STEP Support Programme

Hints and Partial Solutions for Assignment 20

Warm-up

1 You need to be quite careful when writing out this sort of argument. Note that you cannot find the limit of the numerator and denominator separately.

(i) You know:

$$\sin(x+h) = \sin x \cos h + \sin h \cos x$$

and, since h is small we can use the small angle approximations for $\cos h$ and $\sin h$[1]:

$$\sin(x+h) \approx \sin x \times \left(1 - \tfrac{1}{2}h^2\right) + h\cos x\,.$$

We can now say that, in the limit as $h \to 0$:

$$\begin{aligned}\lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} &= \lim_{h\to 0} \frac{\cancel{\sin x} - \frac{h^2}{2}\sin x + h\cos x - \cancel{\sin x}}{h} \\ &= \lim_{h\to 0}\left(\cos x - \frac{h}{2}\sin x\right) = \cos x\,.\end{aligned}$$

You then need to do something very similar when differentiating $\cos x$:

$$\begin{aligned}\lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} &= \lim_{h\to 0} \frac{\cos x\cos h - \sin x \sin h - \cos x}{h} \\ &= \lim_{h\to 0} \frac{\cancel{\cos x} - \frac{h^2}{2}\cos x - h\sin x - \cancel{\cos x}}{h} \\ &= \lim_{h\to 0}\left(-\frac{h}{2}\cos x - \sin x\right) = -\sin x\,.\end{aligned}$$

(ii) Here we can write:

$$\begin{aligned}\ln(x+h) - \ln x &= \ln\left(\frac{x+h}{x}\right) \\ &= \ln\left(1 + \frac{h}{x}\right) \\ &= \frac{h}{x} - \frac{1}{2}\left(\frac{h}{x}\right)^2 + \frac{1}{3}\left(\frac{h}{x}\right)^3 + \cdots \qquad \text{as long as } \left|\frac{h}{x}\right| < 1\,.\end{aligned}$$

[1] Which are $\sin h \approx h$ and $\cos h \approx 1 - \frac{1}{2}h^2$ where h is in **radians**.

Since $x \neq 0$ (necessary for $\ln x$ to be defined) and h is small we can pick h so that $|\frac{h}{x}| < 1$, but it is still necessary to **state** that the expansion is only true when $|\frac{h}{x}| < 1$.

Then we have:

$$\lim_{h\to 0} \frac{\frac{h}{x} - \frac{1}{2}\left(\frac{h}{x}\right)^2 + \frac{1}{3}\left(\frac{h}{x}\right)^3 + \cdots}{h} = \frac{1}{x}\,.$$

(iii) If we square the given approximation we get:

$$1 + t \approx 1 + 2kt + k^2t^2$$

and since t^2 can be ignored we get $k = \frac{1}{2}$ and so $\sqrt{1+t} \approx 1 + \frac{1}{2}t$.

Writing $\sqrt{x+h}$ as $\sqrt{x} \times \sqrt{1 + \frac{h}{x}}$ and using the approximation derived above gives the derivative as:

$$\begin{aligned}
\lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} &= \lim_{h\to 0} \frac{\sqrt{x} \times \sqrt{1+\frac{h}{x}} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x}\left(\cancel{1} + \frac{1}{2} \times \frac{h}{x} - \cancel{1}\right)}{h} \qquad \text{where } \frac{h}{x} \ll 1 \\
&= \lim_{h\to 0} \left(\sqrt{x} \times \frac{1}{2} \times \frac{1}{x}\right) = \frac{1}{2}x^{-\frac{1}{2}}\,.
\end{aligned}$$

The symbol $\ll$ means "much less than".

Using the binomial expansion $\left(1 + \frac{h}{x}\right)^{\frac{1}{2}} = 1 + \left(\frac{1}{2}\right) \times \left(\frac{h}{x}\right) + \left(\frac{1}{2!}\right) \times \left(\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) \times \left(\frac{h}{x}\right)^2 + \cdots$ is perhaps a little more satisfying as you then have a h term which can tend to zero, but we tried to write this question so that people who have not met the binomial expansion can still do it!

For $x^{-\frac{1}{2}}$ you can use $(1+t)^{-\frac{1}{2}} \approx 1 + kt$ to give $1 \approx (1+kt)^2(1+t)$. This gives $1 \approx 1 + t + 2kt + at^2 + bt^3$ (where we are going to ignore the t^2 and t^3 terms) and so $k = -\frac{1}{2}$.

Alternatively, using the binomial expansion gives $(1+t)^{-\frac{1}{2}} = 1 + \left(-\frac{1}{2}\right)t + \left(\frac{1}{2!}\right) \times \left(\frac{1}{2}\right) \times \left(-\frac{1}{2}\right) \times t^2 + \cdots$ and hence $(1+t)^{-\frac{1}{2}} \approx 1 - \frac{1}{2}t$ for small t. The argument is then:

$$\begin{aligned}
\lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} &= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x}} \times \left(\frac{1}{\sqrt{1+\frac{h}{x}}} - 1\right)}{h} \\
&= \lim_{h\to 0} \frac{\frac{1}{\sqrt{x}}\left(\cancel{1} - \frac{1}{2} \times \frac{h}{x} - \cancel{1}\right)}{h} \qquad \text{where } \frac{h}{x} \ll 1 \\
&= \lim_{h\to 0} \left(-\frac{1}{\sqrt{x}} \times \frac{1}{2} \times \frac{1}{x}\right) = -\frac{1}{2}x^{-\frac{3}{2}}\,.
\end{aligned}$$

Preparation

2 **(i)** Let $n = 1$ then $4^n + 6n - 1 = 4 + 6 - 1 = 9$ which is divisible by 9 and hence the statement is true for $n = 1$.

Assume the statement is true for $n = k$, so that $4^k + 6k - 1 = 9M$ for some integer M.

Now consider the case $n = k + 1$. We have:

$$\begin{aligned} 4^{k+1} + 6(k+1) - 1 &= 4 \times 4^k + 6k + 6 - 1 \\ &= 4 \times (9M - 6k + 1) + 6k + 6 - 1 \\ &= 36M - 18k + 9 \\ &= 9(4M - 2k + 1) \end{aligned}$$

and since M and k are both integers, $4M - 3k + 1$ is an integer and $4^{k+1} + 6(k+1) - 1$ is divisible by 9. Hence if it is true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$ it is therefore true for all integers $n \geqslant 1$.

(ii) Let $n = 1$. We have $1^3 = 1$ and $\frac{1}{4} \times 1^2 \times (1+1)^2 = \frac{1}{4} \times 1 \times 4 = 1$ and so the statement is true when $n = 1$.

Assume the statement to be true when $n = k$ so we have $\displaystyle\sum_{i=1}^{k} i^3 = \frac{1}{4}k^2(k+1)^2$.

Now consider the case $n = k+1$. We want to show that $\displaystyle\sum_{i=1}^{k+1} i^3 = \frac{1}{4}(k+1)^2([k+1]+1)^2$.

We have:

$$\begin{aligned} \sum_{i=1}^{k+1} i^3 &= \sum_{i=1}^{k} i^3 + (k+1)^3 \\ &= \tfrac{1}{4}k^2(k+1)^2 + (k+1)^3 \\ &= \tfrac{1}{4}(k+1)^2 \left[k^2 + 4(k+1)\right] \\ &= \tfrac{1}{4}(k+1)^2 \left[k^2 + 4k + 4\right] \\ &= \tfrac{1}{4}(k+1)^2(k+2)^2 \quad \text{as required.} \end{aligned}$$

Hence if it is true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$ it is therefore true for all integers $n \geqslant 1$.

The STEP question (1996 STEP II Q3)

3 To **show that** $F_2 = 1$ etc. you should write something like:

$$F_2 = F_1 + F_0 = 1 + 0 = 1\,.$$

You should also obtain $F_5 = 5$, $F_6 = 8$ and $F_7 = 13$.

For $F_{n+1}F_{n-1} - F_n^2$ it helps to be systematic

n	$F_{n+1}F_{n-1} - F_n^2$		
1	$F_2F_0 - F_1^2$	$1 \times 0 - 1^2$	$= -1$
2	$F_3F_1 - F_2^2$	$2 \times 1 - 1^2$	$= 1$
3	$F_4F_2 - F_3^2$	$3 \times 1 - 2^2$	$= -1$
4	$F_5F_3 - F_4^2$	$5 \times 2 - 3^2$	$= 1$

From this, we make the conjecture that $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. We know that this is true when $n = 1$ (from the first line in the table), so now assume that it is true for $n = k$ i.e. we have $F_{k+1}F_{k-1} - F_k^2 = (-1)^k$.

Consider the case $n = k + 1$. We want to show that $F_{k+2}F_k - F_{k+1}^2 = (-1)^{k+1}$. There are many ways you could approach it, one way is to start with the left hand side:

$$\begin{aligned}
F_{k+2}F_k - F_{k+1}^2 &= [F_{k+1} + F_k]\,F_k - F_{k+1}\,[F_k + F_{k-1}] \\
&= \cancel{F_kF_{k+1}} + F_k^2 - \cancel{F_{k+1}F_k} - F_{k+1}F_{k-1} \\
&= -1 \times \left[F_{k+1}F_{k-1} - F_k^2\right] \\
&= -1 \times (-1)^k = (-1)^{k+1} \qquad \text{as required.}
\end{aligned}$$

Hence if it is true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 1$ it is therefore true for all integers $n \geqslant 1$.

Several of these steps make use of the definition of the Fibonacci numbers, $F_{n+1} = F_n + F_{n-1}$.

For the last part the suggested method is to use induction on k. Throughout this part, think of n as being a fixed (but unknown) integer. Starting with the case $k = 1$ we have:

$$F_{n+1} = F_1F_{n+1} + F_0F_n = 1 \times F_{n+1} + 0 \times F_n$$

which is true. When $k = 2$[2] we have:

$$F_{n+2} = F_2F_{n+1} + F_1F_n = 1 \times F_{n+1} + 1 \times F_n$$

which is also true, from the definition of Fibonacci numbers.

[2]The reason for doing two base cases will be more obvious later. When doing this sort of question it will often only be when you are completing the induction step that you realise that two base cases are necessary.

Now assume that the statement is true for $k = m$, for some integer m, and also for $k = m-1$. We then have $F_{n+m} = F_m F_{n+1} + F_{m-1} F_n$ and also $F_{n+m-1} = F_{m-1} F_{n+1} + F_{m-2} F_n$ [3]. We want to show that the $k = (m+1)^{\text{th}}$ case is true, i.e. we are trying to prove that $F_{n+(m+1)} = F_{(m+1)} F_{n+1} + F_{(m+1)-1} F_n$.

$$\begin{aligned}
F_{n+m+1} &= F_{n+m} + F_{n+m-1} && \textit{(using the Fibonacci number definition)}\\
&= (F_m F_{n+1} + F_{m-1} F_n) + (F_{m-1} F_{n+1} + F_{m-2} F_n) && \textit{(using the inductive steps)}\\
&= F_{n+1}(F_m + F_{m-1}) + F_n(F_{m-1} + F_{m-2}) && \textit{(rearranging)}\\
&= F_{n+1} F_{m+1} + F_n F_m && \textit{(using the Fibonacci number definition)}\\
&= F_{(m+1)} F_{n+1} + F_{(m+1)-1} F_n && \textit{(as required.)}
\end{aligned}$$

Hence if it is true for $k = m$ and $k = m-1$ then it is true for $k = m+1$ and as it is true for $k = 1$ and $k = 2$ it is therefore true for all integers $k \geqslant 1$.

This answer was the result of quite a bit of messing around, trying things and them not working out etc. before a route through the induction was found. Whilst trying things for the second part, I found that I would like to use the "$k = m - 1$" case as well as the "$k = m$" case which meant that I then had to go back and show that both the $k = 1$ and $k = 2$ cases were true. Although the answer here looks like I realised that I need both of those cases before I started, this was not what happened.

[3] We needed two base cases as we want to assume the proposition is true for both $k = m$ and $k = m - 1$.

4 (2006 STEP III Question 8)

Do have a bit of a play around with this first. Try some different functions and see what you can discover.

Start by taking $\mathrm{f}(x) = 1$. We then can use **(iv)** with $\mathrm{f}(x) = \mathrm{g}(x) = 1$. This gives:

$$\begin{aligned} \triangle(1 \times 1) &= 1\triangle 1 + 1\triangle 1 \\ \triangle 1 &= 2\triangle 1 \\ 0 &= \triangle 1 \qquad \text{i.e.} \quad \triangle 1 = 0\,. \end{aligned}$$

Then we can use **(iii)** with $\mathrm{f}(x) = c$ to get:

$$\begin{aligned} \triangle c &= \triangle(c \times 1) \\ &= c \times \triangle 1 \\ &= c \times 0 = 0 \qquad \text{as required.} \end{aligned}$$

For $\triangle x^2$ use **(iv)** (with $\mathrm{f}(x) = \mathrm{g}(x) = x$) and **(i)** to get:

$$\begin{aligned} \triangle x^2 &= \triangle(x \times x) \\ &= x\triangle x + x\triangle x \\ &= x + x = 2x\,. \end{aligned}$$

For $\triangle x^3$ use **(iv)** (with $\mathrm{f}(x) = x$ and $\mathrm{g}(x) = x^2$) to get:

$$\begin{aligned} \triangle x^3 &= \triangle(x \times x^2) \\ &= x\triangle x^2 + x^2\triangle x \\ &= x \times 2x + x^2 \times 1 = 3x^2\,. \end{aligned}$$

At this point, it does appear that $\triangle$ differentiates the polynomial it is applied to. To prove that this is true for all polynomials in x you need to take a bit of care with the argument.

Proposition:

$$\triangle x^n = nx^{n-1}. \tag{*}$$

We know that this is true for $n = 0, 1, 2$ and 3 from the previous work. Assume it is true for $n = k$ (i.e. we assume $\triangle x^k = kx^{k-1}$) and then consider the case $n = k + 1$. We want to show that $\triangle x^{k+1} = (k+1)x^k$.

$$\begin{aligned}
\triangle x^{k+1} &= \triangle\big(x \times x^k\big) \\
&= x\triangle x^k + x^k\triangle x \\
&= x \times kx^{k-1} + x^k \times 1 \\
&= kx^k + x^k = (k+1)x^k \quad \text{as required.}
\end{aligned}$$

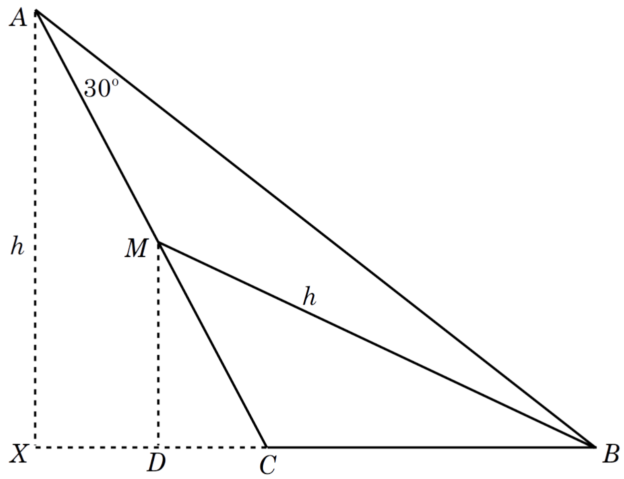
Hence if it is true for $n = k$ then it is true for $n = k + 1$ and as it is true for $n = 0$ it is therefore true for all integers $n \geqslant 0$.

Now consider a general polynomial, $\mathrm{h}(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots a_2x^2 + a_1x + a_0$.

$$\begin{aligned}
\triangle\big(\mathrm{h}(x)\big) &= \triangle\big(a_nx^n + a_{n-1}x^{n-1} + \cdots a_2x^2 + a_1x + a_0\big) \\
&= \triangle\big(a_nx^n\big) + \triangle\big(a_{n-1}x^{n-1}\big) + \cdots + \triangle\big(a_2x^2\big) + \triangle\big(a_1x\big) + \triangle\big(a_0\big) && \text{using } \textbf{(ii)} \\
&= a_n\triangle\big(x^n\big) + a_{n-1}\triangle\big(x^{n-1}\big) + \cdots + a_2\triangle\big(x^2\big) + a_1\triangle\big(x\big) + 0 && \text{using } \textbf{(iii)} \\
&= a_n \times nx^{n-1} + a_{n-1} \times (n-1)x^{n-2} + \cdots + a_2 \times 2x + a_1 && \text{using } (*) \\
&= na_nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + 2a_2x + a_1 = \frac{\mathrm{d}}{\mathrm{d}x}\mathrm{h}(x) && \text{as required.}
\end{aligned}$$

Warm down

5 The following diagram is the same as in the question, with one dotted line added and some extra points labelled.



(i) Using two similar triangles ($\triangle AXC$ and $\triangle MDC$ — similar as the three angles are the same) we can show that $MD = \frac{1}{2}h$ (since M is the midpoint of AC the lengths of $\triangle AXC$ are double those in $\triangle MDC$).

Using $\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}$ in $\triangle MDB$ gives $\angle MBD = \angle MBC = 30°$.

(ii) We have $\angle CAB = \angle MBC = 30°$ and $\angle ACB = \angle MCB$ as this is a shared angle. Hence the third angles in $\triangle ACB$ and $\triangle MCB$ are the same and we have $\angle ABC = \angle BMC$.

(iii) Let $\angle ABC = \theta$. We then have (using the similar triangles of **(ii)** above) $\angle BMC = \theta$, and $\angle AMB = 180° - \theta$. The sine rule in $\triangle AMB$ give us:

$$\frac{AB}{\sin(180° - \theta)} = \frac{h}{\sin 30°}$$

which simplifies to $AB = 2h\sin\theta$.

Using $\triangle AXB$ gives $AB\sin\theta = h$. Equating the two expressions for AB gives us:

$$2h\sin\theta = \frac{h}{\sin\theta}$$

and so $\sin^2\theta = \frac{1}{2}$ and so $\theta = 45°$ (as we must have $0 < \theta < 90°$).