STEP Support Programme

Hints and partial solutions for Assignment 23

Warm-up

1 **(i)** Note that $f'(x) = 2x$ and $g(x) = 2x^3+1$, so $f'(g(x)) = 2\left(2x^3+1\right)$. $F(x) = 4x^6+4x^3+1$ go $F(x) = 24x^5 + 12x^2 \neq f'(g(x))$.

(ii) $\sin 2\alpha = 2\sin\alpha\cos\alpha$ and $\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$. Differentiating $\sin 2x = 2\sin x\cos x$ with the product rule gives:

$$\begin{aligned} F'(x) &= 2\cos^2 x - 2\sin^2 x \\ &= 2\cos 2x \end{aligned}$$

(iii) For all these parts, you must make sure that you use a $\approx$ sign (rather than $=$) when you are using an approximation.

(a) $g(x+h) \approx g(x) + hg'(x)$

(b) $f\big(g(x)+t\big) \approx f\big(g(x)\big) + tf'\big(g(x)\big)$

(c) $F(x+h) = f\big(g(x+h)\big) \approx f\big(g(x)+hg'(x)\big) \approx f\big(g(x)\big) + hg'(x)f'\big(g(x)\big)$ The last step here comes from letting $t = hg'(x)$ in part **(b)**.

(d) You can either compare the result for part **(c)** to $F(x+h) \approx F(x) + hF'(x)$ or use the definition of differentiation as a limit.

(iv) The derivative of $\left(2x^3+1\right)^2$ is $12x^2\left(2x^3+1\right)$. The derivative of $\sin 2x$ is $2\cos 2x$. These should be same as your answers to part **(i)** and **(ii)**.

(v) When doing this without the chain rule, you will find the following results useful:

$$\ln(2x) = \ln 2 + \ln x \text{ and } \ln(x^2) = 2\ln(x)\,.$$

Don't forget that $\ln 2$ is a constant and so will disappear upon differentiation.

The derivative of $\ln(2x)$ is $\dfrac{1}{x}$ and the derivative of $\ln\left(x^2\right)$ is $\dfrac{2}{x}$.

Preparation

2 **(i)** Using a table to multiply out the two brackets will help ensure you consider all the terms. Answer: $4t^2 + \frac{16}{3}ty - 20t + \frac{16}{9}y^2 - \frac{40}{3}y + 25$. This might be useful later!

(ii) **(a)** If you eliminate y you will get $(x-4)^2 + (3x-2)^2 = 9$ which simplifies to the quadratic equation $10x^2 - 20x + 11 = 0$. The discriminant here is negative, so there are no points of intersections.

(b) Here we have $(x-5)^2 + (2x-5)^2 = 5 \implies 5x^2 - 30x + 45 = 0$ and so there is one point of intersection, $(3,3)$. Here the line is a tangent to the circle.

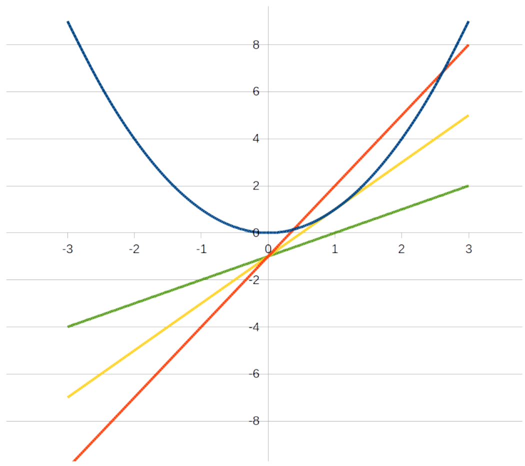
(c) To sketch the circle it helps to rewrite the equation as $(x-3)^3 + (y-1)^2 = 25$ (complete the square on both the x and the y terms separately). Eliminating x gives the equation $y^2 - 5y + 4 = 0$ and the points of intersection are $(-2,1)$ and $(7,4)$.

To check your sketches you could use Desmos.

(iii) Where the line meets the curve x will satisfy $x^2 = ax - 1$ i.e. $x^2 - ax + 1 = 0$. There will be two distinct points of intersection iff[1] $(-a)^2 - 4 \times 1 \times 1 > 0$. Hence we have $a < -2$ or $a > 2$.

Be careful when using $b^2 - 4ac > 0$; here a already appears in the equation (but not as the coefficient of x^2). You could put the formula in quotation marks ("$b^2 - 4ac$") or — perhaps a safer option — you could write $B^2 - 4AC$ or similar.

It is often helpful to draw a graph to get an idea of what is happening. The graph below shows $y = x^2$ and the three straight lines corresponding to $a = 1, 2, 3$. You can see that, as (positive) a increases, the line goes from not intersecting, to touching, to intersecting twice. The picture for negative a is similar, but the lines have negative gradient.



Setting $a^2 - 4 = 0$ will find the two values of a for which the straight line touches the curve. The two tangents are $y = 2x - 1$ and $y = -2x - 1$.

[1] iff means "if and only if", i.e. the implication works both ways. See Assignment 10 for more on this.

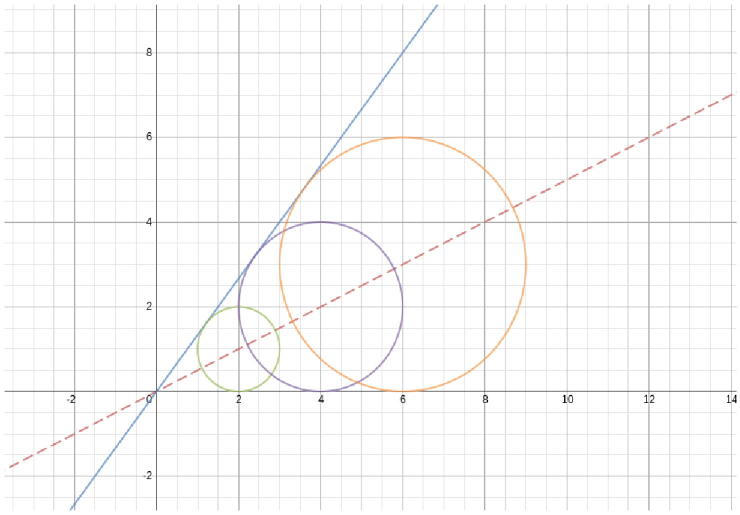
(iv)

$$\begin{aligned}
\tan 2\alpha &= \frac{\sin 2\alpha}{\cos 2\alpha} \\
&= \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha} \\
&= \frac{2\frac{\sin\alpha}{\cos\alpha}}{1 - \left(\frac{\sin\alpha}{\cos\alpha}\right)^2} \\
&= \frac{2\tan\alpha}{1 - \tan^2\alpha}
\end{aligned}$$

The STEP question (2009 STEP I Q8)

3 **(i)** You know that this equation is a circle, radius t and centre $(2t, t)$. A nice diagram is then enough to show that the circle touches the x axis (i.e. the line $y = 0$). You could, if you prefer, substitute $y = 0$ into the first equation and show that there is only one solution, so only one point of intersection and therefore the circle touches the line.

The line joining the origin to the centre has gradient $\frac{t}{2t} = \frac{1}{2}$, so $\tan\alpha = \frac{1}{2}$. By drawing a sketch, you can see that the line that makes angle 2α with the x axis will touch the circle. The circles on the sketch below are the ones obtained by setting $t = 1, 2$ and 3, but you can sketch a general case.



If $\tan\alpha = \frac{1}{2}$ you can use the result from question **2 (iv)** to show that $\tan 2\alpha = \frac{4}{3}$ and hence the equation of this line is $y = \frac{4}{3}x \implies 3y = 4x$.

(ii) There are many ways of doing this! The following suggestion is the low-tech approach.

Since the circle required touches both the lines $y = 0$ and $3y = 4x$ then it must have the same form as in part **(i)**. The question now is: what is t?

If it *touches* $4y+3x = 15$, then there is only one solution of the simultaneous equations:

$$(x-2t)^2 + (y-t)^2 = t^2 \quad \text{and} \quad 4y + 3x = 15\,.$$

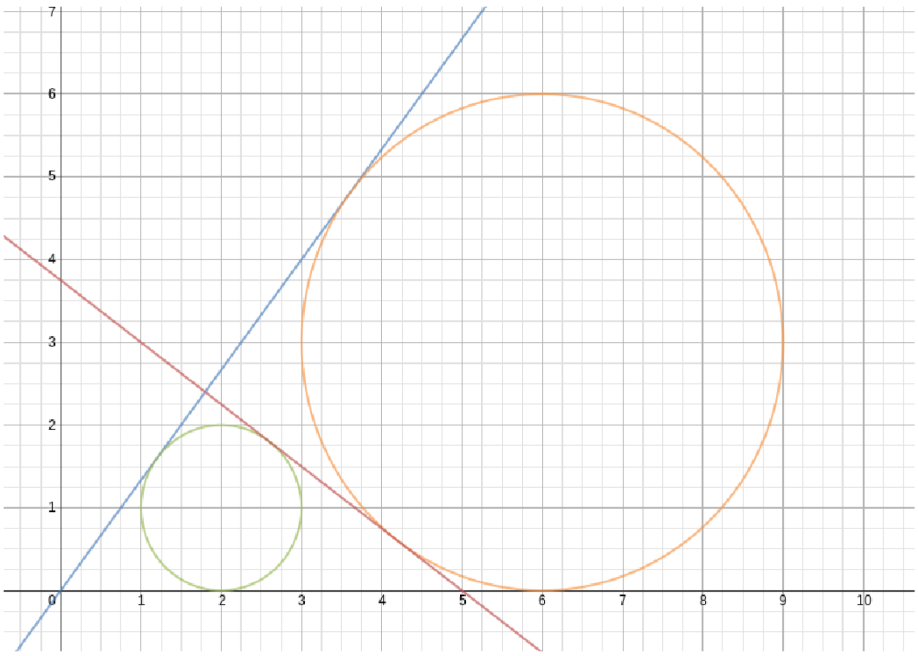
You can then substitute for x or y. The algebra gets a bit complicated, but hang on in there.

$$\begin{aligned}\left(5 - \tfrac{4}{3}y - 2t\right)^2 + (y-t)^2 &= t^2\\ 4t^2 + \tfrac{16}{3}ty - 20t + \tfrac{16}{9}y^2 - \tfrac{40}{3}y + 25 + y^2 - 2ty + \cancel{t^2} &= \cancel{t^2}\\ 36t^2 + 48ty - 180t + 16y^2 - 120y + 225 + 9y^2 - 18ty &= 0\\ 25y^2 + (30t - 120)y + 225 - 180t + 36t^2 &= 0\end{aligned}$$

Using "$b^2 - 4ac = 0$" gives:

$$\begin{aligned}
(30t - 120)^2 - 4 \times 25 \times (225 - 180t + 36t^2) &= 0 \\
30^2(t-4)^2 - 4 \times 25 \times 9(25 - 20t + 4t^2) &= 0 \\
(t-4)^2 - (25 - 20t + 4t^2) &= 0 \\
t^2 - 8t + 16 - 25 + 20t - 4t^2 &= 0 \\
3t^2 - 12t + 9 &= 0 \\
t^2 - 4t + 3 &= 0
\end{aligned}$$

This will gives two values of t, $t = 1$ or $t = 3$, and both of these circles **touch** the three lines, but (as the picture below shows) only one of the circles lies *inside* the triangle.



Hence the equation of the incircle is $(x-2)^2 + (y-1)^2 = 1$.

Warm down

4 This question requires you to think logically and to process a lot of information.

It might help to start by thinking what would happen if they all had brown eyes — when they looked into each others eyes they would only see brown-eyed people.

(i) If one islander leaves on the boat on the first day then he or she must have worked out that she has blue eyes. She/he knows that "*at least one islander has blue eyes*" so if he/she sees that none of the other islanders has blue eyes, then she/he concludes that he/she herself must have blue eyes and must leave. If she/he saw one or more islanders with blue eyes, she could not deduce anything about her own eye-colour.

(ii) There are at least two islanders with blue eyes, as if there is only one islander with blue eyes he or she would see that everyone else has brown eyes, conclude that they have blue eyes and leave on day one.

You can write this out as a proof by contradiction if you like!

(iii) If no-one leaves on the first day then we must have two or more with blue eyes. If two islanders leave on the second day they must have deduced that they have blue eyes. If they noticed that one person (other than themselves) had blue eyes on the first day, but that that person did not leave on the first day then there must be another person with blue eyes, namely themselves.

(iv) There are exactly two islanders with blue eyes (and that no-one did leave on the first day as that would mean we only had one islander with blue eyes who would have left on day one).

(v) There are n islanders with blue eyes.

This can be shown by using proof by induction, and the argument I will use involves *Strong induction*. The argument might go something like the following.

Proposition: If there are n blue-eyed islanders then they will all leave on the nth day.

Base case: We know that this is true for $n = 1$ (and $n = 2$) from the above work.

Inductive step: Assume that the proposition is true for $n = 1, 2, 3, 4, \ldots, k$, so if there are k islanders with blue eyes then they will all leave on the kth day (and if there are less than k they will have left before the kth day). Now consider the $n = k + 1$ case. No-one will have left on days $1, 2, 3, \ldots, k$. On the $k + 1$th day every person with blue eyes will see exactly k others with blue eyes. Since no-one left on the kth day they can conclude that there must be more than k islanders with blue eyes hence they must also have blue eyes and so they need to leave. The same is true for every islander with blue eyes so all $k + 1$ leave on the $k + 1$th day.

Conclusion Therefore if there are n islanders with blue eyes they will all leave on the nth day.