STEP Support Programme

Hints and Partial Solutions for Assignment 2

Warm-up

1 **(i)** You could either expand the brackets and then factorise, or use the difference of two squares to get $[(2x-3)+(x-1)][(2x-3)-(x-1)]$ and then simplify each bracket to get $(3x-4)(x-2)$.

You might like to consider whether verifying the result for two values of x is sufficient to show that the answer is correct.

(ii) It is usually a good idea to ensure that all the denominators are fully factorised before you start to combine them. If you use the lowest common denominator it will make your life *much* easier (and you won't have to cancel common factors at the end).

$$\begin{aligned}\frac{x}{x^2-y^2}-\frac{y}{(x-y)^2}-\frac{1}{x+y} &= \frac{x}{(x-y)(x+y)}-\frac{y}{(x-y)^2}-\frac{1}{(x+y)}\\ &= \frac{x(x-y)-y(x+y)-(x-y)^2}{(x-y)^2(x+y)}\\ &= \frac{-2y^2}{(x-y)^2(x+y)}\end{aligned}$$

In a similar way to the first part, you can substitute some values for x and y to check your simplification.

For the final part $y=0$ and x can be anything except 0 (if both x and y are equal to zero then the fractions are *undefined*).

(iii) There are lots of ways of doing this.

You could start by noticing that $A = 1/B \iff AB = 1$, so all you have to do is show that $(\sqrt{1+x^2}-x)(\sqrt{1+x^2}+x) = 1+x^2-x^2 = 1$.

You can also rationalise the denominator on the right hand side by multiplying both numerator and denominator by $\sqrt{1+x^2}-x$.

For the last part, you can just say that $\sqrt{1+x^2} \approx x$ when x is much larger than 1. This gives:

$$\begin{aligned}\sqrt{1+x^2}-x &= \frac{1}{\sqrt{1+x^2}+x}\\ &\approx \frac{1}{x+x} = \frac{1}{2x}\end{aligned}$$

Even if you hadn't done the first part of this question, it would have been worth having a go at the last part.

(iv) Using the difference of two squares, $(A+B)(A-B) = A^2 - B^2$, is slightly more efficient that expanding the brackets by hand.

$$\begin{aligned}(x^2 - \sqrt{2}\,x + 1)(x^2 + \sqrt{2}\,x + 1) &= ((x^2+1) - \sqrt{2}\,x)((x^2+1) + \sqrt{2}\,x) \\ &= (x^2+1)^2 - 2x^2 \\ &= x^4 + 2x^2 + 1 - 2x^2 \\ &= x^4 + 1\end{aligned}$$

Hence to solve $x^4 + 1 = 0$ you can solve the two quadratic equations $x^2 \pm \sqrt{2}x + 1 = 0$ and it will make your answers a little neater if you write $\sqrt{-2} = i\sqrt{2}$.

For example, solving $x^2 + \sqrt{2}x + 1 = 0$ gives:

$$\begin{aligned}x &= \frac{-\sqrt{2} \pm \sqrt{2-4}}{2} \\ &= \frac{-\sqrt{2} \pm i\sqrt{2}}{2}\end{aligned}$$

The four answers here are the four fourth roots of -1 and are $x = \frac{\pm\sqrt{2} \pm i\sqrt{2}}{2}$.

Preparation

You can use Desmos to check your sketches (you can even use parameters — such as a c in part **(ii)** — and add a “slider” to see what happens as c changes). However you need to make it plain on **your** sketches which portion of the graph you are considering.

2 **(i)** You should remember to label the important points on the graph including the limits of the domain (i.e. the end-points of the graph — the graph should clearly start and end at $(-2, -1)$ and $(2, 3)$). The greatest value is therefore 3.

(ii) It doesn’t matter if you sketch a graph with a positive or negative value of c — the greatest value will be when $x = -2$ and the least when $x = 2$ as the gradient is negative. The least value is $-4 + c$.

(iii) Unlike the previous part, you do need to consider both negative and positive m separately. However, you can combine both cases with use of the modulus function, e.g. the greatest value is $2|m| + 1$ for all m and the least value is $-2|m| + 1$.

(iv) The vertex of the parabola is where the least value of $(x - 1)^2$ is, so the least value is 0 (when $x = 1$). The greatest value is 9 which occurs when $x = 2$.

(v) In this case, the vertex lies outside of the domain of x. The greatest value is at one end of the domain and the least value is at the other end. One way to sketch this graph is to sketch the parabola (including the vertex) with a dotted line and then identify the part which you are restricted to with a solid line.

From your sketch you should be able to deduce that the greatest value is when $x = -2$ and is equal to $(-2 - 3)^2 = 25$, and that the least value is when $x = 2$ and is equal to $(2 - 3)^2 = 1$.

(vi) This part was to get you thinking about how to find the vertex and it was meant as a hint for the next part. The vertex lies in the given domain of x so this is where the least value will be. The curve can be written as $y = (x - 4)^2 + 5$ and so the least value is 5 (when $x = 4$) and the greatest value is 21 (when $x = 0$).

(vii) You need to find a *general* answer in terms of k. Whilst using a particular value of k might give you some insight on how to solve the problem you still need to consider the general case. You should complete the square to locate the vertex, and a sketch will be helpful. The curve can be written as $y = (x + k)^2 - k^2$ which means that the vertex is at $(-k, -k^2)$.

The question was asking for the maximum and minimum values of the graph for a **fixed** yet **unknown** value of k. The least value is $-k^2$ (since the vertex lies within the domain of x if $-2 < k < 2$) and the greatest value is $4 + 4|k|$ (or you can write down two versions, one for $k > 0$ and one for $k < 0$).

When $k > 2$ the vertex lies to the left of the given domain of x. The greatest value is $4 + 4k$ and the least is $4 - 4k$.

The STEP question (1999 STEP I Q6)

3 The greatest and least values should have been given in terms of a, b and c; these parameters were **fixed** but **unknown**. The question was **not** asking you for the maximum and minimum values over all possible values of a, b and c.

You must make sure that you fully justify your answers, you are unlikely to get much credit for just writing down the answers (even if they are correct!).

(i) In the first part of the STEP question, you are asked: for a general $y = bx + a$ what are the greatest and least values in the given domain? Really it is not much harder than questions **2i, ii, iii**, but the greatest and least values are expressions involving a and b (and different for positive and negative b). Remember that a and b are **fixed** and **unknown**.

You will need three sketches, one for each case.

Answers:

$$\begin{array}{llll} b>0 & \text{greatest value } = 10b+a & & \text{least value } = -10b+a \\ b<0 & \text{greatest value } = -10b+a & & \text{least value } = 10b+a \\ b=0 & \text{greatest value } = a & & \text{least value } = a \end{array}$$

Or (if you prefer!):

$$\begin{aligned} \text{greatest value} &= 10|b|+a \\ \text{least value} &= -10|b|+a \end{aligned}$$

(ii) In the second part of the STEP question, you need to consider $c = 0$ as a separate case; but it is fine to say that when $c = 0$ the situation is just the same as in part **(i)**.

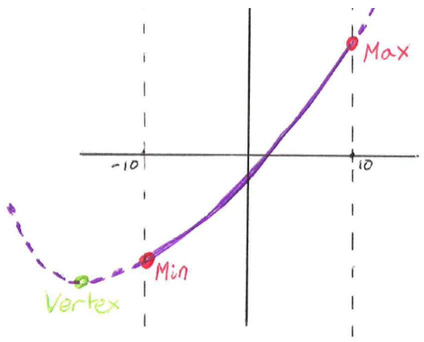
When $c > 0$ the crucial thing is the location of the vertex of the parabola, which needs some very careful completing of the square to locate. The maximum and minimum values (which will be in terms of a, b and c) will vary depending on whether the x coordinate of the vertex (v say — it will be in terms of b and c) lies in the range $v < -10$, or in the range $-10 \leqslant v \leqslant 0$, etc.

Completing the square gives:

$$\begin{aligned} cx^2 + bx + a &= c\left[x^2 + \tfrac{b}{c}x\right] + a \\ &= c\left[\left(x + \tfrac{b}{2c}\right)^2 - \tfrac{b^2}{4c^2}\right] + a \\ &= c\left(x + \tfrac{b}{2c}\right)^2 + a - \tfrac{b^2}{4c} \end{aligned}$$

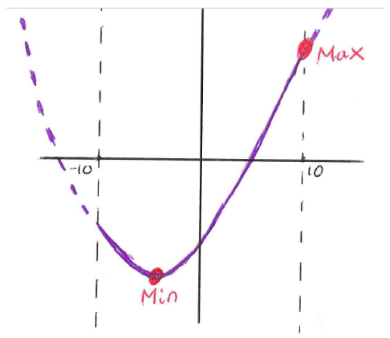
So the vertex is at $x = -\frac{b}{2c}$, $y = a - \frac{b^2}{4c}$.

Case 1: $v \leqslant -10$, i.e. $-\frac{b}{2c} \leqslant -10 \implies \frac{b}{2c} \geqslant 10$



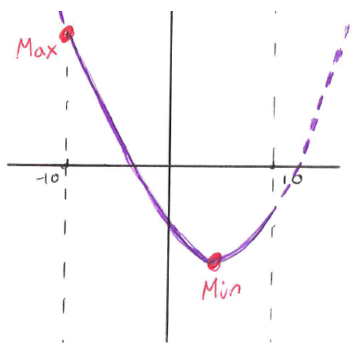
Maximum is at $x = 10, y = 100c + 10b + a$. Minimum is at $x = -10, y = 100c - 10b + a$.

Case 2: $-10 \leqslant v \leqslant 0$ i.e. $-10 \leqslant -\frac{b}{2c} \leqslant 0 \implies 0 \leqslant \frac{b}{2c} \leqslant 10$



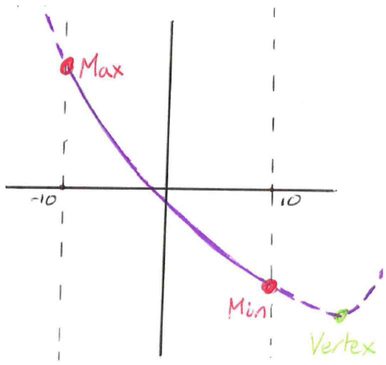
Maximum is at $x = 10, y = 100c + 10b + a$. Minimum is at $x = v = -\frac{b}{2c}, y = a - \frac{b^2}{4c}$.

Case 3: $0 \leqslant v \leqslant 10$ i.e. $0 \leqslant -\frac{b}{2c} \leqslant 10 \implies -10 \leqslant \frac{b}{2c} \leqslant 0$



Maximum is at $x = -10, y = 100c - 10b + a$. Minimum is at $x = v = -\frac{b}{2c}, y = a - \frac{b^2}{4c}$.

Case 4: $v \geqslant 10$ i.e. $-\frac{b}{2c} \geqslant 10 \implies \frac{b}{2c} \leqslant -10$



Maximum is at $x = -10, y = 100c - 10b + a$. Minimum is at $x = 10, y = 100c + 10b + a$.

The answers for these cases can be summarised as follows:

$$\text{greatest value} = \begin{cases} 100c - 10b + a & \text{for } \frac{b}{2c} \leqslant 0 \\ 100c + 10b + a & \text{for } \frac{b}{2c} \geqslant 0 \end{cases}$$

$$\text{least value} = \begin{cases} -\frac{b^2}{4c} + a & \text{for } -10 \leqslant \frac{b}{2c} \leqslant 10 \\ 100c + 10b + a & \text{for } \frac{b}{2c} \leqslant -10 \\ 100c - 10b + a & \text{for } \frac{b}{2c} \geqslant 10 \end{cases}$$

Note that summarising them would not be necessary for full marks, the work on the four separate cases would be enough!

Warm down

4 Start by calling the person in part **(ii)** Xander (X), the person in part **(iii)** Yolanda (Y), and the person in part **(iv)** Zak (Z). You can then put all the information from the first 5 parts of the question in a table:

Race	1st	2nd	3rd	4th	5th
There	Y		Z	A	X
Back	X	Z	Y	B	

You now know that Ahmed and Bachendri are **not** any of X, Y and Z. The missing space for the race to the tree must be Bachendri and the missing space on the race back must be Ahmed. Then use the information from the last two parts to work out which of X, Y and Z must be Charlie.

Final answer:

Race	1st	2nd	3rd	4th	5th
There	E	B	C	A	D
Back	D	C	E	B	A