STEP Support Programme

Hints and Partial Solutions for Assignment 3

Warm-up

1 **(i)** Remember that when we say "evaluate" or "find the value of" then we are not interested in approximations — when asked for the "value" you are required to give the **exact** value, no rounding or truncating!

Writing the terms out in full:

$$\begin{aligned}&\sin 0 + \sin\frac{\pi}{6} + \sin\frac{\pi}{3} + \sin\frac{\pi}{2} + \sin\frac{2\pi}{3} + \sin\frac{5\pi}{6} + \sin\pi\\ =&0 + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} + 0\\ =&2 + \sqrt{3}\end{aligned}$$

(ii) It makes things clearer if you align like terms:

$$\begin{aligned}rS_n - S_n = &\quad\ \ r + r^2 + \cdots + r^{n-1} + r^n\\ &-1 - r - r^2 - \cdots - r^{n-1}\end{aligned}$$

or even more nicely:

$$\begin{aligned}rS_n - S_n = r^n &+ r^{n-1} + \cdots + r^2 + r\\ &- r^{n-1} - \cdots - r^2 - r - 1\end{aligned}$$

So $rS_n - S_n = r^n - 1$, and $S_n = \frac{r^n-1}{r-1}$ for $r \neq 1$.

When $r = 1$ you cannot use the formula derived for S_n (as this would involve dividing by 0). However, when $r = 1$ we have $S_n = 1 + 1 + \cdots + 1 = n$.

For the second part, all the terms are multiplied by a, so just multiply the formula for the sum by a.

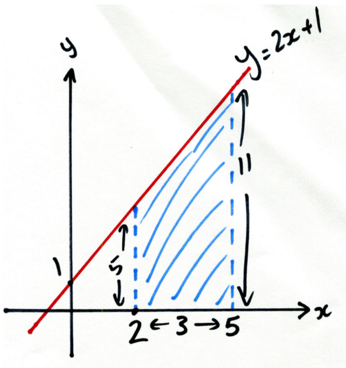
For the last part, if $-1 < r < 1$ then $r^n \to 0$ as $n \to \infty$. Hence we have $\displaystyle\sum_{i=0}^{\infty} ar^i = \frac{a}{1-r}$.

(iii) Again: no approximations, please! Also it is perfectly doable to evaluate this by hand (i.e. without a calculator — and calculators are not allowed in STEP).

You could just apply the formula you have found earlier with $a = 3$, $n = 10$ and $r = \frac{1}{2}$ to get $3 \times \dfrac{\left(1 - \dfrac{1}{1024}\right)}{\frac{1}{2}} = 6 \times \left(\dfrac{1023}{1024}\right) = \dfrac{3069}{512}$.

Preparation

2 **(i)** This purpose of this question was to try to ensure that you were thinking about the integral as an **area** under a graph rather than something to evaluate using an integration formula.



Using the formula for the area of a trapezium we have an area of:

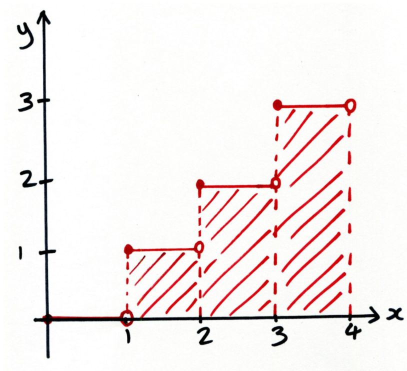
$$\frac{3}{2} \times (5 + 11) = 24$$

For this sort of integral (with limits) you don't need the "$+c$" because it will cancel out. Make sure that your final answer does not include $+c$, it should just be 24.

(ii) Here we were introducing you to the "floor" function. I particularly like the fact that $[10\pi] \neq 10 \times [\pi]$. Answers: 10, 8, 6, 31.

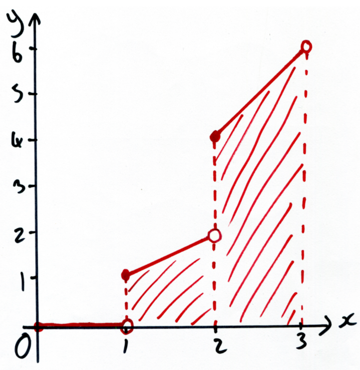
(iii) A question to get you thinking about graph sketching. You should have found that you got a series of steps (the first step having height 0). You should **not** have vertical lines at $x = 1$, $x = 2$ etc since the function takes only one value at these points. The graph is **not** continuous. Either don't put in anything at all, or if you must have vertical lines, make them dotted.

If you are being particularly careful, you can indicate on your graph that the value of y when $x = 2$ is 2 by having an empty circle at the (right hand) end of the step for $1 \leq x < 2$ and a solid circle at the (left hand) beginning of the step for $2 \leq x < 3$ etc.



(iv) Here you need to find the sum of the areas of three rectangles (or 4 if you count the first one with area 0). The total area is 6.

(v) This part is harder. It is best to consider different regions of x separately; for example, when $1 \leq x < 2$ then $[x] = 1$ and so in this region $y = x \times 1 = x$. You can show similarly that when $2 \leq x < 3$ then $y = x \times 2$, so you get sections of straight line graphs which get steeper as x gets larger. Be careful at the points where x is an integer as the graph is not continuous — do **not** draw solid vertical lines connecting the disparate line segments (again, dotted lines are ok and this helps with finding the area). There will be "jumps" at each integer value of x.



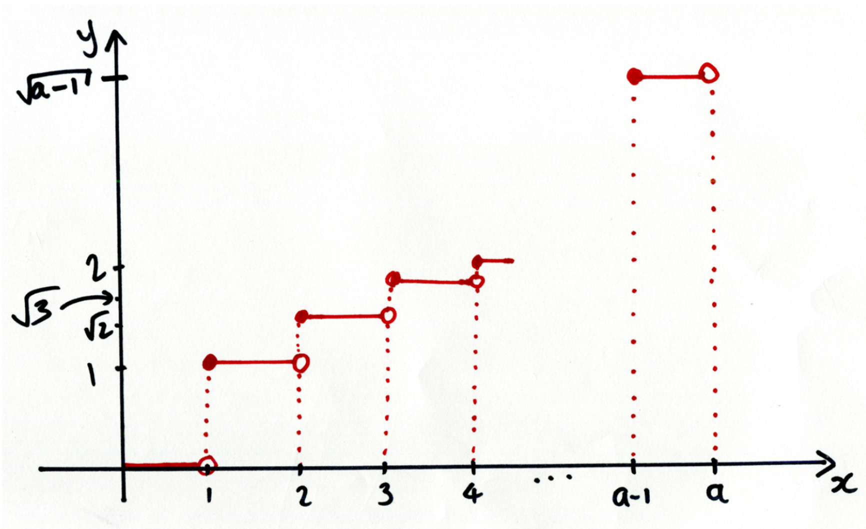
The area is given by:

$$0 + \tfrac{1}{2}(1+3) + \tfrac{1}{2}(4+6) = \frac{3}{2} + 5 = \frac{13}{2}$$

The STEP question (2004 STEP 1 Q2)

3 In all of your sketches, do not draw solid vertical lines joining up the horizontal bits: the graphs are not continuous.

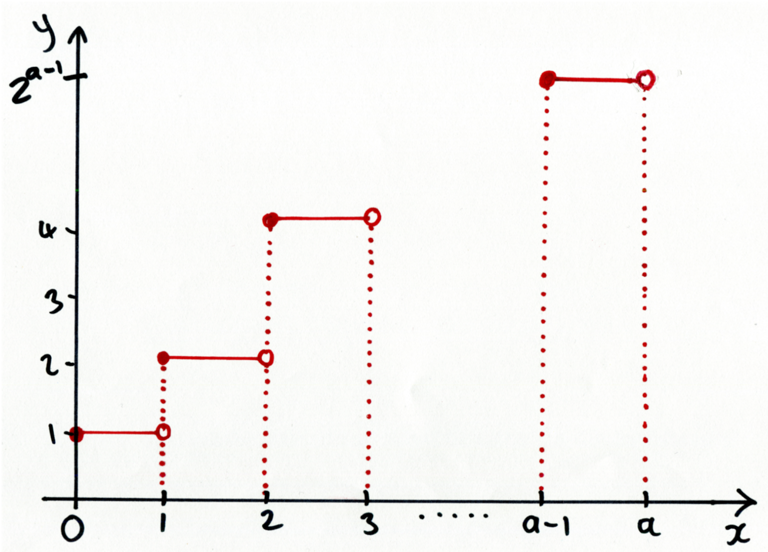
(i) What is needed is a clear diagram showing the first few rectangles (with heights $1, \sqrt{2}, \sqrt{3}, \cdots$), three dots to show there are some in the middle and then the last rectangle (preferably showing that it starts at $x = a - 1$ and ends at $x = a$, and with the height, $\sqrt{a-1}$, written on it). Make sure that there is a "gap" to show that there are missing rectangles in the middle.



Looking at the sum of the areas of the rectangles this gives:

$$\int_0^a \sqrt{[x]}\,\mathrm{d}x = 0 + 1 + \sqrt{2} + \sqrt{3} + \cdots + \sqrt{a-1}$$
$$= \sum_{r=0}^{a-1} \sqrt{r}$$

(ii) Start in the same way as in part **(i)**, i.e. draw a **sketch** (a sketch is often useful, even when you are not told to do it!).



The heights of the rectangles are $1, 2, 4, 8, \cdots, 2^{a-1}$, so we have:

$$\int_0^a 2^{[x]}\,\mathrm{d}x = 2^0 + 2^1 + 2^2 + \cdots + 2^{a-1}$$

The RHS is a geometric sequence with first term 1, common ratio 2, and a terms, so its sum is

$$\frac{1(2^a - 1)}{2 - 1} = 2^a - 1$$

It is a common feature of STEP questions that letters are used to represent things other than what you are used to using them for. This is partly to check that you actually understand what the formula is for rather than just substituting values in. Here a is the number of terms, rather than being the first term.

(iii) In the previous parts you were considering the sum of rectangles all of width 1. In this part, the last rectangle will be narrower.

If you are stuck, a good starting point would be to draw a few sketches with particular values of a to see if working with these special cases helps you to understand the general case. You could start, say, by considering $a = 3.2$.

The key idea is that up to $x = [a]$ the graph looks just like the one in part **(ii)**, and that $[a]$ is the last integer before a, so the last rectangle goes from $x = [a]$ to $x = a$. A careful sketch (and you can use the result from part (ii) to sum up all the "whole" rectangles) and you should be there.

A mathematical way of writing this would be to write:

$$\int_0^a 2^{[x]}\,\mathrm{d}x = \int_0^{[a]} 2^{[x]}\,\mathrm{d}x + \int_{[a]}^a 2^{[x]}\,\mathrm{d}x.$$

This becomes:

$$\begin{aligned}\int_0^{[a]} 2^{[x]}\,\mathrm{d}x + \int_{[a]}^{a} 2^{[x]}\,\mathrm{d}x &= \frac{1(2^{[a]}-1)}{2-1} + (a-[a]) \times 2^{[a]} \\ &= (2^{[a]}-1) + 2^{[a]}(a-[a])\end{aligned}$$

Note that the first integral can be done by using part **(ii)** (replacing a with $[a]$), whereas the second one is a single rectangle of width $a - [a]$.

Warm down

4 **(i)** Start off with N bananas. You should be able to show that after Arthur has been at them there are $\frac{2}{3}(N-1)$ left. Do this again for Brenda and Chandrima (I find it easier to simplify at each stage). The number of bananas left after Chandrima has had a go is the same as $3m+1$ so you should now be able to derive the given equation.

$$\begin{aligned}
\text{after Arthur :}\quad & \tfrac{2}{3}(N-1) && \text{bananas left}\\
\text{after Brenda :}\quad & \tfrac{2}{3}\left[\tfrac{2}{3}(N-1)-1\right] &&\\
&= \tfrac{4}{9}N-\tfrac{10}{9} && \text{bananas left}\\
\text{after Chandrima :}\quad & \tfrac{2}{3}\left[\tfrac{4}{9}N-\tfrac{10}{9}-1\right] &&\\
&= \tfrac{8}{27}N-\tfrac{38}{27} && \text{bananas left}
\end{aligned}$$

Then setting $\frac{8}{27}N-\frac{38}{27}=3m+1$ gives the required equation.

(ii) The important idea here is that we are looking for pairs of integers (N, m) which satisfy the equation $8N = 81m + 65$. Not every integer N will work; for example if $N = 20$ we get $m = \frac{95}{81}$, which is not an integer.

First off, assume that N is a possible number of bananas, i.e. for this particular **integer** N we have a corresponding **integer** m so that $8N = 81m+65$. Now consider N' which is defined as $N' = N + 81$ (and since N is an integer, N' is an integer). We then have:

$$\begin{aligned}
8N' &= 8(N+81)\\
&= 8N+8\times 81\\
&= 81m+65+8\times 81\\
&= 81m+81\times 8+65\\
&= 81(m+8)+65
\end{aligned}$$

(the penultimate line was found by substituting in $8N = 81m + 65$).

Hence we can write $N' = 81m' + 65$ where $m' = m + 8$, so if N is a solution so is $N' = N + 81$.

(iii) It is quite tricky to find an integer value of N such that when we substitute N into $8N = 81m+65$ we also get an integer for m. You could start by trying $N = 1$, $N = 2$, etc, but you would be going a very long time before you found a value of N that worked.

If we step back a bit from the context of the problem (forget about bananas, just think about the abstract equation $8N = 81m + 65$) we can see that substituting the integer $N = -2$ does indeed give an integer value of m as then we have $8\times -2 = 81m+65 \implies m = -1$.

We now have an integer value of N for which m is also an integer, namely $N = -2$. However we also know that if N is an integer solution, then $N + 81$ is an integer solution, so we can use this to find a **positive** value of N from our initial value of -2. Now we have a positive value of N (i.e. $-2 + 81 = 79$) we can think about bananas again.

The general formula is $N = -2 + 81k$, $m = -1 + 8k$.

You can read more about the original "Monkeys and Coconuts" problem on this Wikipedia page, or watch this Numberphile video about the problem