STEP Support Programme

Hints and Partial Solutions for Assignment 1

Warm-up

1 You can check many of your answers to this question by using Wolfram Alpha. Only use this as a check though — and if your answer was wrong try again!

(i)

$$\begin{aligned}\sqrt{50}+\sqrt{18} &= \sqrt{25\times 2}+\sqrt{9\times 2}\\ &= 5\sqrt{2}+3\sqrt{2}\\ &= 8\sqrt{2}\end{aligned}$$

(ii) You can use $(x+y)^3 = x^3+3x^2y+3xy^2+y^3$, but if you set $y = 2\sqrt{5}$ you must remember that $y^2 = (2\sqrt{5})^2$ etc — those brackets are very important!

$$\begin{aligned}(3+2\sqrt{5})^3 &= 3^3+3\times 3^2\times(2\sqrt{5})+3\times 3\times(2\sqrt{5})^2+(2\sqrt{5})^3\\ &= 27+54\sqrt{5}+180+40\sqrt{5}\\ &= 207+94\sqrt{5}\end{aligned}$$

(iii) I quite like to lay out this sort of thing as a table, I tend to find that I make fewer mistakes this way.

	1	$-\sqrt{2}$	$\sqrt{6}$
1	1	$-\sqrt{2}$	$\sqrt{6}$
$-\sqrt{2}$	$-\sqrt{2}$	2	$-\sqrt{12}$
$\sqrt{6}$	$\sqrt{6}$	$-\sqrt{12}$	6

Using $\sqrt{12} = 2\sqrt{3}$ gives a final answer of $9-4\sqrt{3}+2\sqrt{6}-2\sqrt{2}$.

(iv) "Hidden" quadratics crop up quite a lot. Just make sure that your answers make sense in the original equation. For example, if you let $y = x^2$ and you get (say) $y = -2$ as one possibility, you need to discard this solution if you want real values of x. Note that $\sqrt{a+b} \neq \sqrt{a}+\sqrt{b}$ [1].

(a)

$$(1+\sqrt{2})^2 = 1+2\sqrt{2}+2 = 3+2\sqrt{2}$$

[1] Your teachers have probably been going on about how $(a+b)^2 \neq a^2+b^2$ (unless a and/or b is zero), and I am sure that you rarely make this mistake. For some reason students are far more likely to make the analogous mistake of writing $\sqrt{a+b} = \sqrt{a}+\sqrt{b}$. Don't.

(b)

$$\begin{aligned} x^2 + \frac{4}{x^2} &= 12 \\ x^4 - 12x^2 + 4 &= 0 \\ x^2 &= \frac{12 \pm \sqrt{144-16}}{2} \\ x^2 &= 6 \pm \sqrt{36-4} \\ x^2 &= 6 \pm 4\sqrt{2} \end{aligned}$$

From part **(a)** we know that $(1+\sqrt{2})^2 = 3 + 2\sqrt{2}$, so $\pm\sqrt{3+2\sqrt{2}} = \pm(1+\sqrt{2})$. Therefore we have:

$$\begin{aligned} \pm\sqrt{6+4\sqrt{2}} &= \pm\sqrt{2(3+2\sqrt{2})} \\ &= \pm\sqrt{2}(1+\sqrt{2}) \\ &= \pm(2+\sqrt{2}) \end{aligned}$$

Similarly we have $\pm\sqrt{6-4\sqrt{2}} = \pm(2-\sqrt{2})$. Hence the solutions are given by $x = \pm(2 \pm \sqrt{2})$ — which represents 4 different solutions.

Preparation

2 **(i)**

$$\begin{aligned}\frac{2}{x+3} + \frac{1}{x+1} &= 1\\ 2(x+1) + (x+3) &= (x+1)(x+3)\\ 3x+5 &= x^2+4x+3\\ x^2+x-2 &= 0\\ (x+2)(x-1) &= 0\end{aligned}$$

Therefore we have $x = 1, -2$. It is a good idea to substitute your answers back into the original equation to check that they work.

(ii) The word "value(s)" occurs in the question: this is just a way of not giving anything away about the number of solutions that are expected (unlike saying "value" or "values"). Setting the discriminant ("$b^2 - 4ac$") equal to 0 gives $b^2 - 4 \times 4 \times 9 = 0$ and so $b = \pm 12$.

(iii) There are various ways to solve a quadratic inequality. I quite like sketching a graph, but you can also identify the "critical values" (where the expression is equal to zero) and then test values in each separate region.

Be *really* careful about using a standard expression such as $b^2 - 4ac$ when these letters already occur in the question. You can write it in inverted commas to remind yourself what you are doing: "Using '$b^2 - 4ac$' gives ...". But it is probably better to write something like $B^2 - 4AC$ (provided these letters do not also occur in the question!) and then you can write $B = 5c$ etc without confusing yourself.

When you have an inequality, you must be very careful not to divide (or multiply) by something which may, or may not, be negative. For example, if you have $x^2 < 4x$ you should **not** divide by x to get $x < 4$ as this inequality is not equivalent to the previous one (if $x = -10$, the second equality holds but the first one does not). It is far better to rearrange to get $x^2 - 4x < 0$ and then sketch (or factorise the right hand side of) $y = x^2 - 4x$.

We have:

$$\begin{aligned}(5c)^2 - 4 \times 3 \times c &\leqslant 0\\ 25c^2 - 12c &\leqslant 0\\ c(25c-12) &\leqslant 0\end{aligned}$$

Therefore $0 < c < \frac{12}{25}$.

The STEP question (2005 STEP I Q3)

3 Comments

Do remember that whenever you are asked to prove a given result then you must do so clearly with a full explanation.

(i) The easiest approach is probably to solve the equation to get $x = \sqrt{ab}$ or $x = -\sqrt{ab}$, but you could also consider the discriminant.

You must clearly explain **why** there are two solutions if a and b are either both positive or both negative. It might seem obvious to you, but you need to **write it down!**

(ii) For the first part, rearrange to get a quadratic equation and then set the discriminant equal to 0.

You were then asked to show something of the form: the condition $c^2 = A$ is equivalent to the condition $c^2 = B$. All you have to do is show, one way or another, that $A = B$.

You can go forwards or backwards. You can start with $c^2 = B$ and then work backwards to get $c^2 = A$ (for added Brownie points put a $\iff$ between each bit of working to show that the argument can be followed both ways — provided you are sure that it really is $\iff$). Perfectionists can do the working from $c^2 = B$ in rough and then reverse the argument starting from $c^2 = A$ on neat paper. Please do not start with $A = B$ (because that is what you are trying to show); though you could start with $A - B = \cdots$, hoping to show that $A - B = 0$.

For the last part, you do need to show that $c \neq 0$; i.e. you need to show that c^2 is **strictly** greater than zero (not just $c^2 \geqslant 0$). This is easiest if you use the first expression you have for c^2 and remember the information that you were given at the beginning of the question.

A possible solution outline

(i) Let $\frac{x}{x-a} + \frac{x}{x-b} = 1$. Then

$$\begin{aligned} x(x-b) + x(x-a) &= (x-a)(x-b) \\ x^2 - bx + x^2 - ax &= x^2 - ax - bx + ab \\ x^2 &= ab \\ x &= \pm\sqrt{ab} \end{aligned}$$

For the square root to give real solutions, ab must be positive so, a and b need to be either both positive or both negative.

(ii)

$$\begin{aligned}\frac{x}{x-a}+\frac{x}{x-b}&=1+c\\ \Leftrightarrow x^2-bx+x^2-ax&=x^2-ax-bx+ab+cx^2-acx-bcx+abc\\ \Leftrightarrow (c-1)x^2-c(a+b)x+ab(c+1)&=0\end{aligned}$$

This quadratic in x has a repeated root when the discriminant is 0, that is, when $c^2(a+b)^2-4(c-1)ab(c+1)=0$.

$$\begin{aligned}c^2(a+b)^2&=4ab(c-1)(c+1)\\ c^2(a+b)^2&=4abc^2-4ab\\ c^2\left[(a+b)^2-4ab\right]&=-4ab\\ c^2\left[a^2+b^2+2ab-4ab\right]&=-4ab\\ c^2(a-b)^2&=-4ab\end{aligned}$$

Therefore

$$\begin{aligned}c^2&=\frac{-4ab}{(a-b)^2}\\ &=\frac{(a-b)^2-(a+b)^2}{(a-b)^2}\\ &=1-\left(\frac{a+b}{a-b}\right)^2\end{aligned}$$

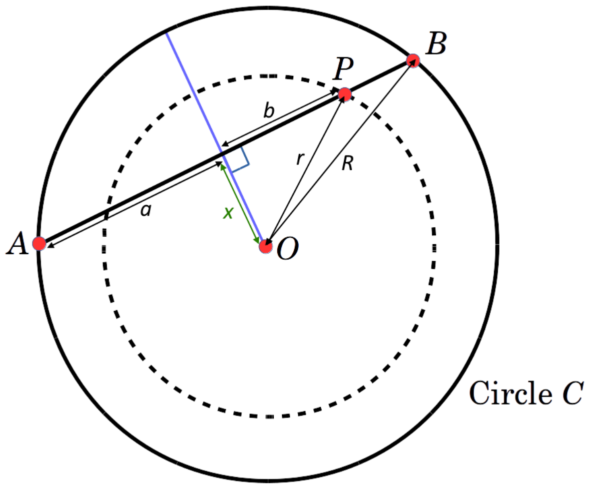
as required.

$c^2\geq 0$ since c is a real number. As a and b are real, $\left(\frac{a+b}{a-b}\right)^2\geq 0$, so $c^2\leq 1$.

If $c^2=0$, $-4ab=0$ so either $a=0$ or $b=0$, but it is given that a and b are non-zero, so $c^2>0$.

Warm down

4 The first thing to do is draw your own diagram (big!) and put the lengths you have been given (R, a, b and r) on your diagram.



Let x be the length of the line from O to the midpoint of AB.

By Pythagoras, $x^2 + b^2 = r^2$, and $x^2 + a^2 = R^2$.

The area between the circles is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2)$.

Substituting, we get $\pi((x^2 + a^2) - (x^2 + b^2)) = \pi(a^2 - b^2)$ as required.