STEP Support Programme

Hints and Partial Solutions for Assignment 6

Warm-up

1 **(i)** It is far, far easier to cancel the fractions before doing any multiplication (in fact if you cancel completely no multiplication is necessary). Start by writing each bracket as a single fraction.

$$\frac{(1+\frac{1}{2})(1+\frac{1}{4})(1+\frac{1}{6})(1+\frac{1}{8})}{(1-\frac{1}{2})(1-\frac{1}{4})(1-\frac{1}{6})(1-\frac{1}{8})} = \frac{\frac{3}{2}\times\frac{5}{4}\times\frac{7}{6}\times\frac{9}{8}}{\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\frac{7}{8}} = \frac{3\times5\times7\times9}{1\times3\times5\times7} = 9$$

The second part needs you to cancel out all the common factors and appreciate which ones are left — the first part should act as a guide.

Note: You should make sure your cancellations do not obliterate the underlying number — you might like to cancel in pencil.

Whilst working with some special cases if you are stuck is a good idea, you also need to show how the general case works (and not just by generalising from the special cases). This requires something like:

$$\frac{\frac{3}{2}\times\frac{5}{4}\times\cdots\times\frac{2n-1}{2(n-1)}\times\frac{2n+1}{2n}}{\frac{1}{2}\times\frac{3}{4}\times\frac{5}{6}\times\cdots\times\frac{2n-1}{2n}} = \frac{3\times5\times\cdots\times(2n-1)\times(2n+1)}{1\times3\times\cdots\times(2n-3)\times(2n-1)} = 2n+1$$

(ii) You can use matrices to solve these equations, but this is not the easiest method and it is very easy to make mistakes when finding the inverse matrix.

I personally can never remember the inverse of a 3 by 3 matrix in terms of co-factors etc — I tend to Gauss-Jordan elimination[1] if I have to find an inverse (and this method has the advantage that it can be used with any square matrix).

However, I always consider the alternatives before messing about with matrices as there is far too much room for error for my liking.

The easiest way for the first set of equations is to add pairs of equations, in each case two unknowns will cancel out and you can find the third one. It is quick to check whether your three values satisfy the three original equations, so do it!

Adding together $a+b-c=2$ and $a-b+c=0$ gives $2a=2 \Rightarrow a=1$.

Adding together $a-b+c=0$ and $-a+b+c=8$ gives $2c=8 \Rightarrow c=4$.

Adding together $a+b-c=2$ and $-a+b+c=8$ gives $2b=10 \Rightarrow b=5$

For the second set of equations, adding the second and third equations gives $c=4$ as above.

Adding the first and second: $(k+1)a=2 \Rightarrow a=\frac{2}{k+1}$

[1] If you are interested this website has some nice examples

Substituting for a and c in the second equation then gives $\frac{2}{k+1} - b + 4 = 0 \Rightarrow b = \frac{2}{k+1} + 4$.

When you have a in terms of k you should be able to see that $k = -1$ will be problematic (and if you add the first two equations after setting $k = -1$ you get $0 = 2$ which shows that in this case there is no solution to the equations). When $k = -1$ the first two equations are *inconsistent* i.e. they cannot both be true at the same time.

Preparation

2 In general, if you have n objects where r_1 are identical, another r_2 are identical (but different to the first lot), and so on until the last r_k identical objects then there are:

$$\frac{n!}{r_1! \times r_2! \times \cdots \times r_k!}$$

distinct ways of arranging the objects.

You could have left your answers in terms of factorials here, but do be aware that A-levels may require you to find the actual number (it is probably safest to do both unless told otherwise).

(i) $3!(= 6)$, $4!(= 24)$ and $7!(= 5040)$.

(ii) $\frac{4!}{2!} = 12$, $\frac{5!}{2!} = 60$, $\frac{7!}{2!} = 2520$, $\frac{5!}{3!} = 20$ and $\frac{8!}{4!} = 1680$.

(iii) $\frac{6!}{2! \times 2!}$, $\frac{6!}{2! \times 2! \times 2!}$ and $\frac{11!}{2! \times 2! \times 2!}$.

(iv) The best way to start is by considering the letters $L_1, I_1, L_2, L_3, I_2, A, N$. "Show carefully" means that you must fully explain your argument and leave no gaps for an examiner (or other relevant person) to have to fill in. Your argument might look something like this:

There are 7! ways of arranging the seven letters $L_1, I_1, L_2, L_3, I_2, A, N$. If Lillian's name had seven *distinct* letters, there would be 7! arrangements. However, L_1, L_2 and L_3 can be arranged in 3! ways, and in Lillian's name they are indistinguishable, so we need to divide by 3!. Similarly, I_1 and I_2 can be arranged in 2! ways, so we also need to divide by 2!. Hence, the number of ways is $\frac{7!}{3! \times 2!}$

(v) $\frac{11!}{4! \times 4! \times 2!}$.

The STEP question (2005 STEP I Q1)

3 **(i)** As the answer is given, you need to show carefully that you have all the possible numbers and not just stop when you get to 15 — you need to show there are no more. The key is to be systematic, and it is probably easiest to consider the different cases that arise according to the numbers of nines.

With the number 99999 the digit sum is 45, so this is out. However we can have 4 nines and a seven. There are 5 places in which we can put the seven, so 5 different numbers here.

With 3 nines, the other two digits need to sum to 16 so they must be 2 eights. You can use $\dfrac{5!}{3! \times 2!}$ to show that there are 10 different numbers with 3 nines and 2 eights.

If we then consider 2 nines then the highest digit sum we can make is $9+9+8+8+8 = 42$ so 2 nines is not possible. Similarly for 1 nine and 0 nines.

You cannot just stop after getting 15 possibilities, you do need to explain why there are no more!

(ii) Again, start with 4 nines; this gives 36 so the remaining digit is a 3.
$3, 9, 9, 9, 9$ can be written in 5 ways.

With 3 nines, the remaining two digits sum to 12:
$6, 6, 9, 9, 9$ gives $\frac{5!}{3! \times 2!} = 10$ ways.
$5, 7, 9, 9, 9$ gives $\frac{5!}{3!} = 20$ ways.
$4, 8, 9, 9, 9$ also gives 20 ways.

(Making 12 with nine and three has already been covered in the 4 nines case).

With 2 nines, the remaining three digits sum to 21. We could use two eights: $5, 8, 8, 9, 9$ gives $\frac{5!}{2! \times 2!} = 30$ ways.
We could use one eight: $6, 7, 8, 9, 9$ gives $\frac{5!}{2!} = 60$ ways.
We could use no eights: $7, 7, 7, 9, 9$ gives $\frac{5!}{3! \times 2!} = 10$ ways.

With just one nine, the remaining digits sum to 30. We could use three eights: $6, 8, 8, 8, 9$ gives $\frac{5!}{3!} = 20$ ways.
We could use two eights: $7, 7, 8, 8, 9$ gives $\frac{5!}{2! \times 2!} = 30$ ways.

With no nines, there is only one possible set of digits, $7, 8, 8, 8, 8$ which can be arranged in 5 ways.

Therefore, the number of five-digit numbers whose digits sum to 39 is $5 + 10 + 20 + 20 + 30 + 60 + 10 + 20 + 30 + 5 = 210$.

Warm down

4 **(i)** **(a)** $\frac{18}{100}$

(b) $\frac{62}{100}$

(c) $\frac{3}{10}$

(d) One way of tackling this question is to use *Bayes' Theorem* (which has come in and out of different modules over the years):

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

So the probability that a person is a woman given that they are a smoker is given by:

$$P(W|S) = \frac{P(\text{Woman and a Smoker})}{P(\text{Smoker})} = \frac{0.18}{0.38}$$

However, you did not need to know Bayes' Theorem formally for this question (in fact we were assuming that you didn't know it).
If you assume a population of 100 people you find that out of these 100, 38 are smokers and of these smokers 18 are women (i.e. there are 18 women smokers). Then the probability if you are considering the 38 smokers that you pick is women is therefore $\frac{18}{38}$.

(e) $\frac{10}{31}$

(ii) This seems to be a very surprising result, given that (at first glance) the test appears to be very accurate.

It turns out that if you test positive you still have a less than 5% chance ($99/2097 = 0.04721$ to 4s.f.) of having the disease, which can be shown by either using Bayes' theorem or by considering a population as in part **(i)**.

If we have a population of $100,000$ people, then 100 have "Mathmotitus" and $99,900$ don't have "Mathmotitus". Out of the 100 with "Mathmotitus", 99 will test positive and 1 will test negative. Out of the $99,900$ who don't have the disease, $97,902$ will test negative and 1998 will test positive. Therefore the total number of people who test positive is $99 + 1998 = 2097$ and so the probability that someone who tests positive has "Mathmotitus" is $\dfrac{99}{2097}$.

Misunderstandings of probability can have huge implications, especially in criminal trials. The two probabilities P(Matching DNA evidence found | accused is innocent) and P(accused is innocent | matching DNA evidence found) can be very different, but it is the second one that we must consider!

For more on this visit https://en.wikipedia.org/wiki/Prosecutor%27s_fallacy.