STEP Support Programme

Hints and partial solutions for Assignment 25

Warm-up

1 **(i)** **(a)** The integral becomes:

$$\begin{aligned}\int \frac{u+2}{u} \times 1\,\mathrm{d}u &= \int 1 + \frac{2}{u}\,\mathrm{d}u \\ &= u + 2\ln|u| + c \\ &= x - 2 + 2\ln|x-2| + c\,.\end{aligned}$$

You can, if you wish, combine the "-2" with the constant of integration to get a final answer of $x + 2\ln|x-2| + k$. This is not a necessary step, but does look a little nicer.

(b) Here we have:

$$\begin{aligned}\int \frac{3\left(u^2-1\right)}{\cancel{u}} \times \cancel{u}\,\mathrm{d}u &= 3\int \left(u^2-1\right)\,\mathrm{d}u \\ &= 3\left(\tfrac{1}{3}u^3 - u\right) + c \\ &= u^3 - 3u + c \\ &= (2x+1)^{\frac{3}{2}} - 3(2x+1)^{\frac{1}{2}} + c\end{aligned}$$

The final answer could also be written as $2(x-1)\sqrt{2x+1} + c$ (by factorising out $(2x+1)^{\frac{1}{2}}$).

(ii) In the case the integral becomes:

$$\begin{aligned}\int_0^{\frac{1}{6}\pi} \frac{1}{\sqrt{4-4\sin^2\theta}} \times 2\cos\theta\,\mathrm{d}\theta &= \int_0^{\frac{1}{6}\pi} \frac{1}{\cancel{2\cos\theta}} \times \cancel{2\cos\theta}\,\mathrm{d}\theta \\ &= \int_0^{\frac{1}{6}\pi} 1\,\mathrm{d}\theta \\ &= \Big[\theta\Big]_0^{\frac{1}{6}\pi} \\ &= \tfrac{1}{6}\pi\end{aligned}$$

Preparation

2 **(i)** Since you are given A and B in the question you should stick with these variables (rather than using a and b). We have:

$$\begin{aligned}\tan(A-B) &= \frac{\sin(A-B)}{\cos(A-B)}\\ &= \frac{\sin A\cos B-\sin B\cos A}{\cos A\cos B+\sin A\sin B}\\ &= \frac{\frac{\sin A\,\cancel{\cos B}}{\cos A\,\cancel{\cos B}}-\frac{\sin B\,\cancel{\cos A}}{\cancel{\cos A}\,\cos B}}{\frac{\cancel{\cos A}\,\cancel{\cos B}}{\cancel{\cos A}\,\cancel{\cos B}}+\frac{\sin A}{\cos A}\frac{\sin B}{\cos B}}\\ &= \frac{\tan A-\tan B}{1+\tan A\tan B}\end{aligned}$$

(ii) One approach is:

$$\begin{aligned}\ln\left(1+\frac{\frac{1}{2}-x}{\frac{1}{2}+x}\right) &= \ln\left(\frac{\frac{1}{2}+\cancel{x}+\frac{1}{2}-\cancel{x}}{\frac{1}{2}+x}\right)\\ &= \ln\left(\frac{1}{\frac{1}{2}+x}\right)\\ &= \ln 1-\ln\left(\tfrac{1}{2}+x\right)\\ &= -\ln\left(\tfrac{1}{2}+x\right)\end{aligned}$$

(iii) Dividing throughout by $\cos^2\theta$ gives $\tan^2\theta+1\equiv\sec^2\theta$.

This is a result worth "knowing by heart", or at least being able to derive fairly quickly.

(iv) The quotient rule (using a prime to denote differentiation)

$$\left(\frac{\mathrm{u}}{\mathrm{v}}\right)'=\frac{\mathrm{u}'\mathrm{v}-\mathrm{v}'\mathrm{u}}{\mathrm{v}^2}$$

follows immediately from the product rule applied to uv^{-1} using the chain rule to differentiate v^{-1} (which gives $-\mathrm{v}^{-2}\mathrm{v}'$).

Using this we have:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{\sin\theta}{\cos\theta}\right) &= \frac{\cos\theta\times\cos\theta-(-\sin\theta)\times\sin\theta}{\cos^2\theta}\\ &= \frac{1}{\cos^2\theta}\\ &= \sec^2\theta\,.\end{aligned}$$

Another result which is good to "know by heart"!

(v) One approach is:

$$\begin{aligned}\frac{1+\sin 2\alpha}{1+\cos 2\alpha} &= \frac{\cos^2\alpha + \sin^2\alpha + 2\sin\alpha\cos\alpha}{\cos^2\alpha + \cancel{\sin^2\alpha} + \cos^2\alpha - \cancel{\sin^2\alpha}} \\ &= \frac{(\sin\alpha + \cos\alpha)^2}{2\cos^2\alpha} \\ &= \tfrac{1}{2}\left(\frac{\sin\alpha + \cos\alpha}{\cos\alpha}\right)^2 \\ &= \tfrac{1}{2}\left(\tan\alpha + 1\right)^2\end{aligned}$$

(vi) The answer is given, so you do need to show some working:

$$\begin{aligned}\int_0^{84} \frac{x^2}{x^2+(84-x)^2}\,\mathrm{d}x &= \int_{84}^{0} \frac{(84-u)^2}{(84-u)^2+u^2} \times -1\,\mathrm{d}u \\ &= -1 \times -\int_0^{84} \frac{(84-u)^2}{u^2+(84-u)^2}\,\mathrm{d}u \\ &= \int_0^{84} \frac{(84-x)^2}{x^2+(84-x)^2}\,\mathrm{d}x\end{aligned}$$

where the last step involves a substitution of $u = x$.

We then have:

$$\begin{aligned}I + I &= \int_0^{84} \frac{x^2}{x^2+(84-x)^2}\,\mathrm{d}x + \int_0^{84} \frac{(84-x)^2}{x^2+(84-x)^2}\,\mathrm{d}x \\ &= \int_0^{84} \frac{x^2+(84-x)^2}{x^2+(84-x)^2}\,\mathrm{d}x \\ &= \int_0^{84} 1\,\mathrm{d}x = 84\end{aligned}$$

so $2I = 84$ and hence $I = 42$.

The STEP question (1994 STEP I Q8)

3 Using the given substitution we have:

$$\begin{aligned}
I = \int_0^{\frac{1}{4}\pi} \ln(1+\tan\theta)\,\mathrm{d}\theta &= \int_{\frac{1}{4}\pi}^{0} \ln\left(1+\tan\left(\tfrac{1}{4}\pi-\phi\right)\right)\times -1\,\mathrm{d}\phi \\
&= \int_0^{\frac{1}{4}\pi} \ln\left(1+\frac{1-\tan\phi}{1+\tan\phi}\right)\mathrm{d}\phi \qquad \text{using } \tan\left(\tfrac{1}{4}\pi\right)=1 \\
&= \int_0^{\frac{1}{4}\pi} \ln\left(\frac{1+\cancel{\tan\phi}+1-\cancel{\tan\phi}}{1+\tan\phi}\right)\mathrm{d}\phi \\
&= \int_0^{\frac{1}{4}\pi} \left(\ln 2 - \ln(1+\tan\phi)\right)\mathrm{d}\phi \\
&= \int_0^{\frac{1}{4}\pi} \ln 2\,\mathrm{d}\phi - I \\
&= \Big[\phi\ln 2\Big]_0^{\frac{1}{4}\pi} - I \\
&= \tfrac{1}{4}\pi\ln 2 - I.
\end{aligned}$$

Don't forget that $\ln 2$ is a constant so when we integrate we get $\phi\times\ln 2$. We now have $I=\frac{1}{4}\pi\ln 2 - I$ and so we have $I=\frac{1}{8}\pi\ln 2$.

(i) Looking at the limits, and thinking that we would like something that looks a little like the first integral, try $x=\tan\theta$.

$$\begin{aligned}
\int_0^1 \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x &= \int_0^{\frac{1}{4}\pi} \frac{\ln(1+\tan\theta)}{\cancel{1+\tan^2\theta}}\times\cancel{\sec^2\theta}\,\mathrm{d}\theta \\
&= \tfrac{1}{8}\pi\ln 2\,.
\end{aligned}$$

(ii) You can get a hint of what to use by looking at the upper limit of the integral. Starting with $x=2u$ gives:

$$\begin{aligned}
\int_0^{\frac{1}{2}\pi} \ln\left(\frac{1+\sin x}{1+\cos x}\right)\mathrm{d}x &= \int_0^{\frac{1}{4}\pi} \ln\left(\frac{1+\sin 2u}{1+\cos 2u}\right)\times 2\,\mathrm{d}u \\
&= \int_0^{\frac{1}{4}\pi} \ln\left(\tfrac{1}{2}(1+\tan u)^2\right)\times 2\,\mathrm{d}u \qquad \text{using } \mathbf{2(v)} \\
&= 2\int_0^{\frac{1}{4}\pi} \left[\ln\left(\tfrac{1}{2}\right)+\ln\left(1+\tan u\right)^2\right]\mathrm{d}u \\
&= 2\int_0^{\frac{1}{4}\pi} \ln\left(\tfrac{1}{2}\right)\mathrm{d}u + 4\int_0^{\frac{1}{4}\pi} \ln\left(1+\tan u\right)\mathrm{d}u \\
&= 2\Big[u\times\ln\left(\tfrac{1}{2}\right)\Big]_0^{\frac{1}{4}\pi} + 4\times\tfrac{1}{8}\pi\ln 2 \\
&= 2\times\tfrac{1}{4}\pi\times(-\ln 2)+\tfrac{1}{2}\pi\ln 2 = 0
\end{aligned}$$

If you hadn't done the preparation questions first, you will have had to do some more lines of working to get to the expression in $\tan u$. However, knowing what sort of expression you are aiming for is a big help, and using the limits to suggest a suitable substitution is usually a good idea.

Other methods are also possible!

Warm down

4 **(i)** Use $t = a - x$ to give:

$$\begin{aligned} I = \int_0^a \frac{\mathrm{f}(x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\,\mathrm{d}x &= \int_a^0 \frac{\mathrm{f}(a-t)}{\mathrm{f}(a-t)+\mathrm{f}(t)} \times -1\,\mathrm{d}t \\ &= \int_0^a \frac{\mathrm{f}(a-t)}{\mathrm{f}(a-t)+\mathrm{f}(t)}\mathrm{d}t \\ &= \int_0^a \frac{\mathrm{f}(a-x)}{\mathrm{f}(a-x)+\mathrm{f}(x)}\mathrm{d}x \end{aligned}$$

Then we have $2I = \displaystyle\int_0^a 1\mathrm{d}x = a$ and hence $I = \frac{1}{2}a$.

For the last part, first note that $\cos x + \sin x \neq 0$ in the range $0 \leqslant x \leqslant \frac{1}{2}\pi$.

We can use $\sin x = \cos(\frac{1}{2}\pi - x)$ to get:

$$\int_0^{\frac{1}{2}\pi} \frac{\cos x}{\cos x + \cos(\frac{1}{2}\pi - x)}\,\mathrm{d}x$$

which is now in the same form as the previous integral, so this integral is equal to $\frac{1}{2}a = \frac{1}{4}\pi$.

(ii) The result $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$ is well worth memorising!

(a) Here $\mathrm{f}(x) = \sin x$, so the integral is $\Big[\ln|\sin(x)|\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \ln 1 - \ln\dfrac{1}{\sqrt{2}} = \frac{1}{2}\ln 2$.

(b) Writing the integral in the form $\displaystyle\int \frac{\frac{1}{x}}{\ln x}\,\mathrm{d}x$ gives the result $\displaystyle\int \frac{1}{x\ln x}\,\mathrm{d}x = \ln|\ln x| + c$.

(c) $S + T = x + c_1$ and $S - T = \ln(\cos x + \sin x) + c_2$ (using (‡) for the second of these integrals). You can then solve simultaneous equations to give:

$$\begin{aligned} S &= \tfrac{1}{2}x + \tfrac{1}{2}\ln(\cos x + \sin x) + k_1 \\ T &= \tfrac{1}{2}x - \tfrac{1}{2}\ln(\cos x + \sin x) + k_2\,. \end{aligned}$$