STEP Support Programme

Hints and partial solutions for Assignment 24

Warm-up

1 **(i)** You can use the website Wolfram Alpha to check your answers (only as a check though!). It can work out definite and indefinite integrals.

(a) $\left[\frac{1}{4}e^{4t}\right]_0^{\ln 2} = \frac{1}{4}e^{\ln 16} - \frac{1}{4} = \frac{15}{4}$. Note that $4\ln 2 = \ln 2^4$.

(b) $\left[-\frac{1}{2}\cos(2t)\right]_0^{0.5\pi} = -\frac{1}{2} \times -1 + \frac{1}{2} \times 1 = 1$.

(c) The derivative of $x\sin x$ is $x\cos x + \sin x$ and so the first integral is $[x\sin x]_0^{\pi} = 0$

To evaluate the first integral first note that $\dfrac{\mathrm{d}}{\mathrm{d}x}(x\cos x) = \cos x - x\sin x$. Hence the integral is $-x\cos x + c$.

Integration by inspection can be a surprisingly useful technique in STEP.

(ii) **(a)** Answer: $xe^x - e^x + c$

(b) Answer: $[-x\cos x + \sin x]_0^{0.5\pi} = 1$

(c) Answer: $x\ln x - x + c$

(d) This is a nice application of the technique of integration by parts which you may not have seen before. It does not matter which of e^x and $\sin x$ you choose to integrate for the first integration by parts, but whichever you choose you must integrate this term again on the second integration by parts. Otherwise you will end up with the true, but not very useful, statement $I = I$.

The final answer is $I = \frac{1}{2}e^x(\sin x - \cos x) + c$.

Preparation

2 **(i)** Be careful with negative signs. Answer: $2\sin B\cos A$.

(ii) You may have met cosec before, but if you have not you might like to sketch the graph of $y = \operatorname{cosec} x$. It will help to sketch $y = \sin x$ first. Answers: **(a)** $\sqrt{2}$ and **(b)** 2.

(iii) You should find that most of the terms cancel leaving $\sqrt{n}$. Write out the first couple of terms and the last term or two (with "…" in between). I find it slightly easier to start with the "$k = n$" case and work backwards to "$k = 1$".

(iv) $\cos\pi = -1$, $\cos 2\pi = 1$ and $\cos 3\pi = -1$ suggesting that $\cos n\pi = (-1)^n$.

(v) Answer: $\frac{1}{k}\sin kx + c$. Don't forget the constant of integration!

(vi) $I = x\sin x(+c)$ (from question **1 (i)(c)**) and $J = -\cos x(+c')$, so $I - J = x\sin x + \cos x + k$.

The STEP question (1998 STEP II Q4)

3

$$\begin{aligned}
I_n - I_{n-1} &= \int_0^{\pi} (\tfrac{1}{2}\pi - x)\text{cosec}(\tfrac{1}{2}x)\left(\sin(nx + \tfrac{1}{2}x) - \sin(nx - \tfrac{1}{2}x)\right)\, \mathrm{d}x \\
&= \int_0^{\pi} (\tfrac{1}{2}\pi - x)\text{cosec}(\tfrac{1}{2}x)\Big[\Big(\sin(nx)\cos\left(\tfrac{1}{2}x\right) + \cos(nx)\sin\left(\tfrac{1}{2}x\right)\Big) \\
&\qquad\qquad - \Big(\sin(nx)\cos\left(\tfrac{1}{2}x\right) - \cos(nx)\sin\left(\tfrac{1}{2}x\right)\Big)\Big]\, \mathrm{d}x \\
&= \int_0^{\pi} (\tfrac{1}{2}\pi - x)\cancel{\text{cosec}(\tfrac{1}{2}x)}\Big[2\cos(nx)\cancel{\sin\left(\tfrac{1}{2}x\right)}\Big]\, \mathrm{d}x \\
&= \int_0^{\pi} (\pi - 2x)\cos(nx)\, \mathrm{d}x \\
&= \pi\int_0^{\pi} \cos(nx)\, \mathrm{d}x - 2\int_0^{\pi} x\cos(nx)\, \mathrm{d}x\,.
\end{aligned}$$

The first integral becomes $\frac{\pi}{n}\left[\sin nx\right]_0^{\pi} = 0$. Evaluating the second integral (by parts) gives:

$$\begin{aligned}
I_n - I_{n-1} &= -2\Big[\tfrac{1}{n}x\sin(nx) + \tfrac{1}{n^2}\cos(nx)\Big]_0^{\pi} \\
&= -2\Big[\Big(0 + \tfrac{1}{n^2}(-1)^n\Big) - \Big(0 + \tfrac{1}{n^2}\Big)\Big] \\
&= 2\left(\frac{1 - (-1)^n}{n^2}\right)\,.
\end{aligned}$$

To evaluate I_n, note that $I_n = (I_n - I_{n-1}) + (I_{n-1} - I_{n-2}) + \cdots + (I_1 - I_0) + I_0$ and $I_0 = \displaystyle\int_0^{\pi} \left(\tfrac{1}{2}\pi - x\right)\, \mathrm{d}x = \left[\tfrac{1}{2}\pi x - \tfrac{1}{2}x^2\right]_0^{\pi} = 0$[1].

This gives the final answer of $I_n = 2\displaystyle\sum_{i=1}^{n} \frac{1 - (-1)^i}{i^2}$.

[1] Or you can sketch a graph to show that $\displaystyle\int_0^{\pi} \left(\tfrac{1}{2}\pi - x\right)\, \mathrm{d}x = 0$.

Discussion

When n is even you should find that $I_n - I_{n-1} = 0$ as then $1 - (-1)^n = 0$, and when n is odd we have $I_n - I_{n-1} = \dfrac{4}{n^2}$. This means that you can write[2] $I_n = 4\displaystyle\sum_{i=1}^{r} \frac{1}{(2i-1)^2}$, where $r = \frac{1}{2}n$ if n is even and $r = \frac{1}{2}(n+1)$ if n is odd[3].

Then from the result

$$\sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} = \tfrac{1}{8}\pi^2$$

given at the start of question **4** it follows that $I_n \to \frac{1}{2}\pi^2$ as $n \to \infty$.

[2] You are not asked to do this bit in the STEP question!

[3] We could write this in the form $r = \lfloor \frac{1}{2}(n+1) \rfloor$, i.e. the "floor" of $\frac{1}{2}(n+1)$, that is the biggest integer less than or equal to $\frac{1}{2}(n+1)$.

Warm down

4 **(i)**

$$S_{\text{even}} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \tfrac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} = \tfrac{1}{4}S\,.$$

Then $S = S_{\text{even}} + S_{\text{odd}}$ therefore $S = \frac{1}{4}S + \frac{1}{8}\pi^2$ and you can rearrange to get $S = \frac{1}{6}\pi^2$.

We also have $S_{\text{even}} = \frac{1}{6}\pi^2 - \frac{1}{8}\pi^2 = \frac{1}{24}\pi^2$.

(ii) The first sum is equivalent to $-S_{\text{odd}} + S_{\text{even}}$ and so is equal to $-\frac{1}{8}\pi^2 + \frac{1}{24}\pi^2 = -\frac{1}{12}\pi^2$.

For the second sum, it is probably helpful to write out some terms, such as:

$$\sum_{n=1}^{\infty} \frac{\cos \frac{1}{2}n\pi}{n^2} = \frac{0}{1^2} + \frac{-1}{2^2} + \frac{0}{3^2} + \frac{1}{4^2} + \cdots\,.$$

This sum is the same as:

$$-\sum_{n=1}^{\infty} \frac{1}{(2n)^2} + 2 \times \sum_{n=1}^{\infty} \frac{1}{(4n)^2}\,.$$

This can then be simplified to $-\frac{1}{8}\sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{1}{48}\pi^2$.

Alternatively (and probably more easily) this sum can be written as:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^2}{(2n)^2} &= \tfrac{1}{4}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \\ &= \tfrac{1}{4} \times \left(-\tfrac{1}{12}\pi^2\right) \\ &= -\tfrac{1}{48}\pi^2\end{aligned}$$

(iii) Again, it is a good idea to write down some terms to get a feeling for what is going on. You should get:

$$\left(\frac{1}{1^2} + \frac{1}{5^2}\right) + \left(\frac{1}{7^2} + \frac{1}{11^2}\right) + \left(\frac{1}{13^2} + \frac{1}{17^2}\right) + \cdots\,.$$

This sum excludes the denominators which are the squares of multiples of two and three. You might think then that the required sum would be $S - S_{\text{even}} - S_{\text{multiples of 3}}$ but then the multiples of 6 would be subtracted twice, so we to add them back on. The required sum is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} - \sum_{n=1}^{\infty} \frac{1}{(3n)^2} + \sum_{n=1}^{\infty} \frac{1}{(6n)^2}\,.$$

This simplifies to $\frac{2}{3}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{9}\pi^2$.