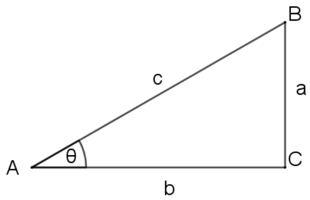
STEP Support Programme

Hints and Partial Solutions for Assignment 5

Warm-up

1 **(i)** As always, a diagram might be useful:



$\cos\theta = \frac{b}{c}$ and $\sin\theta = \frac{a}{c}$, so (using Pythagoras' Theorem),

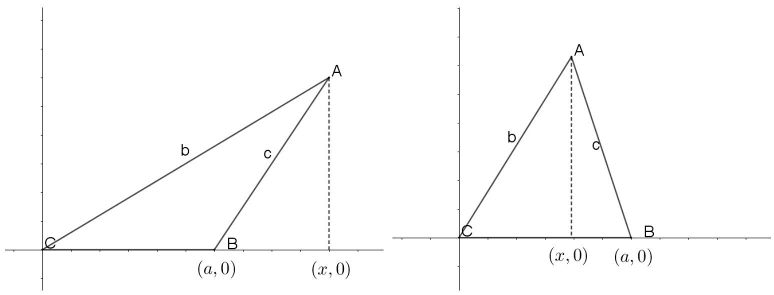
$$\cos^2\theta + \sin^2\theta = \frac{b^2+a^2}{c^2} = \frac{c^2}{c^2} = 1$$

Remember that although $\cos^2\theta + \sin^2\theta = 1$, it is **not** generally the case that $\cos\theta + \sin\theta = 1$. For what values of $\cos\theta$ does this equality hold?

As mentioned previously, in general $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.

What can you say about a and b if $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$?

(ii) There are two possible diagrams, depending on whether you assume $x > a$ or $x < a$:



Make sure that you don't assume too much: in this question, it is OK to assume that $x > a$ (or vice versa); but in other situations you might have to consider the two cases separately. You certainly can't assume that $x = 0$ (so that A is on the y-axis) because that would be restricting yourself to a right-angled triangle.

In both cases we have $b^2 = x^2 + y^2$.

When $x > a$, $c^2 = (x-a)^2 + y^2$, and when $x < a$, $c^2 = (a-x)^2 + y^2$. Since $(a-x)^2 = (x-a)^2$, the two expressions are equivalent.

Eliminating y^2,

$$\begin{aligned} & b^2 - x^2 = c^2 - (x-a)^2 \\ \Rightarrow\ & b^2 - x^2 = c^2 - x^2 + 2ax - a^2 \\ \Rightarrow\ & b^2 + a^2 = c^2 + 2ax \end{aligned}$$

Since $x = b\cos C$, $c^2 = a^2 + b^2 - 2ab\cos C$.

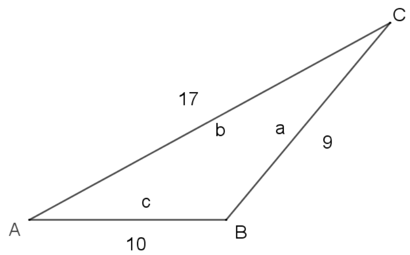
Preparation

2 **(i)** Note that the bit in italics said *leave your answers as fractions*. This implies that you should **not** be working with decimals (even if you convert your final answer into a fraction — your exact answers are unconvincing if approximate decimals were used along the way).

There is actually no need to find angle C, you can get $\sin C$ from the value of $\cos C$ and the result you demonstrated in question **1(i)**.

To find the three altitudes, use the formula for the area of the triangle — $\frac{1}{2}bh$ — using AB, BC and CA as the base in turn.

Note also that we ask for the *value*, not the *exact value*; we don't have to use the word *exact* (even though we want the exact value) because asking for the *value* is unambiguous (the *value* means the actual value, not an approximation!).



Using the cosine rule, $100 = 81 + 289 - 2 \times 9 \times 17 \cos C$, so $\cos C = \frac{270}{2\times 9\times 17} = \frac{15}{17}$.

Since $\sin^2 C + \cos^2 C = 1$, and we know that $\sin C$ will be positive as $0 \leqslant C \leqslant \pi$,we have $\sin C = \frac{8}{17}$.

The area of the triangle is $\frac{1}{2}ab\sin C = \frac{1}{2}(9)(17)\frac{8}{17} = 36$.

Since twice the area of the triangle is equal to the base multiplied by the altitude, each altitude must be twice the area (72) divided by each base, so the altitudes are $\frac{36}{5}$, 8 and $\frac{72}{17}$.

(ii) This part provides a simple example of the method required for the main part of the STEP question, i.e. using the volume of the pyramid to find the height.

The fourth vertex of the base is $(4, 6, 0)$, so the rectangular base has an area of 24. Volume of a pyramid is given by $\frac{1}{3} \times$ base area $\times$ height, so $\frac{1}{3} \times 24 \times$ height $= 40$, so the height is 5 and the apex is at $(2, 3, 5)$.

(iii) It is probably easiest to start by simplifying $1 - \dfrac{1}{1+x^2}$ (by which we mean write it as a single fraction).

$$\begin{aligned}\left(1-\frac{1}{1+x^2}\right)^{\frac{1}{2}}\sqrt{1+x^2} &= \left(\frac{1+x^2-1}{1+x^2}\right)^{\frac{1}{2}}\sqrt{1+x^2}\\ &= \left(\frac{x^2}{1+x^2}\right)^{\frac{1}{2}}\sqrt{1+x^2}\\ &= \frac{x}{\sqrt{1+x^2}}\sqrt{1+x^2}\\ &= x\end{aligned}$$

By definition $\sqrt{x}$ is positive or zero (i.e. it is non-negative). So $\sqrt{9}=3$ and not -3, and $\sqrt{(-5)^2}=5$. This means that $\sqrt{(x^2)}=x$ if x is positive but $\sqrt{(x^2)}=-x$ if x is negative. We can put these two cases together if we write $\sqrt{(x^2)}=|x|$.

The STEP question (2006 STEP I Q8)

3 A *tetrahedron* is a triangular based pyramid, i.e. a three dimensional shape with 4 faces, each of which is a triangle. A *regular* tetrahedron is one where all the faces are the same and are regular polygons — so for a regular tetrahedron all the faces are equilateral triangles.

(i) If you take the x, y plane to be where the base of the tetrahedron is then the area of the base is $\frac{1}{2}ab$, so the volume is $\frac{1}{6}abc$.

(ii) You must be careful here not to use a for two different things. In the question, a is defined to be the length OA. In the Cosine Rule you usually use a for the length BC but as a has already been used you have to choose a different letter for BC (or just call it BC). Writing the Cosine Rule as $AB^2 = BC^2 + AC^2 - 2 \times BC \times AC \times \cos\theta$ will help to avoid confusion. You can then use $BC = \sqrt{b^2 + c^2}$ etc.

The Cosine Rule gives us:

$$(a^2 + b^2) = (b^2 + c^2) + (a^2 + c^2) - 2\sqrt{b^2 + c^2}\sqrt{a^2 + c^2}\cos\theta$$

which rearranges to give

$$\cos\theta = \frac{b^2 + c^2 + a^2 + c^2 - a^2 - b^2}{2\sqrt{b^2 + c^2}\sqrt{a^2 + c^2}}$$

which simplifies to the required result for $\cos\theta$.

Once you have found $\cos\theta$ you can find $\sin\theta$ using the result from question **1(i)**:

$$\begin{aligned}
\sin\theta &= \sqrt{1 - \cos^2\theta} \\
&= \sqrt{1 - \frac{c^4}{(a^2 + c^2)(b^2 + c^2)}} \\
&= \sqrt{\frac{a^2b^2 + b^2c^2 + a^2c^2 + c^4 - c^4}{(a^2 + c^2)(b^2 + c^2)}} \\
&= \sqrt{\frac{a^2b^2 + b^2c^2 + a^2c^2}{(a^2 + c^2)(b^2 + c^2)}}
\end{aligned}$$

Using the Sine formula for the area of triangle ABC gives:

$$\frac{1}{2} \times BC \times AC \sin C = \frac{1}{2} \times \sqrt{a^2 + c^2}\sqrt{b^2 + c^2}\sqrt{\frac{a^2b^2 + b^2c^2 + a^2c^2}{(a^2 + c^2)(b^2 + c^2)}}$$

which simplifies to $\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}$.

Using the volume we have:

$$\begin{aligned}
&\frac{1}{3} \times d \times \left(\frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}\right) = \frac{1}{6}abc \\
\Rightarrow\ &\frac{d\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}{6} = \frac{abc}{6} \\
\Rightarrow\ &d\sqrt{a^2b^2 + b^2c^2 + c^2a^2} = abc \\
\Rightarrow\ &d^2(a^2b^2 + b^2c^2 + c^2a^2) = a^2b^2c^2 \\
\Rightarrow\ &\frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{d^2}
\end{aligned}$$

Warm down

4 **(i)** Removing 2 socks does not ensure a pair (I might take out RB); 3 socks are enough to ensure a pair. The simplest way to show that 3 socks are enough is to list all the possibilities — RRR, RRB, RBB, BBB.

Note that order is not important, so RBB is the same as BRB etc.

(ii) 4 is not necessarily enough as I might take out $RRRB$. However, 5 is always enough; the first three socks I take out guarantee a pair, from part **(i)**. So I now have a pair of socks and a third leftover sock. The next two socks are either both the same, giving me a second pair, or one of each colour, in which case one will definitely match the third leftover sock.

(iii) Looking at the answers to the previous two parts would suggest that you need $2n + 1$ socks, but you cannot just extrapolate from a couple of results — you need to justify why this is the answer in the general case.

$2n$ socks would be enough if there are an even number of blue socks and an even number of red socks.

But if there are an odd number of red socks and an odd number of blue socks $2n$ socks are not enough. You can show this by letting the number of red socks you pick be $2r+1$ and the number of blue socks be $2b+1$, in which case you have only $n-1$ pairs. But you do have a spare red and a spare blue sock, so you can then consider what happens when you add on one more sock.

A justification on the above lines would be acceptable (with the gaps filled in) but there are many other ways of thinking about it.