 2 and 3 as we have:

$$\begin{aligned}b=2\quad&\Longrightarrow\quad 3b^3-b^2-7b-7=24-4-14-7=-3\\ b=3\quad&\Longrightarrow\quad 3b^3-b^2-7b-7=81-9-21-7=44\end{aligned}$$

Hence the root lies between 2 and 3.

Note that the question stated "with the help of a sketch", so your solution should include one — even though ours does not!

(iii) There are lots of ways of doing this. One way is to write $a=\frac{p-q}{2}$ and $b=\frac{p+q}{2}$ and substitute these in. If you divide by q or $b-a$ (which will be necessary if you don't start at $(*)$) you do need to state that you can do this "as $q\neq 0$". We have:

$$\begin{aligned}4\left(b^2+ab+a^2\right)&=3(b+a)\left(b^2-a^2\right)\\ 4\left(\frac{(p+q)^2}{4}+\frac{p^2-q^2}{4}+\frac{(p-q)^2}{4}\right)&=3p^2q\\ p^2+\cancel{2pq}+q^2+p^2-q^2+p^2-\cancel{2pq}+q^2&=3p^2q\\ 3p^2+q^2&=3p^2q\end{aligned}$$

You can rearrange $3p^2+q^2=3p^2q$ to get $p^2=\frac{q^2}{3(q-1)}$. Since $p^2>0$ and $q^2>0$ we must have $q-1>0$ and so $q>1$ (i.e. $b-a>1$). Since we know that $a\geqslant 0$ we have $p\geqslant q$ and hence $p^2\geqslant q^2$ (as both p and q are positive). We therefore have $\frac{q^2}{3(q-1)}\geqslant q^2$ and so $\frac{1}{3(q-1)}\geqslant 1$, and since $q>1$ we can write this as $3(q-1)\leqslant 1 \implies q\leqslant\frac{4}{3}$. Hence we have $1<b-a\leqslant\frac{4}{3}$.

[1] The difference of two cubes identity from Assignment 14 will be useful.

Warm down

4 Fractals were very trendy about 20 years ago, but they seemed to have fallen out of the public consciousness more recently[2]. There are practical applications of fractals, including image compression, computer generated landscapes and compact antennas (produced by a company called "Fractenna").

One of the defining characteristics of fractals is *self-similarity*, i.e. if you zoom in, the small portion resembles the whole thing. If you zoom in on one side of the Koch Snowflake what you see looks like the original side.

Another characteristic of fractals is that they can have a non-integer dimension. For example, the Koch Snowflake has dimension $\log_3(4) \approx 1.2619$ — somewhere between a line and a plane, but closer to a line. Fractal dimensions have been estimated for various coastlines, and Great Britain has an estimated fractal dimension of 1.25. Others include Australia (1.13) and Norway (1.52 - it has lovely crinkly edges).

(i) The number of edges is multiplied by 4 at each stage, so the total number is 3×4^n. We are taking the original triangle as $n = 0$ and the "first iteration" as $n = 1$.

(ii) The length of each edge is divided by 3 at each stage. In the first iteration ($n = 1$) each edge has length $\frac{1}{3}$ and at the n^{th} iteration they each have length $\left(\frac{1}{3}\right)^n$.

(iii) Multiplying the answers to parts **(i)** and **(ii)** gives the total length of the curve as:

$$\left(\tfrac{1}{3}\right)^n \times 3 \times 4^n = 3 \times \left(\tfrac{4}{3}\right)^n .$$

This means that the total length edge is a geometric progression with common ratio $\frac{4}{3}$. Hence as the number of iterations tends to infinity, the total edge length also tends to infinity.

(iv) For the first iteration, the small triangles each have area $\frac{A}{9}$; try splitting the original triangle into smaller triangles each with side length $\frac{1}{3}$ of the original one to see why this is true. The area of the first iteration is therefore $A + 3 \times \left(\frac{A}{9}\right)$.

In the second iteration, each of the 12 sides will have a triangle of area $\frac{A}{9^2}$ added on, so the area is $A + 3 \times \left(\frac{A}{9}\right) + 12 \times \left(\frac{A}{9^2}\right)$.

The area of the n^{th} iteration is:

$$\begin{aligned} A_n &= A + 3 \times \left(\tfrac{A}{9}\right) + 3 \times 4 \times \left(\tfrac{A}{9^2}\right) + 3 \times 4^2 \times \left(\tfrac{A}{9^3}\right) + \cdots + 3 \times 4^{n-1} \times \left(\tfrac{A}{9^n}\right) \\ &= A + 3 \times \left(\tfrac{A}{9}\right) \times \left(1 + \tfrac{4}{9} + \tfrac{4^2}{9^2} + \cdots + \tfrac{4^{n-1}}{9^{n-1}}\right) \\ &= A + 3 \times \left(\tfrac{A}{9}\right) \times \left(\frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}}\right) \end{aligned}$$

[2] My hopes that the popularity of "Frozen" would result in fractals becoming well-known again seems to have been unfounded.

And hence as $n \to \infty$ we have $A_n \to A + \frac{1}{3}A \times \frac{9}{5} = \frac{8}{5}A$, which is finite.

You could also find A, as this is the area of an equilateral triangle with side length 1.

One interesting conclusion is that, as $n \to \infty$, the total length of the curve tends to infinity, but the area remains bounded. This implies that you would not be able to draw along all the edges of the infinite fractal, but you would be able to colour it in.