STEP Support Programme

Hints and Partial Solutions for Assignment 8

Warm-up

1 **When you are asked to prove something it is best not to start with what you are trying to prove**. The question states "by considering $(x-y)^2$" so the first line of your proof should be $(x-y)^2 \geqslant 0$.

Working backwards can be OK if the implication works both ways each time (so that you can put a $\iff$ in between each line of working). If you are working backwards then you do need to put the $\iff$ signs in, for example:

$$\begin{aligned} & x^2+y^2 \geqslant 2xy \\ \iff & x^2+y^2-2xy \geqslant 0 \\ \iff & (x-y)^2 \geqslant 0 \end{aligned}$$

However, this is not as elegant as "doing it the right way around", i.e. making your first statement $(x-y)^2 \geqslant 0$:

$$\begin{aligned} & (x-y)^2 \geqslant 0 \\ \iff & (x^2-2xy+y^2) \geqslant 0 \\ \iff & x^2+y^2 \geqslant 2xy \end{aligned}$$

Equality holds when $(x-y)^2 = 0$ (i.e. when $x = y$).

Once you have shown that $x^2+y^2 \geqslant 2xy$ you can use the substitution $a = x^2$ and $b = y^2$, which is allowed as both a and b are non-negative:

Let $a = x^2$ and $b = y^2$. Then $a+b \geqslant 2\sqrt{ab}$, so

$$\frac{a+b}{2} \geqslant \sqrt{ab}. \qquad (*)$$

Equality holds when $a = b$.

(i) Again, you should not start with the result you are trying to prove. Multiplying by 2 should give you a hint for how you might approach this and hopefully suggests that could start by considering $(x-y)^2+(y-z)^2+(z-x)^2$. Since this is the sum of non-negative terms, we know that $(x-y)^2+(y-z)^2+(z-x)^2 \geqslant 0$.

$$\begin{aligned}(x-y)^2+(y-z)^2+(z-x)^2 &\geqslant 0\\ \Rightarrow x^2-2xy+y^2+y^2-2yz+z^2+z^2-2zx+x^2 &\geqslant 0\\ \Rightarrow 2x^2+2y^2+2z^2 &\geqslant 2xy+2yz+2zx\\ \Rightarrow x^2+y^2+z^2 &\geqslant xy+yx+zx\end{aligned}$$

(ii) Here you use AM-GM (the starred equation in introduction to the question) three times: twice to obtain the first inequality then once to obtain the second inequality.

Since $\frac{p+q}{2} \geqslant \sqrt{pq}$ and $\frac{r+s}{2} \geqslant \sqrt{rs}$, by $(*)$,

$$\frac{p+q+r+s}{4} = \frac{\frac{p+q}{2}+\frac{r+s}{2}}{2} \geqslant \frac{\sqrt{pq}+\sqrt{rs}}{2}$$

Applying $(*)$ again, with $a=\sqrt{pq}$ and $b=\sqrt{rs}$ gives

$$\frac{\sqrt{pq}+\sqrt{rs}}{2} \geqslant \sqrt{\sqrt{pqrs}} = \sqrt[4]{pqrs}$$

(iii) This was rather tricky. The first part is not so bad — use the result in part **(ii)** with $s = \dfrac{p+q+r}{3}$.

$$\begin{aligned}\frac{p+q+r}{3} &= \frac{p+q+r}{4}+\frac{p+q+r}{12}\\ &= \frac{p+q+r+\frac{(p+q+r)}{3}}{4}\\ &\geqslant \sqrt[4]{pqr\left(\frac{p+q+r}{3}\right)}\end{aligned}$$

Let $z=\frac{p+q+r}{3}$. Then the inequality can be written

$$z \geqslant \sqrt[4]{pqrz}.$$

Raising both sides to the power 4 gives

$$z^4 \geqslant pqrz \Rightarrow z^3 \geqslant pqr.$$

Finally, taking the cube root of both sides and substituting $\frac{p+q+r}{3}$ back in for z gives

$$\frac{p+q+r}{3} \geqslant \sqrt[3]{pqr}$$

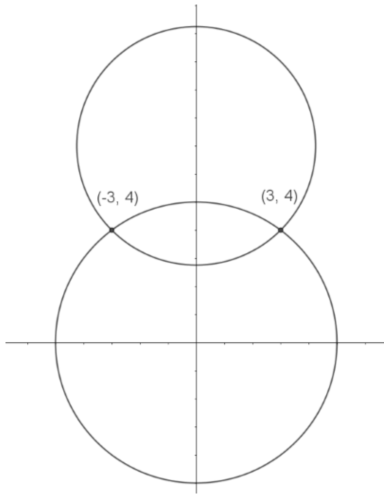
as required.

Preparation

2 **(i)** The midpoint of the line is at $(-2, 9)$. The diameter is of length $\sqrt{6^2 + 8^2} = 10$, so the radius is 5. Hence the equation of the circle is $(x + 2)^2 + (y - 9)^2 = 25$.

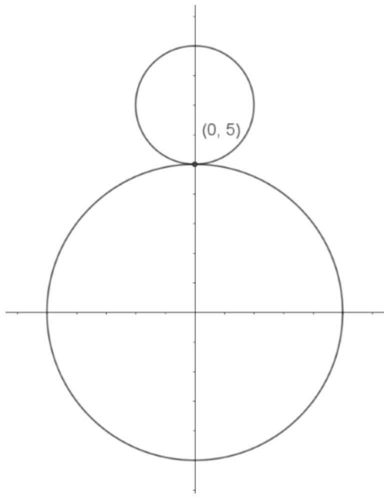
(ii) **(a)** $x^2 + y^2 = 25$ and $x^2 + (y - 7)^2 = 18$
Substituting $x^2 = 25 - y^2$ and simplifying gives $56 - 14y = 0 \Rightarrow y = 4$.

Therefore $x^2 = 9, x = \pm 3$, and so there are two intersections between the circles, at $(\pm 3, 4)$

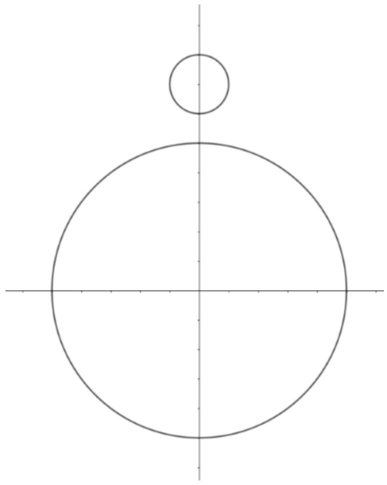


(b) $x^2+y^2 = 25$ and $x^2+(y-7)^2 = 4$ This time, simplifying gives $70-14y = 0 \Rightarrow y = 5$.

Therefore $x^2 = 0$ and so there is one intersection at $(0, 5)$.



(c) $x^2 + y^2 = 25$ and $x^2 + (y - 7)^2 = 1$ This time, $73 - 14y = 0 \Rightarrow y = \frac{73}{14}$ which is greater than 5, so there are no real solutions and no intersection.



(iii) You can do this by pure geometry — the two points of intersection are symmetric about the y-axis and since the centre of the circle must lie on the perpendicular bisector of any chord the centre must be on the y-axis.

You can instead use coordinate geometry. For this second approach substitute each of the two pairs of coordinates of the points of intersection (i.e. $x = \pm 3$ and $y = 4$) into $(x-a)^2 + (y-b)^2 = r^2$ which gives:

$$(3-a)^2 + (4-b)^2 = r^2 \qquad \text{and}$$
$$(-3-a)^2 + (4-b)^2 = r^2$$

Subtracting these gives $(3-a)^2 - (-3-a)^2 = 0 \implies a = 0$.

If the centre is at $(0, 2)$ and the circle passes through $(3, 4)$ (which we know as we know is passes through $(\pm 3, 4)$) then $x^2 + (y-2)^2 = 13$.

(iv) Rearranging the first equation and substituting $x^2 = 4 - y^2$ in the second equation gives

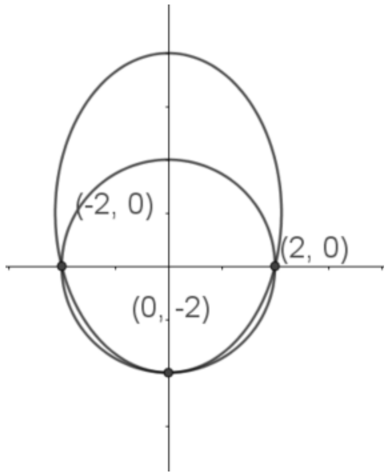
$$32 - 8y^2 + 4y^2 - 8y + 4 = 36$$

which simplifies to $4y(y+2) = 0$, so $y = 0$ or $y = -2$.

When $y = 0, x^2 = 4$ so $x = \pm 2$.

When $y = -2, x^2 = 0$ so $x = 0$.

Here we have a circle and ellipse (you don't have to know anything about ellipses) meeting in three different places, $(\pm 2, 0)$ and $(0, -2)$. At one of the intersections the two curves are touching, i.e. their gradients are the same at this point (for the other two points the curves are crossing).



Try using Desmos to sketch the curves.

The STEP question

3 Eliminate y first and then solve the resulting quadratic equation in x. You should find that one of your values of x is possible, but the other (the larger one) gives imaginary values for y. You can use the position of the points of intersection to show that the centre of the circle lies on the x axis, so at $(a, 0)$ say (and you can do this algebraically or by using a geometrical argument). You can then write the equation of the circle as $(x-a)^2 + y^2 = r^2$ and use the distance from $(a, 0)$ to one of your points of intersection to write r^2 in terms of a.

If the answer is given then you **must** ensure that there are no gaps in your argument.

Worked solution:

$(x+2)^2 + 2y^2 = 18 \Rightarrow 8(x+2)^2 + 16y^2 = 144$.

Substituting $16y^2 = 144 - 8(x+2)^2$ in $9(x-1)^2 + 16y^2 = 25$ gives

$$9(x-1)^2 + 144 - 8(x+2)^2 = 25$$

Expanding and rearranging:

$$\begin{aligned}9x^2 - 18x + 9 + 144 - 8x^2 - 32x - 32 - 25 = 0\\ \Rightarrow x^2 - 50x + 96 = 0\\ \Rightarrow (x-48)(x-2) = 0\end{aligned}$$

When $x = 48$, $50^2 + 2y^2 = 18 \Rightarrow y^2 = 9 - \frac{50^2}{2} < 0$ so there is no real solution for y when $x = 48$.

When $x = 2$, $4^2 + 2y^2 = 18 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$.

Hence, the ellipses intersect at $(2, 1)$ and $(2, -1)$.

Note that at no point did we have to know what an ellipse actually is!

Any circle which passes through $(2, 1)$ and $(2, -1)$ must have its centre on the x axis, since the x axis is the perpendicular bisector of the chord joining $(2, 1)$ and $(2, -1)$. Let the circle's centre be at the point $(a, 0)$. Then the circle's equation is $(x-a)^2 + y^2 = r^2$. The radius of the circle is the length of the line joining $(a, 0)$ to $(2, 1)$:

$$r^2 = (a-2)^2 + 1^2$$

Hence, the equation of the circle is

$$\begin{aligned}(x-a)^2 + y^2 = (a-2)^2 + 1\\ \Rightarrow x^2 - 2ax + a^2 + y^2 = a^2 - 4a + 4 + 1\\ \Rightarrow x^2 - 2ax + y^2 = 5 - 4a\end{aligned}$$

Warm down

4 **(i)** 3 socks might not be enough (as you can have 1 red, 1 blue and 1 green). But then as the next sock you take must be red, blue or green you will have a matching pair if you take 4 socks.

(ii) As before, with 3 socks you can have 1 red, 1 blue and 1 green. Suppose that the next sock is red (it doesn't matter what colour it is) so with 4 socks you now have 1 pair of (red) socks. If the fifth sock is blue or green you now have 2 pairs and are finished, however the fifth sock may be red again so 5 socks are not enough to ensure that you have two pairs. In this case, the sixth sock will be either red, green or blue and hence match up with one of the other socks.

(iii) You could argue as follows, perhaps putting in a little more detail.

Suppose you have $2n$ socks. Since $2n$ is even, there are two possibilities:

(1) either you have an even number of each colour;

(2) or you have an even number of one colour (green, say) and an odd number of red socks and odd number of blue socks.

Why can you not have an odd number of each colour?

In case **(1)**, you have n pairs, so you are done.

In case **(2)**, you have $n-1$ pairs and two left-over socks, one red and one blue. Now if you pick one more sock, it may be green, in which case you still wouldn't have n pairs. So you must pick two more and this gives n pairs whatever colour they are.