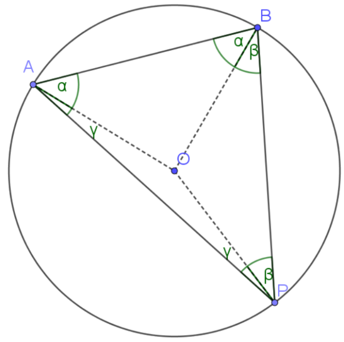
STEP Support Programme

Hints and Partial Solutions for Assignment 4

Warm-up

1 **(i)** This question was asking you to prove that the angle at the centre of a circle is twice the angle at the circumference. When drawing a diagram try not to draw a "special" triangle, i.e. try not to make APB equilateral or isosceles (unless of course you have been told that this is true!).



The triangle APO must be isosceles as AO and OP are both radii. Labelling the angles we have:

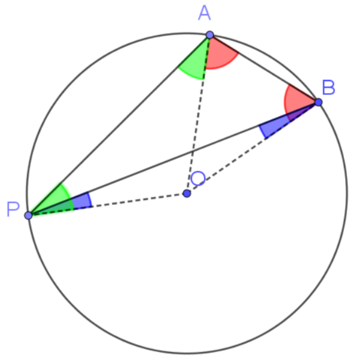
$$\begin{aligned}\angle ABO &= \angle BAO = \alpha\\ \angle BPO &= \angle PBO = \beta\\ \angle APO &= \angle PAO = \gamma\end{aligned}$$

We then have $\angle AOB = 180° - 2\alpha$. Since $2\alpha + 2\beta + 2\gamma = 180°$:

$$\begin{aligned}\angle AOB &= 2\beta + 2\gamma\\ &= 2(\beta + \gamma)\\ &= 2\angle APB\end{aligned}$$

as required.

(ii) In this case $APBO$ does not form an arrowhead (sometimes called a *delta*) and two of the lines cross.



For clarity, I have coloured the angles in my diagram; let the red angles be r, the blue angles b and the green angles g. Then $\angle AOB = 180^\circ - 2r$, and $2g + 2r - 2b = 180^\circ$ So:

$$\begin{aligned}\angle AOB &= 2g - 2b \\ &= 2(g - b) \\ &= 2\angle APB\end{aligned}$$

as required.

Can you prove the case where $\angle APB$ is obtuse? You may wish to prove that opposite angles in a cyclic quadrilateral are *supplementary*[1] first.

[1] Supplementary means that the angles add up to 180°, so we can say that "the angles in a triangle are supplementary". Angles that add up to 90° are called *complementary*.

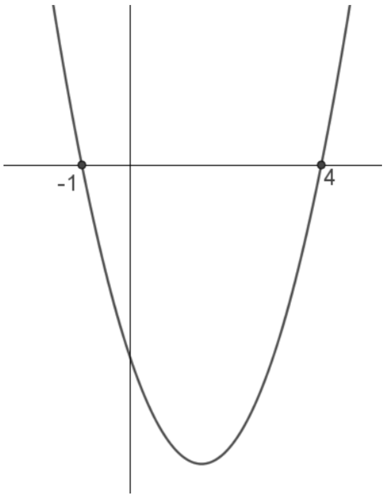
Preparation

2 **(i)** Two points to watch out for here.

(a) You have to be careful about wording. In this part of the question, x could lie in one of two ranges, so you should write $x < a$ **or** $x > b$ (say) rather than $x < a$ **and** $x > b$, as any single value of x cannot be in both ranges at once.

(b) You have to be careful to use the appropriate type of inequality — if the question involves **strict** inequalities ($<$, $>$) then the answer should involve strict inequalities, and similarly for **weak** inequalities ($\leq$, $\geq$).

$x^2 - 3x - 4 = (x - 4)(x + 1)$



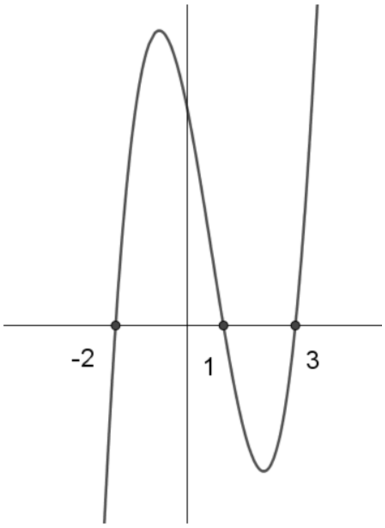
From the graph, $x^2 - 3x - 4 > 0$ when $x < -1$ or $x > 4$.

(ii) $f(x) = x^3 - 2x^2 - 5x + 6$

$f(3) = 27 - 18 - 15 + 6 = 0$, so $x = 3$ is a root. Hence, $(x - 3)$ is a factor.

You don't know that the roots are going to be integers; but if they are integers there are not many possibilities (they can only be factors of 6), so it might have been worth simply testing these until you have found the three roots. Alternatively (and if the other roots had not been integers this is the approach you would have had to use) you can divide the cubic by the known factor $(x - 3)$: $f(x) = (x - 3)(x^2 + x - 2)$. Then factorise the resulting quadratic to give $f(x) = (x - 3)(x - 1)(x + 2)$.

Don't forget to answer the question — it asks you to find the roots so you do actually need to write "$x = \cdots$" as well as writing the cubic as a product of three linear factors. If you had divided the cubic by $(x - 3)$ then you will probably have written the cubic as a product of three linear factors first — you still need to write down the three roots (which are $x = 3$, $x = 1$ and $x = -2$).



The graph shows that $f(x) \leqslant 0$ if $x \leqslant -2$ or if $1 \leqslant x \leqslant 3$.

(iii) $x^2 - 3x + 2 = (x-2)(x-1)$, so $x^2 - 3xy + 2y^2 = (x-2y)(x-y)$. If $x^2 - 3xy + 2y^2 = 0$ then either $x - 2y = 0$ or $x - y = 0$, so $y = \frac{1}{2}x$ or $y = x$.

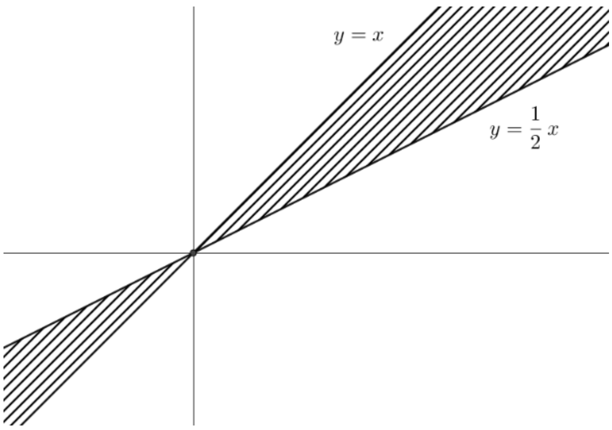
(iv) There are two main approaches here.

(a) You can write the expression as the product of two linear factors and then consider the signs of each bracket, which will result in two inequalities for each case (for example, $x - 2y \geqslant 0$ and $x - y \leqslant 0$ for the first bracket to be positive and the second bracket negative).

(b) A perhaps simpler line of attack is note that the two straight lines drawn in part **(iii)** divide the plane into 4 regions. You then only need to determine the sign of the quadratic expression in each of the 4 regions, for example by calculating its value at a single point in each region. So, for example, using the point $(3, 2)$ we have

$$x^2 - 3xy + 2y^2 = 3^2 - 3 \times 3 \times 2 + 2 \times 2^2 = -1$$

which is negative, so the region in which this point lies is one of the required regions.



The shaded region shows where $x^2 - 3xy + 2y^2 \leqslant 0$.

Be careful to pick points which lie **within** the regions — the point $(1, 1)$ is no good as it lies on one of the boundary lines.

Note that there are only 4 regions — the x and y axes are **not** boundaries of regions. It is not a problem if you test in 8 regions rather than 4; you are just making life harder for yourself!

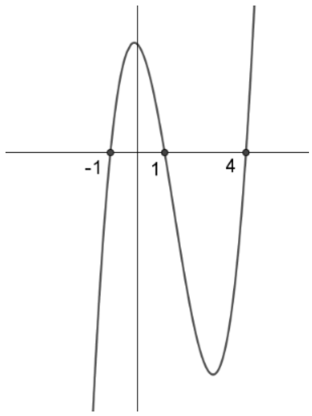
The STEP question (1995 STEP I Q1)

3 **(i)** As in the last question, you have to be careful to distinguish between **strict** or **weak** inequalities. The three roots of $x^3 - 4x^2 - x + 4 = 0$ are all integers.

Let $f(x) = x^3 - 4x^2 - x + 4$.

$f(1) = 1 - 4 - 1 + 4 = 0$ so $(x - 1)$ is a factor.

So $f(x) = (x - 1)(x^2 - 3x - 4) = (x - 1)(x - 4)(x + 1)$.



From the sketch, $f(x) \geqslant 0$ if $-1 \leqslant x \leqslant 1$ or if $x \geqslant 4$.

(ii) You can realise that $x = y$ is a solution and then divide throughout by $(x - y)$, but the simplest approach is to use your answer to part **(i)** and essentially replace 1 with y:

$x^3 - 4x^2y - xy^2 + 4y^3 = (x - y)(x - 4y)(x + y)$, so $x^3 - 4x^2y - xy^2 + 4y^3 = 0$ on the lines $y = x$, $y = \frac{1}{4}x$ and $y = -x$.

(iii) You can consider where the product of the three factors was positive (i.e. where all three are positive, or where two are negative and one positive), or test points in all the 6 regions that the three lines create to see where the inequality is true. Both ways are fine **but** you must fully justify your answer. The second approach is simpler and probably easier to explain fully.

Examples of points in each of the six regions:

$(1, 0) : x^3 - 4x^2y - xy^2 + 4y^3 = 1 > 0$

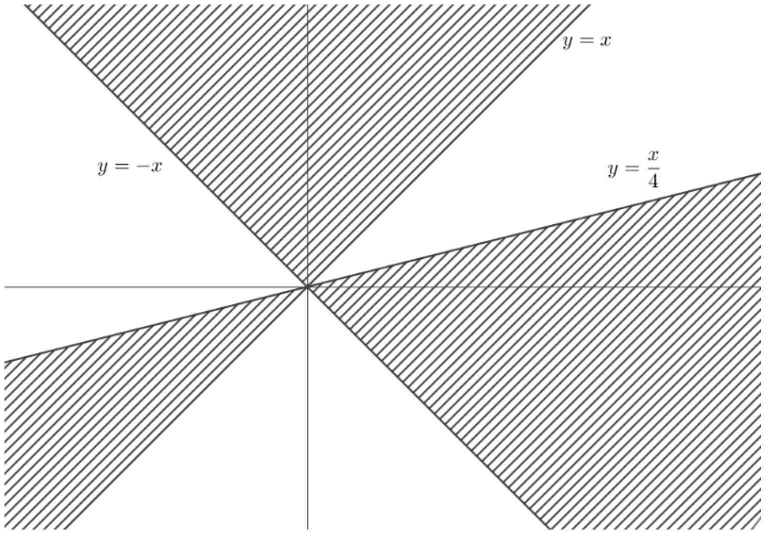
$(2, 1) : x^3 - 4x^2y - xy^2 + 4y^3 = 8 - 16 - 2 + 4 = -6 < 0$

$(0, 1) : x^3 - 4x^2y - xy^2 + 4y^3 = 4 > 0$

$(-2, 1) : x^3 - 4x^2y - xy^2 + 4y^3 = -8 - 16 + 2 + 4 = -18 < 0$

$(-2, -1) : x^3 - 4x^2y - xy^2 + 4y^3 = -8 + 16 + 2 - 4 = 6 > 0$

$(0, -1) : x^3 - 4x^2y - xy^2 + 4y^3 = -4 < 0$



Warm down

4 **(i)** You need to think very carefully about the question. It says that **if** there is an even number **then** the other side is a vowel. It does not follow that if there is a vowel on one side then there must be an even number on the other side; furthermore if there is an odd number on one side there can be any letter on the other side.

It is fairly obvious that we must check the card showing the 6. There is one other card that we must check, and this card could show the statement to be false. Think about the different implications there might be depending on what is on the second sides of the cards.

In the first version of this assignment we neglected to include the information that the cards had a number on one side and a letter on the other. If we didn't know this then we would actually have to turn over three cards — the only one we would not have had to check is the "E". (Why?)

Answer: We must check "6" to see if there is a vowel on the other side. The other card we must check is "Q", as if this has an even number on the other side it will disprove my claim.

(ii) The fact that the ropes are "non-uniform" means that we cannot cut one in half and so time 30 mins.

However, we can time 30 mins by setting light to both ends of one rope at the same time and seeing how long it takes for the rope to burn through.

If we light both ends of one rope and one end of the second rope we can then light the second end of the second rope once the first rope has burnt out and the second rope will burn through after 45 mins.