STEP Support Programme

Hints and Partial Solutions for Assignment 7

Warm-up

You can check your graphs by plotting them using Desmos.

1 **(i)** **(a)** For $y = x + \dfrac{1}{x}$, there are two turning points. For $y = x - \dfrac{1}{x}$, the gradient is positive for all (non-zero) x so there are no stationary points (the graph heads "upwards" at each point).

(b) For both graphs, $y \to +\infty$ as $x \to +\infty$ and $y \to -\infty$ as $x \to -\infty$, but we can do a little better than that. As $x \to \pm\infty$, the $\pm\frac{1}{x}$ term becomes negligible, so we can say that $y \to x$ as $x \to \pm\infty$, in both cases.

To describe this behaviour, we say that $y = x$ is an *asymptote* — which means that the gap between the graph and the line $y = x$ approaches zero as $x \to \pm\infty$. It is good to draw the asymptotic line on the graph, and we usually draw it as a dotted line.

(c) The behaviour as $x \to 0$ depends on whether x is positive or negative. Consider the first graph:

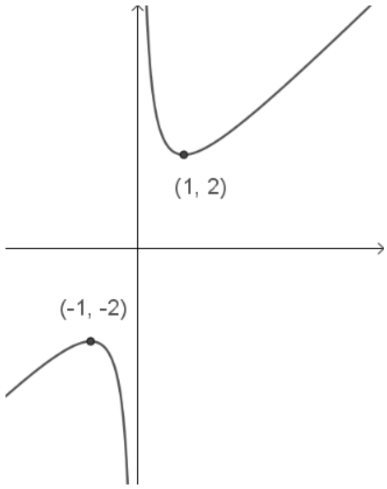
$$y = x + \frac{1}{x}\,.$$

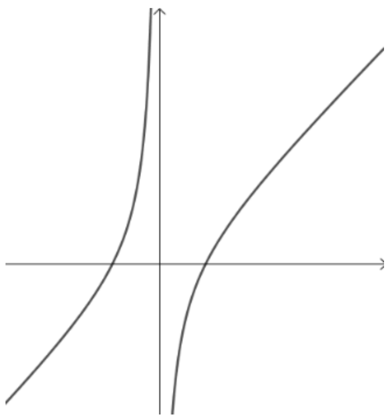
As x tends to 0 from the positive direction, y tends to $+\infty$ (as $x + \frac{1}{x} > 0$ when $x > 0$), but as x tends to 0 from the negative direction, y tends to $-\infty$. We can write this very conveniently as:

$$\text{as } x \to 0_+,\; y \to +\infty \text{ and as } x \to 0_-,\; y \to -\infty\ .$$

(d) Neither graph intersects the y-axis (which would require $x = 0$). For the second graph $y = 0$ when $x = \pm 1$, so it intersects the x-axis at these values of x. For the first graph, there are no intersections with the x-axis since there are no (real) values of x that satisfy the equation $y = 0$.

Here are the graphs:





(ii) **(a)** Multiplying throughout by x is a **bad** idea: when x is negative the inequality sign will be reversed, which makes it rather complicated. Instead draw the line $y = 2$ onto your graph of $y = x + \dfrac{1}{x}$. You can solve $x + \dfrac{1}{x} = 2$ if you want, but you should be able to see that the line $y = 2$ intersects the curve $y = x + \dfrac{1}{x}$ at $(1, 2)$ without doing this.

Since the inequality is strict, you must be careful to exclude $x = 1$ (which gives equality) from your set of values. You can write "$0 < x < 1$ or $x > 1$" or "$x > 0$, $x \neq 1$" (or something else equivalent).

(b) Replace the inequality sign with an equal sign and solve the resulting equation, which can be written as $2x^2 - 3x - 2 = 0$. This has solutions $x = -\frac{1}{2}$ and $x = 2$. Then sketch the line $y = \frac{3}{2}$ onto your graph to see where the inequality holds, leading to $-\frac{1}{2} \leqslant x < 0$ or $x \geqslant 2$.

Preparation

2 **(i)**

$$
\begin{aligned}
&(\alpha^2 + b\alpha + c) - (\beta^2 + b\beta + c) = 0\\
&\Rightarrow \alpha^2 + b\alpha = \beta^2 + b\beta\\
&\Rightarrow \alpha^2 - \beta^2 = b(\beta - \alpha)\\
&\Rightarrow (\alpha - \beta)(\alpha + \beta) = -b(\alpha - \beta)
\end{aligned}
$$

Since $\alpha \neq \beta$, we can divide through by $(\alpha - \beta)$, so $-b = (\alpha + \beta) \Rightarrow b = -(\alpha + \beta)$.
If $\alpha \neq \beta$ were not given, then you would need to consider the case $\alpha - \beta = 0$ separately.

Substituting b back into one of the equations gives $c = \alpha\beta$.

We then have $(x - \alpha)(x - \beta) \equiv x^2 - (\alpha + \beta)x + \alpha\beta \equiv x^2 + bx + c$.

(ii) This part is working in the opposite direction to part **(i)**, i.e. it is starting with $x^2 + bx + c \equiv (x - \alpha)(x - \beta)$.

Let $x^2 + bx + c \equiv (x - \alpha)(x - \beta)$. When $x = 0$, $0^2 + 0b + c = (0 - \alpha)(0 - \beta)$, so $c = \alpha\beta$.
When $x = 1$, $1 + b + c = (1 - \alpha)(1 - \beta)$.
Substituting and expanding gives $1 + b + \alpha\beta = 1 - \alpha - \beta + \alpha\beta$ so $b = -(\alpha + \beta)$.

(iii) $\alpha = 2, \beta = 5$ or the reverse.

(iv) Comparing the given result and what you are trying to show should suggest that substituting $x = 0, 1$ and -1 is probably a good idea:

$x = 0 : d = (0 - \alpha)(0 - \beta)(0 - \gamma) = -\alpha\beta\gamma$, so $-d = \alpha\beta\gamma$.
$x = 1 : 1 + b + c + d = (1 - \alpha)(1 - \beta)(1 - \gamma)$.
$x = -1 : -1 + b - c + d = (-1 - \alpha)(-1 - \beta)(-1 - \gamma)$, which simplifies to give $1 - b + c - d = (1 + \alpha)(1 + \beta)(1 + \gamma)$.

(v) Since $(1 + \alpha)(1 + \beta)(1 + \gamma) = -15$, we need to consider the factors of 15. This gives $(1 + \alpha) = \pm1, \pm3, \pm5$, or ±15 so $\alpha = 0, -2, 2, -4, 4, -6, 14$, or -16.

Since $\alpha\beta\gamma = -16$ we can rule out $0, -6$ and 14.

You also have $(1 - \alpha)(1 - \beta)(1 - \gamma) = 9$, which means that the only possible values of α are $-2, 2$ and 4.

The final answer is $\alpha = -2, \beta = 2, \gamma = 4$.

The STEP question (2002 STEP I Q5)

3 $x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = (x + k_1)(x + k_2)\cdots(x + k_n)$.

When $x = 0, 0 + 0 + 0 + \cdots + 0 + a_n = k_1k_2\cdots k_n$, so $a_n = k_1k_2\cdots k_n$.

When $x = 1, 1 + a_1 + a_2 + \cdots + a_n = (1 + k_1)(1 + k_2)\cdots(1 + k_n)$.

When $x = -1, (-1)^n + a_1(-1)^{n-1} + \cdots + a_n = (k_1 - 1)(k_2 - 1)\cdots(k_n - 1)$.

Looking at specific cases (such as $n = 3$ and $n = 4$) can help you generalise.

To find the roots of $x^4 + 22x^3 + 172x^2 + 552x + 576 = 0$, let the roots be $-k_1, -k_2, -k_3$ and $-k_4$. Then we have:

- $k_1k_2k_3k_4 = 576$
- $(k_1 + 1)(k_2 + 1)(k_3 + 1)(k_4 + 1) = 1 + 22 + 172 + 552 + 576 = 1323$
- $(k_1 - 1)(k_2 - 1)(k_3 - 1)(k_4 - 1) = 1 - 22 + 172 - 552 + 576 = 175$

To find the roots, it is easiest to start with $(k_1-1)(k_2-1)(k_3-1)(k_4-1) = 175 = 1\times5\times5\times7$. There are 12 different possibilities for $k_1 - 1$ (and hence for k_1):

$(k_1 - 1) = \pm1, \pm5, \pm7, \pm25, \pm35$, or ±175, so
$k_1 = 0, 2, 6, -4, 8, -6, 26, -24, 36, -34, 176$ or -174.

We also have $k_1k_2k_3k_4 = 576 = 1 \times 2^6 \times 3^2$. This means that k_1 can only have factors of $1, 2$ and 3, and cannot be 0. We can use this to eliminate 5 values of k_1, leaving $2, 6, -4, 8, -6, -24, 36$ as possible values.

We can then use $(k_1 + 1)(k_2 + 1)(k_3 + 1)(k_4 + 1) = 1323 = 3^3 \times 7^2$ to eliminate everything but 4 values (which are $-4, 2, 6, 8$).

Now note that we must have either 0, 2 or 4 k values equal to -4 (as $k_1k_2k_3k_4$ is positive). We cannot have all four k values equal to -4 (as we need some factors of 3), so consider what might happen if two of them are equal to -4. This would then mean we need $(k_3 - 1)(k_4 - 1) = 1 \times 7$ and $(k_3 + 1)(k_4 + 1) = 3 \times 7^2$. The first of these means that we would need $k_3 = 2, k_4 = 8$, but these then doesn't satisfy the second equation. Hence none of the values of k_i are equal to -4.

Since $k_1k_2k_3k_4 = 2^6 \times 3^2$ and the possible values of k_i are $2, 6, 8$ we must have two values equal to 6. The other two values must multiply to give 2^4 which suggests that the other two values are 2 and 8.

It is a very good idea to check your answer by expanding the brackets and show that this is the given quartic.

Please note that the question asks for the **roots** so the answer is **not** $2, 6, 6, 8$ (which are the values of k_i), but $-2, -6$ (repeated root) and -8.

Warm down

4 This problem is known as "Bachet's Weights Problem", after Claude Gaspard Bachet de Méziriac (1581-1638) who published it in 1624, but it is thought to date back to Fibonacci in 1202, making it one of the earliest problems in integer partitions.

(i) **(a)** If we are only putting weights on one side, then we obviously need a weight of 1oz, and we need either another 1oz or a 2oz to make 2oz. With the 2oz we can make 1oz, 2oz and 3oz (whereas if we had picked a second 1oz we would only be able to make 1oz or 2oz, and no matter what the third weight is, we could only make 5 weights — 1oz, 2oz, n oz, $(n+1)$oz and $(n+2)$oz).
If we have 1oz and 2oz then to make a 4oz weight we must have a 4oz or 3oz weight. Only the 4oz weight will enable us to make 1oz, 2oz, 3oz, 4oz, 5oz, 6oz and 7oz.

(b) 1oz, 2oz, 4oz, 8oz and 16oz.

(c) Each weight is either in the pan or not, so for each weight there are two options and with n weights there are $2 \times 2 \times \cdots \times 2 = 2^n$ options in total. Note that the weights have to be carefully picked if you are going to achieve 2^n *different* weights; for example, if you chose 1oz, 2oz, 3oz and 6oz, then two different combinations, $(1+2+3)$oz or 6oz, give the *same* weight.
There a connection with binary numbers here — the number 10 can be written as 1010, which is one 8, no 4's, one 2 and no 1's. The number 31 is 11111 — i.e. one 16, one 8, one 4, one 2 and one 1. The 1's and 0's correspond to whether a weight is in the pan or not.
If you have the weights $1, 2, 4, \cdots, 2^{n-1}$ then you can make all the numbers up to $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$.

(ii) **(a)** If we have the weights 1oz and 3oz, then we can obviously weigh 1oz and 3oz, if we put the two together in the same pan then we can weigh 4oz and if they are in different pans then we can weigh the difference; i.e. 2oz.

(b) There are **three** options for each weight this time: left pan, right pan or no pan, so there are 3^n different arrangements of the weights.
However we **cannot** actually weigh 3^n **different** weights.
To find the total number of distinct non-zero weights we can weigh, start by considering the 3^n possible distribution of our weights in the two pans. Take off 1 for the case where no weights are in either pan, which leaves us with $3^n - 1$ distributions. Then for each distribution, there is a "mirror image" where the weights swap pans. Both a given distribution and its mirror distribution enable you to weigh the same amount, so we need to divide by 2 giving at most $\dfrac{3^n - 1}{2}$ distinct weights.

(c) Guided by the previous parts, we try 1, 3, 9 and 27 oz, which works (and is in fact the only possibility with four weights).
Bachet proposed that the weights of $1, 3, 3^2, \cdots, 3^n - 1$ ounces enable one to weigh any number of weights from 1oz to $(1+3+3^2+\cdots+3^{n-1}) = \frac{1}{2}(3^n - 1)$ but he did not prove that this was the least possible number of weights. Major Percy MacMahon (1854-1929) proved this and that this solution is unique (so, for example, there is no other set of 4 weights that can weigh all integer weights from 1oz to 40oz).