STEP Support Programme

Hints and Partial Solutions for Assignment 19

Warm-up

1 **(i)** Since OBT is a right angled triangle the length BT will be $r\tan\theta$. Looking at the relevant areas we have:

$$\tfrac{1}{2}r \times r\tan\theta > \tfrac{1}{2}r^2\theta > \tfrac{1}{2}r^2\sin\theta\,.$$

Then we can then divide by $\frac{1}{2}$, r^2 and $\sin\theta$ (all non-zero) to get the required result.

(ii) We have $\lim\limits_{\theta\to 0}\left(\dfrac{1}{\cos\theta}\right) = 1$ so as θ tends to 0, $\left(\dfrac{\theta}{\sin\theta}\right)$ is trapped between 1 and something that is tending to 1. Therefore we must have

$$\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1 \text{ and } \sin\theta \approx \theta \text{ when } \theta\approx 0\,.$$

Things to watch out for:

(1) Don't write "$\sin\theta = \theta$ for small θ"; it should be "$\sin\theta\approx\theta$ for small θ".

(2) Don't write $\lim\limits_{\theta\to 0}\left(\dfrac{\theta}{\sin\theta}\right) = \dfrac{0}{0} = 1$ because $\dfrac{0}{0}$ could be anything. For example, $\lim\limits_{x\to 0}\left(\dfrac{x}{x^2}\right) = \infty$ even though both the numerator and denominator tend to 0. Another example is $\lim\limits_{x\to 0}\left(\dfrac{2x}{x}\right) = 2$.

(3) Don't write "$\sin\theta\to\theta$ as $\theta\to 0$". We should only have a constant on the right hand side of $\to$. You can write $\sin\theta - \theta \to 0$ as $\theta\to 0$.

(4) Note that when taking a limit, you have to replace $>$ with $\geqslant$. For example, $\theta > \sin\theta$ for $\theta > 0$, but $\theta = \sin\theta$ when $\theta = 0$.

(5) The point of this part of the question was to derive the approximation $\sin\theta\approx\theta$, valid for small θ (which means much smaller than 1; we write this as $\theta \ll 1$). We can't therefore use this result earlier in the question to obtain the $\lim\limits_{\theta\to 0}\dfrac{\theta}{\sin\theta}$; that is why the question says firmly 'Using part **(i)** ...'.

(iii) Starting from:

$$\tfrac{1}{2}r \times r\tan\theta > \tfrac{1}{2}r^2\theta > \tfrac{1}{2}r^2\sin\theta$$

we can divide by $\frac{1}{2}$, r^2 to get:

$$1 > \frac{\theta}{\tan\theta} > \cos\theta$$

Then it is more or less the same as part **(ii)**. We have $\tan\theta \approx \theta$ for small θ.

(iv) Writing $\cos\theta = (1-\sin^2\theta)^{\frac{1}{2}}$, we can use the binomial expansion with $x = -\sin^2\theta$ and $n = \frac{1}{2}$. This gives $\cos\theta = 1 + \frac{1}{2}(-\sin^2\theta) + \dfrac{\frac{1}{2}\times -\frac{1}{2}}{2!}(-\sin^2\theta)^2 + \cdots$. Using $\sin\theta \approx \theta$ and ignoring terms of θ^4 and higher powers gives $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

(v) You used radians at the point where you found the area of the sector OBA.

Preparation

2 **(i)** The distance is $\sqrt{x^2 + y^2 - 4x + 4y + 8}$.

(ii) If X is equidistant from P and Q then we have $PX = QX$, or perhaps more usefully $PX^2 = QX^2$. This gives:

$$x^2 + y^2 - 4x + 4y + 8 = x^2 + y^2 - 8x + 16$$

which simplifies to $4x + 4y = 8$ i.e. $x + y = 2$.

There are other ways of obtaining the required equation, but you were told to consider the distances PX and QX, so this is the method you **have** to use.

(iii) If the two equations describe the same line then the coefficients will be in the same ratio, so:

$$\frac{a}{1} = \frac{4}{a} = \frac{b}{2}\,.$$

Using the first equality gives $a^2 = 4$, so $a = \pm 2$. The second equality gives $ab = 8$, and so $b = \pm 4$ (and a and b have the same sign).

(iv)

$$\begin{aligned} 5(1-p) &= 2px - y \\ 5 + y &= 2px + 5p \\ p &= \frac{5+y}{2x+5} \end{aligned}$$

(v) Start by factorising $xy + 2y + 3x$ to get $(x+2)(y+3) - 6$. Alternatively, since the answer is given you could have expanded $(x+2)(y+3) = 60$ and shown that this gives the starting equation.

If both x and y are positive integers there are only some possible values of x and y. A systematic approach is needed, and it is useful to note that $(x+2) \geqslant 3$ and $(y+3) \geqslant 4$ (assuming that 0 is not a “positive integer”).

$x+2$	$y+3$	x	y
3	20	1	17
4	15	2	12
5	12	3	9
6	10	4	7
10	6	8	3
12	5	10	2
15	4	13	1

The STEP question (2005 STEP I Q6)

3 **(i)** It may be easier to use $AP^2 = 4BP^2$ which gives:

$$(5-x)^2 + (16-y)^2 = 4\left[(-4-x)^2 + (4-y)^2\right] .$$

The algebra is a little intimidating, but if you hold your nerve this will simplify to $3x^2+3y^2+42x = 153$. Dividing by 3 and noting that $x^2+y^2+14x = (x+7)^2+y^2-49$ will lead to the required result.

$$\begin{aligned}
(5-x)^2 + (16-y)^2 &= 4\left[(-4-x)^2 + (4-y)^2\right] \\
25 - 10x + x^2 + 256 - 32y + y^2 &= 4\left[16 + 8x + x^2 + 16 - 8y + y^2\right] \\
25 + 256 - 64 - 64 &= 3x^2 + 3y^2 + 42x \\
153 &= 3(x^2 + y^2 + 42x) \\
51 &= x^2 + y^2 + 14x \\
51 &= (x+7)^2 + y^2 - 49 \\
100 &= (x+7)^2 + y^2
\end{aligned}$$

(ii) It is easier to do this question if you write both equations in expanded form, i.e.:

$$\begin{aligned}
x^2 + y^2 + 14x - 51 &= 0 \\
(k^2-1)x^2 + (k^2-1)y^2 + (2a - 2k^2b)x + k^2b^2 - a^2 &= 0
\end{aligned}$$

If these describe the same path (we should really say locus) then the **ratios** of the corresponding coefficients must be the same. Note that the corresponding coefficients are not necessarily equal, so you **cannot** conclude (for example) that $k^2 - 1 = 1$.

Using this you can get two expressions involving k^2. Using the coefficients of the x^2 and x terms gives:

$$k^2 - 1 = \frac{2a - 2k^2b}{14}$$

and using the coefficients of x^2 and the constant term gives:

$$k^2 - 1 = \frac{k^2b^2 - a^2}{-51} .$$

Each of these can be rearranged to get k^2 in terms of a and b. Doing this and equating gives the equation:

$$\frac{a+7}{b+7} = \frac{a^2+51}{b^2+51}.$$

The first equation rearranges as:

$$\begin{aligned}
14k^2 - 14 &= 2a - 2k^2b \\
k^2(14 + 2b) &= 2a + 14 \\
k^2 &= \frac{a+7}{b+7}
\end{aligned}$$

And the second rearranges as:

$$\begin{aligned}-51k^2+51 &= k^2b^2-a^2\\ 51+a^2 &= k^2(b^2+51)\\ k^2 &= \frac{a^2+51}{b^2+51}\end{aligned}$$

The final part requires you to expand $(a+7)(b^2+51)=(b+7)(a^2+51)$. You should find a common factor of $(b-a)$ which can be divided out since it is given that $a \neq b$. Make sure that when you divide by $b-a$ you write explicitly '$a \neq b$ (given)' — or something similar.

$$\begin{aligned}(a+7)(b^2+51)-(b+7)(a^2+51) &= 0\\ ab^2+7b^2+51a+\cancel{7\times 51}-ba^2-7a^2-51b-\cancel{7\times 51} &= 0\\ ab^2-ba^2+7(b^2-a^2)-51(b-a) &= 0\\ ab(b-a)+7(a+b)(b-a)-51(b-a) &= 0 \quad \text{since } b-a\neq 0 \text{ we can divide by } (b-a)\\ ab+7(a+b)-51 &= 0\\ (a+7)(b+7)-49-51 &= 0\\ (a+7)(b+7) &= 100 \quad \text{as required.}\end{aligned}$$

Note that I didn't need to evaluate 7×51.

This final result drops out a little more neatly if you start with $\dfrac{a+7}{b+7}=\dfrac{a^2+51}{b^2+51}$ and then subtract 1 from both sides to get $\dfrac{a+7}{b+7}-1=\dfrac{a^2+51}{b^2+51}-1$ before rearranging to get $\dfrac{a-b}{b+7}=\dfrac{a^2-b^2}{b^2+51}$. However I don't think the subtracting 1 is an obvious step, so I have chosen the "lower-tech" solution above.

Warm down

4 **(i)** There are various ways of doing this, but you have to read the question very carefully to ensure you are working out the required probability.

One way is to use a list of all the possibilities (which are equally likely):

1. Pick coin (H_1, T_1) and look at side H_1.
2. Pick coin (H_1, T_1) and look at side T_1.
3. Pick coin (H_1, H_2) and look at side H_1.
4. Pick coin (H_1, H_2) and look at side H_2.
5. Pick coin (T_1, T_2) and look at side T_1.
6. Pick coin (T_1, T_2) and look at side T_2.

The question tells you that you are looking at a head, so you must have one of the cases 1, 3 or 4. Of these three cases, two of them have a head on the other side, so the probability is $\frac{2}{3}$.

This question is equivalent to the infamous Monty Hall problem about a game show; also, to the Principle of Restricted Choice in bridge (the card game).

(ii) If you know about expectation, then you can find the expected gain per game by:

$$1 \times \mathrm{P}(\text{no sixes}) + (-1) \times \mathrm{P}(\text{one six}) + (-2) \times \mathrm{P}(\text{two sixes}) + (-3) \times \mathrm{P}(\text{three sixes})\,.$$

Which is equal to:

$$\left(\frac{5}{6}\right)^3 - 3 \times \frac{1}{6}\left(\frac{5}{6}\right)^2 - 2 \times 3\left(\frac{1}{6}\right)^2 \frac{5}{6} - 3 \times \left(\frac{1}{6}\right)^3 = \frac{125 - 75 - 30 - 3}{216} = \frac{17}{216}$$

and since this is positive you should accept!

Equivalently, you can consider what you would expect to happen over 216 games and whether you would expect to gain money or not.