STEP Support Programme

Hints and partial solutions for Assignment 21

Warm-up

1 **(i)** Careful algebraic manipulation should enable you to show that these results are true. Make sure you are secure with the index laws — in particular $a^{x+y} \neq a^x + a^y$, and $(a^x)^2 = a^x \times a^x = a^{2x} \neq a^{x^2}$. Also be careful to include the cross term when expanding $(a^x + a^{-x})^2$.

(a)

$$\begin{aligned}
(\mathrm{C}(x))^2 - (\mathrm{S}(x))^2 &= \tfrac{1}{4}\left(a^x + a^{-x}\right)^2 - \tfrac{1}{4}\left(a^x - a^{-x}\right)^2 \\
&= \tfrac{1}{4}\left(a^{2x} + 2 + a^{-2x}\right) - \tfrac{1}{4}\left(a^{2x} - 2 + a^{-2x}\right) \\
&= \tfrac{1}{4}\left(\cancel{a^{2x}} + 2 + \cancel{a^{-2x}} - \cancel{a^{2x}} + 2 - \cancel{a^{-2x}}\right) \\
&= 1
\end{aligned}$$

Notice the cancelling — when the answer is given it is important to show why bits "disappear".

(b)

$$\begin{aligned}
\mathrm{C}(x)\mathrm{C}(y) + \mathrm{S}(x)\mathrm{S}(y) &= \left(\tfrac{1}{2}\left(a^x + a^{-x}\right)\right)\left(\tfrac{1}{2}\left(a^y + a^{-y}\right)\right) + \left(\tfrac{1}{2}\left(a^x - a^{-x}\right)\right)\left(\tfrac{1}{2}\left(a^y - a^{-y}\right)\right) \\
&= \tfrac{1}{4}\left(a^{x+y} + \cancel{a^{x-y}} + \cancel{a^{-x+y}} + a^{-x-y}\right) + \tfrac{1}{4}\left(a^{x+y} - \cancel{a^{x-y}} - \cancel{a^{-x+y}} + a^{-x-y}\right) \\
&= \tfrac{1}{4}\left(2a^{(x+y)} + 2a^{-(x+y)}\right) \\
&= \tfrac{1}{2}\left(a^{x+y} + a^{-(x+y)}\right) \\
&= \mathrm{C}(x+y)
\end{aligned}$$

(c)

$$\begin{aligned}
\mathrm{C}(x)\mathrm{S}(y) + \mathrm{S}(x)\mathrm{C}(y) &= \left(\tfrac{1}{2}\left(a^x + a^{-x}\right)\right)\left(\tfrac{1}{2}\left(a^y - a^{-y}\right)\right) + \left(\tfrac{1}{2}\left(a^x - a^{-x}\right)\right)\left(\tfrac{1}{2}\left(a^y + a^{-y}\right)\right) \\
&= \tfrac{1}{4}\left(a^{x+y} - \cancel{a^{x-y}} + \cancel{a^{-x+y}} - a^{-x-y}\right) + \tfrac{1}{4}\left(a^{x+y} + \cancel{a^{x-y}} - \cancel{a^{-x+y}} - a^{-x-y}\right) \\
&= \tfrac{1}{4}\left(2a^{(x+y)} - 2a^{-(x+y)}\right) \\
&= \mathrm{S}(x+y)
\end{aligned}$$

For the last part:

$$\begin{aligned}
\mathrm{C}(2x) &= \mathrm{C}(x+x) \\
&= \mathrm{C}(x)\mathrm{C}(x) + \mathrm{S}(x)\mathrm{S}(x) \\
&= (\mathrm{C}(x))^2 + (\mathrm{S}(x))^2 \\
&= (\mathrm{C}(x))^2 + \left((\mathrm{C}(x))^2 - 1\right) \\
&= 2\,(\mathrm{C}(x))^2 - 1
\end{aligned}$$

(ii) You should have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}\,.$$

You can then factorise out a^x and take it outside the limit (as a^x does not depend on h) to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

and hence get the required result.

The second result is a bit trickier. You have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{a^{-(x+h)} - a^{-x}}{h}$$

which becomes $a^{-x} \lim\limits_{h\to 0} \frac{a^{-h}-1}{h}$. Unfortunately you cannot use the given limit on this, but using a substitution of $-h = \epsilon$ (and noting that as $h \to 0$, $\epsilon \to 0$) we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a^{-x} \lim_{\epsilon\to 0} \frac{a^\epsilon - 1}{-\epsilon}\,.$$

Take a negative sign outside the limit and then you can use the given result.

(iii) Using the definition in part **(i)** and the results from part **(ii)** we get:

$$\frac{\mathrm{dC}(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\left[\tfrac{1}{2}\left(a^x + a^{-x}\right)\right] = \tfrac{1}{2}\left(Ka^x - Ka^{-x}\right) = K\mathrm{S}(x)$$

Then a second differentiation will get the required result.

Preparation

2 **(i)** We have $|2x-3|=7$ i.e. either $2x-3=7$ or $2x-3=-7$. This gives $x=-2$ or $x=5$. It is always a good idea to substitute your answers into the *original* equation to make sure they work.

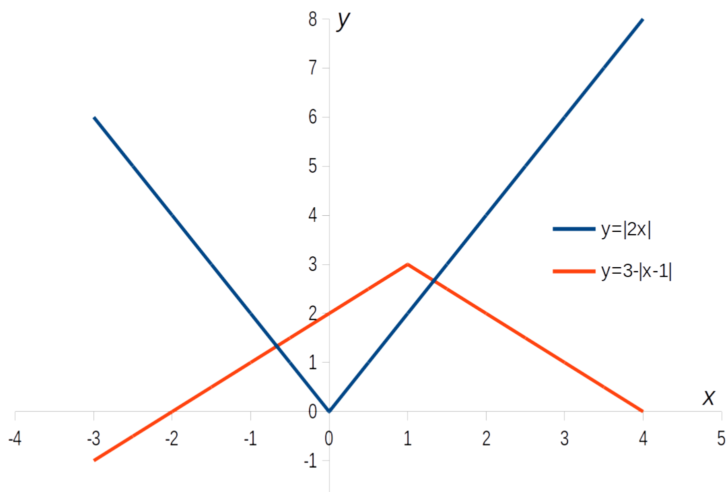
(ii) You should get a "V" shaped graph. The intercepts with the axes can be found in the usual way (i.e. substituting $x=0$ and $y=0$). The website Desmos can be used to check your graphs (amongst the many things it can plot, it can plot modulus graphs).

(iii) As the question says, there are three regions to consider. In the region $0<x<1$, the relevant equation is $2x-(x-1)=3$. But the solution to this is $x=2$ which does not lie in the region we are considering. So this is not a solution. Substituting $x=2$ into the original equation gives $|4|+|1|\neq 3$.

In the region $x<0$ the equation becomes $-2x-(x-1)=3$, which gives $x=-\frac{2}{3}$. This lies in the region we are currently considering, and substituting $x=-\frac{2}{3}$ into the original equation gives $\left|-\frac{4}{3}\right|+\left|-\frac{5}{3}\right|=3$.

The last region is $x>1$ when the equation becomes $2x+(x-1)=3$ and so $x=\frac{4}{3}$.

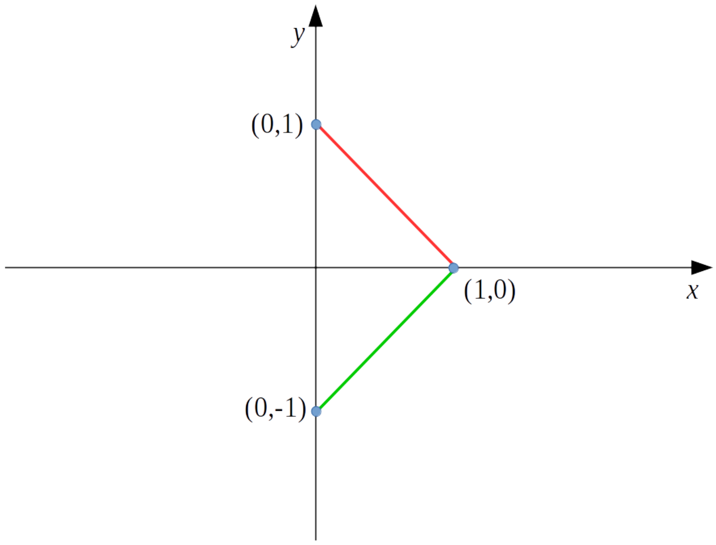
You can also sketch the graphs $y=|2x|$ and $y=3-|x-1|$ to make sure your solutions are sensible. From the graph you can see that one solution lies in the region $x<0$ and one lies in the region $x>1$.



(iv) In the region $x>0$ and $y>0$ you need to sketch $x+y=1$. Make sure that your line does not extend outside this region — it terminates at $(0,1)$ and $(1,0)$. When $x>0$ and $y<0$ the equation becomes $x-y=1$ so you need to sketch the straight line segment from $(1,0)$ to $(0,-1)$.

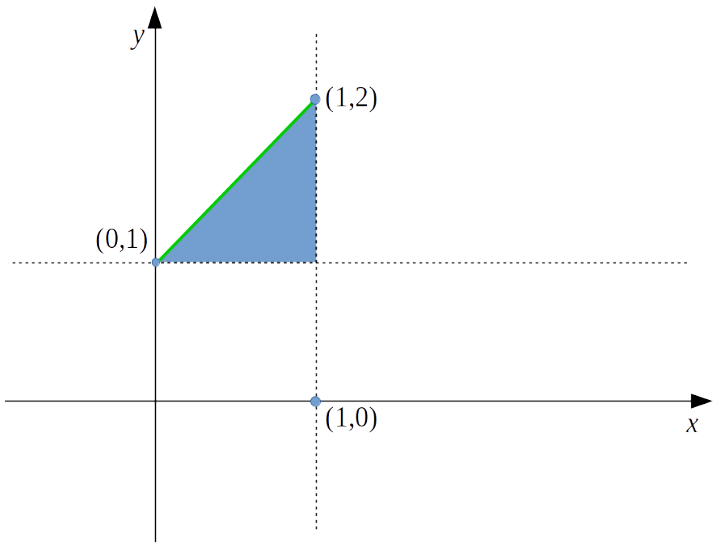
Note that $|x|+|y|=1$ is symmetrical in both axes, so you could instead use this to sketch the graph in the region $x>0$ and $y<0$.

In the sketch below, the red line is the graph of $|x| + |y| = 1$ for $x > 0, y > 0$ and the green line is the graph of $|x| + |y| = 1$ for $x > 0, y < 0$.



(v) It is best to start by drawing two dotted lines for $x = 1$ and $y = 1$, and remember that you are restricting yourself to $x < 1$ and $y > 1$. In the given region the equation becomes $(1 - x) + (y - 1) = 1$, i.e. $y = x + 1$.

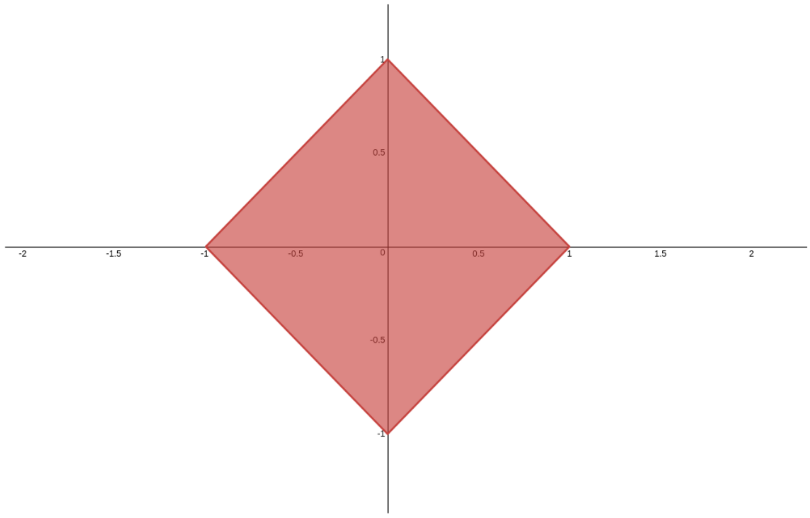
You end up with a straight line segment with end points $(0, 1)$ and $(1, 2)$, and then the shaded region is the triangle with vertices $(0, 1)$, $(1, 2)$ and $(1, 1)$. You could use the point $(0, 2)$ as a test point to see which side of the line $y = x + 1$ is required (or any other point in the region $x < 1, y > 1$ which does not lie on the line $y = x + 1$).



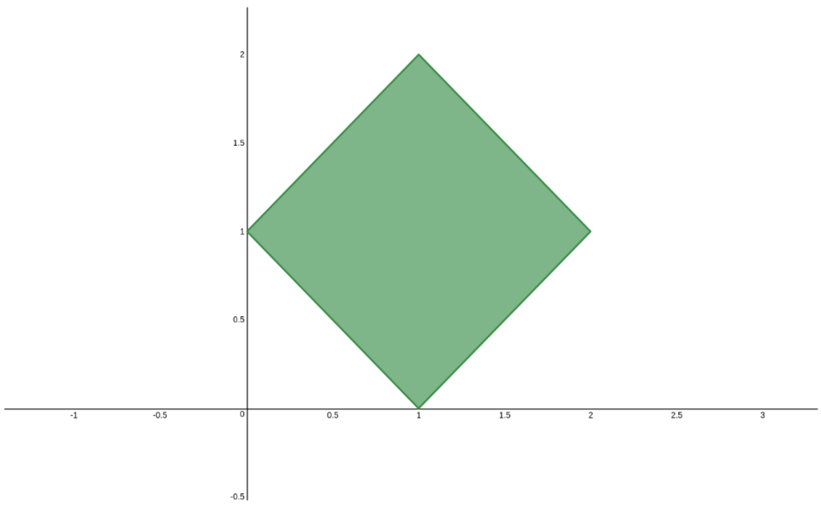
The STEP question (1999 STEP I Q4)

3 **(i)** Start by drawing the boundary, $|x| + |y| = 1$. It might be helpful to note that there is symmetry in the x-axis and the y-axis (since $|-x| = |x|$ etc.).

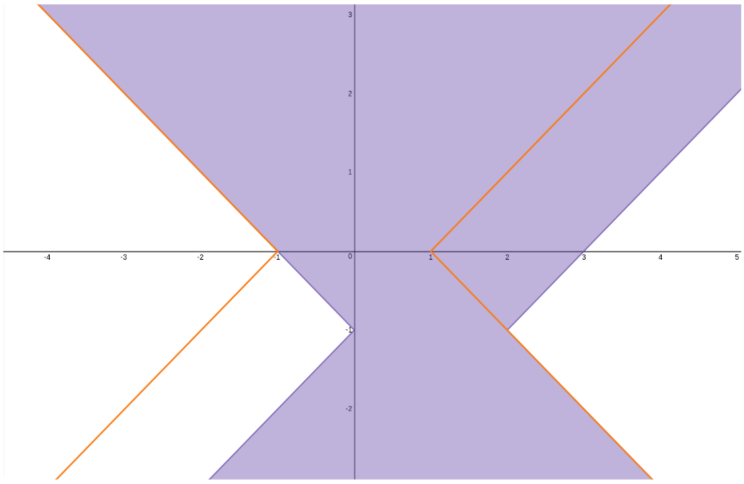
Note that to be really sure that you have the correct region, you should test a point in *each quadrant*.



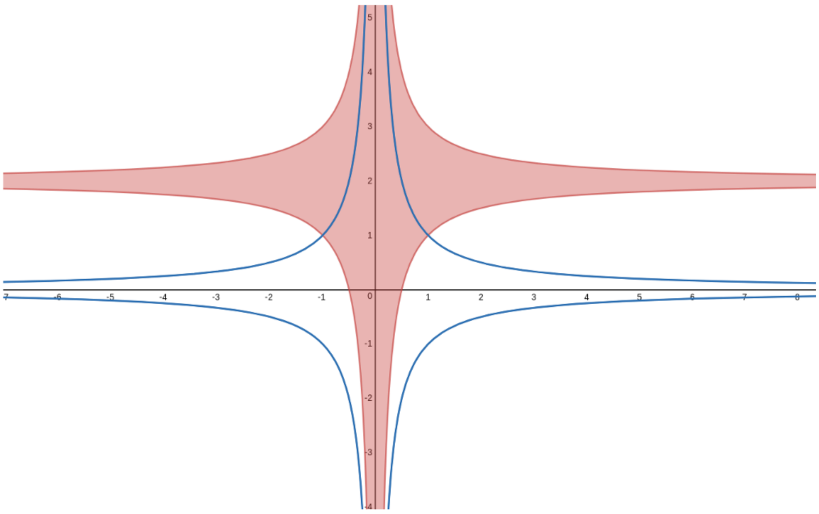
(ii) The easiest way to get this graph is to note that it is a translation of the previous graph with vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.



(iii) Here you can sketch $|x| - |y| = 1$, by sketching $x - y = 1$ in the first quadrant (where x and y are both positive) and then making use of the fact that this graph will be symmetrical on both axes. Then you can apply a translation of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and test points to find out which region of the plane is described by the inequality. In the sketch below the orange line is the graph of $|x| - |y| = 1$.



(iv) Here, start by sketching $xy = 1$ in the first quadrant. You can then use symmetry in the axes and a translation of $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ to get the required graph.

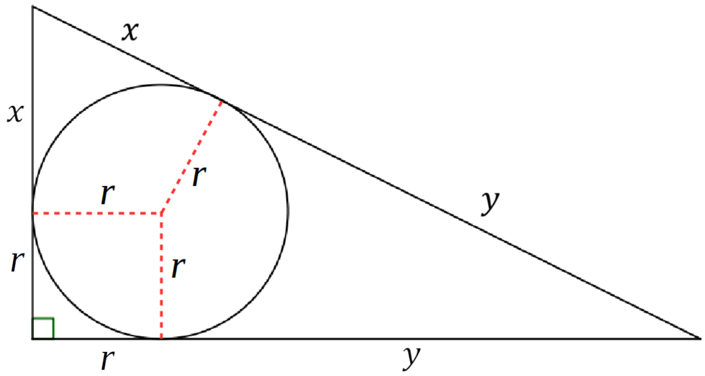


Note that the first two parts are clearly related. The first two parts are trying to suggest a way that you can get the second graph from the first graph. The hope is that you will then go on to use that idea for parts **(iii)** and **(iv)**.

All of your graphs can be checked by using Desmos.

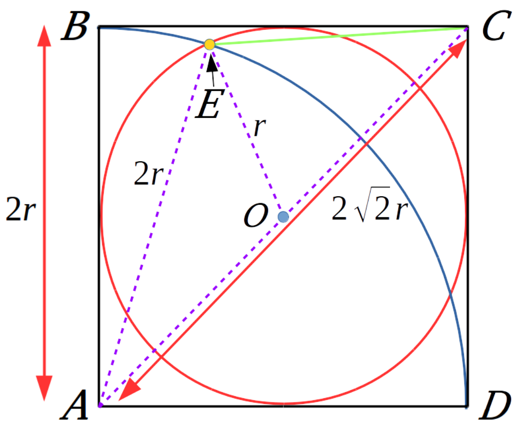
Warm down

4 **(i)** Let the radius of the circle be r. Using the facts that tangents to a circle are equal and that the radii are perpendicular to the tangents we have the height of the triangle as $r + x$ and the base length is $r + y$.



The area of the triangle is then $\frac{1}{2}(r+x)(r+y) = \frac{1}{2}(r^2 + rx + ry + xy)$. You also have $(r+x)^2 + (r+y)^2 = (x+y)^2$ which gives $2r^2 + 2rx + 2ry = 2xy$. Combine the two and you can express the area of the triangle in terms of x and y only giving the area of the triangle to be xy.

(ii) Let $AB = 2r$ (you can actually let AB be anything you like, including 2). The small circle now has radius r, $AC = 2\sqrt{2}\,r$ and $OA = \sqrt{2}\,r$ (where O is the centre of the small circle). Since E lies at the intersection of the circle and the arc we have $OE = r$ and $AE = 2r$.



You can then show that $\triangle AOE$ and $\triangle AEC$ are similar as they share an angle at A and $AE : AC = AO : AE = 1 : \sqrt{2}$. Hence $\triangle AEC$ is an enlargement of $\triangle AOE$ by scale factor $\sqrt{2}$.

We then have $EC = \sqrt{2}\,OE$ and since $OE = r$ we have $EC = \sqrt{2}\,r = \frac{1}{2}AC$.