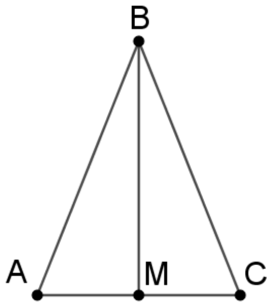
STEP Support Programme

Hints and Partial Solutions for Assignment 9

Warm-up

1 **(i)** When showing that two triangles are congruent, you must fully justify your statements.

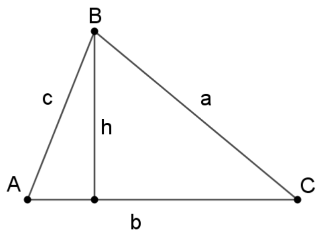
Don't assume things that are not given, especially not if it is the thing you are supposed to be proving! In this case, do not assume that the triangle has two equal angles: the definition of an isosceles triangle is that is has two equal *sides* (and you are supposed to **prove** that it has two equal angles).



Since M is the midpoint, $AM = MC$. It is given that $AB = BC$. BM is common to both triangles. Therefore, the triangles BAM and BMC are congruent (SSS), so $\angle AMB = \angle BMC, \angle ABM = \angle MBC$, and $\angle BAC = \angle BCA$.

As AC is a straight line, the equal angles $\angle AMB$ and $\angle BMC$ must be right angles, so AC and MB are perpendicular.

(ii) A diagram is always helpful:



The height of the triangle is given by $h = a \sin C$ or $c \sin A$.

The area, $\frac{1}{2}bh$ can be written as $\frac{1}{2}ba \sin C$ or $\frac{1}{2}bc \sin A$.

Therefore, $\frac{1}{2}ba \sin C = \frac{1}{2}bc \sin A \Rightarrow a \sin C = c \sin A$.

Having proved that $\dfrac{\sin A}{a} = \dfrac{\sin C}{c}$, you don't have to start all over again to show that $\dfrac{\sin C}{c} = \dfrac{\sin B}{b}$; you can state that this second result follows by relabelling (i.e. swapping the letters around) or by symmetry.

(iii) Considering triangle BMA (which is right-angled by part **(i)**) we have $BM = \cos\alpha$ and $AM = \sin\alpha$ (so $AC = 2\sin\alpha$).

Using the sine rule we have $\dfrac{\sin B}{b} = \dfrac{\sin A}{a}$ which gives $\dfrac{\sin 2\alpha}{AC} = \dfrac{\sin(90^\circ - \alpha)}{1}$.

Since $\sin(90^\circ - \alpha) = \cos\alpha$, $\dfrac{\sin 2\alpha}{2\sin\alpha} = \cos\alpha$.

Therefore, $\sin(2\alpha) = 2\sin\alpha\cos\alpha$.

Preparation

2 **(i)** As a general rule, **do not** divide by something that might be zero. This is quite a simple example, but if you divide by x rather than factorising you will lose one of the solutions.

$x(5x+3) = 0$ and so $x = 0$ or $x = -\frac{3}{5}$.

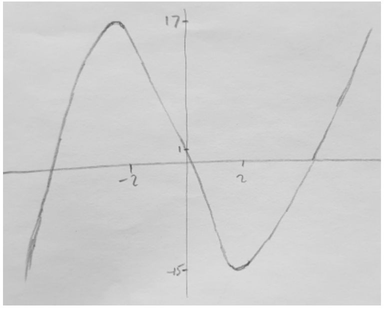
(ii) Remember to show key points on your sketch, but do not work out lots of values and plot the graph.

(a) $y = x^3 - 12x + 1 \Rightarrow \frac{dy}{dx} = 3x^2 - 12$ Stationary points when $\frac{dy}{dx} = 0$ so $x = \pm 2$.

(b) When x is large and positive, y is positive, so the maximum is at $x = -2$, the minimum at $x = 2$, and then the function increases for $x > 2$.

(c) When $x = 0, y = 1$.

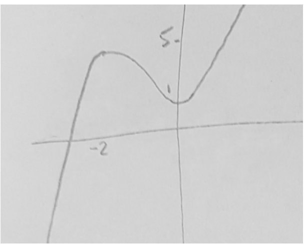
(d) When $x = -2, y = 17$. When $x = 2, y = -15$.



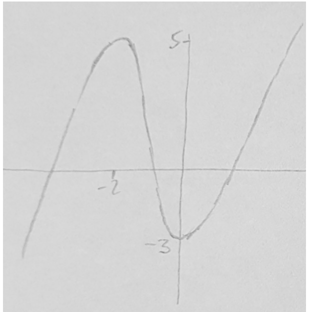
(e) From the graph, it can be seen that the equation has three real roots.

(iii) Note that these cubics (which have no x term) all have a turning point on the y-axis.

(a) Turning points at $(-2, 5)$ and $(0, 1)$.

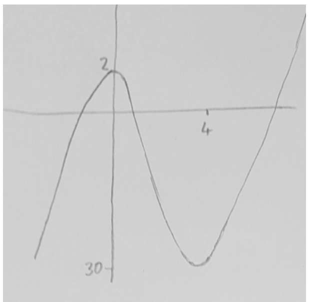


(b) Turning points at $(-2, 5)$ and $(0, -3)$.

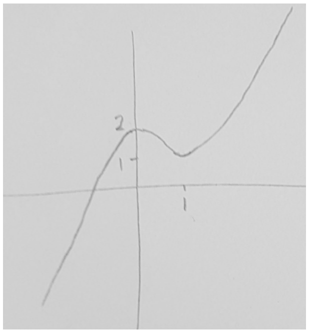


(c) Turning points at $(-1,-1)$ and $(0,-3)$.

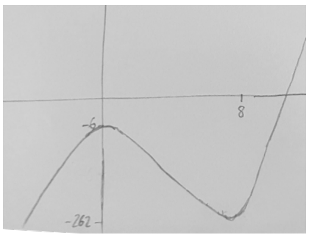
(d) Turning points at $(0,2)$ and $(4,-30)$.



(e) Turning points at $(0,2)$ and $(1,1)$.



(f) Turning points at $(0,-6)$ and $(8,-262)$.



(iv) Graphs **(b)** and **(d)** have three x intercepts. The key point here is: cubics have one turning point on either side of the x-axis if (and only if) they have three intercepts with the x axis (i.e. the equations have three real roots). Note that cubics must always have at least one real root.

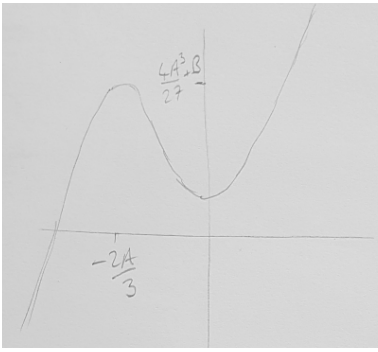
The STEP question

3 $f(x) = x^3 + Ax^2 + b \Rightarrow f'(x) = 3x^2 + 2Ax$.

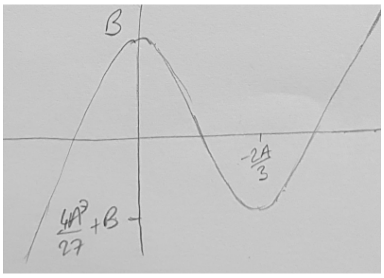
Turning points when $f'(x) = 0$, so $x = 0$ or $x = \frac{-2A}{3}$.

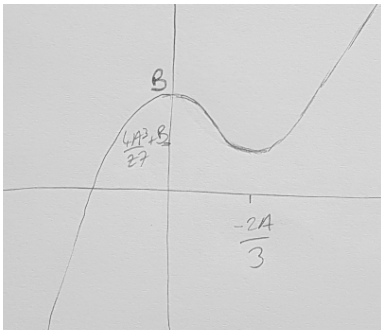
When $x = 0, f(x) = B$, and when $x = \frac{-2A}{3}, f(x) = \frac{4A^3}{27} + B$, so the turning points are at $(-\frac{2}{3}A, \frac{4}{27}A^3 + B)$ and $(0, B)$.

When $A > 0$ and $B > 0$ both the turning points are above the x axis, with the minimum at $(0, B)$. There is only one real root (one intersection with the x axis).



When $A < 0$ and $B > 0$ the maximum will be at $(0, B)$ (which is above the x axis). The minimum will be below the x axis when $\frac{4}{27}A^3 + B < 0$, giving three real roots, and above the x axis when $\frac{4}{27}A^3 + B > 0$ when there is only one real root. This is illustrated in the two sketches below.





For the second part, the letters are now a and b, not A and B, because in the first part there were conditions on the values of A and B.

You need to be careful with the direction of the implication here. The statement

"has three distinct real roots **if** $27b^2 + 4a^3b < 0$"

means

"**if** $27b^2 + 4a^3b < 0$ **then** the equation has three distinct real roots".

It is tempting to show that the converse statement is true instead, i.e.

"**if** the equation has three distinct real roots **then** $27b^2 + 4a^3b < 0$"

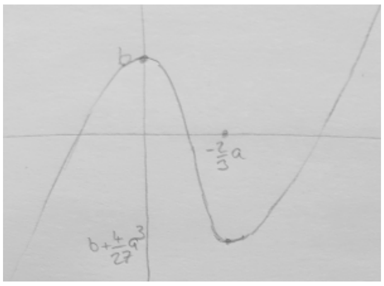
which is **not** what the question is asking for.

From the first part of the question, we know that the turning points of $f(x) = x^3 + ax^2 + b$ have coordinates $(0, b)$ and $(-\frac{2}{3}a, \frac{4}{27}a^3 + b)$

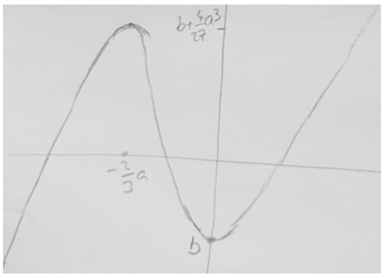
$27b^2 + 4a^3b < 0 \Rightarrow 27b\left(b + \frac{4}{27}a^3\right) < 0$

Therefore, if $27b^2 + 4a^3b < 0$ then exactly one of b and $b + \frac{4}{27}a^3$ is less than zero.

This sketch illustrates $b > 0$ and $b + \frac{4}{27}a^3 < 0$:

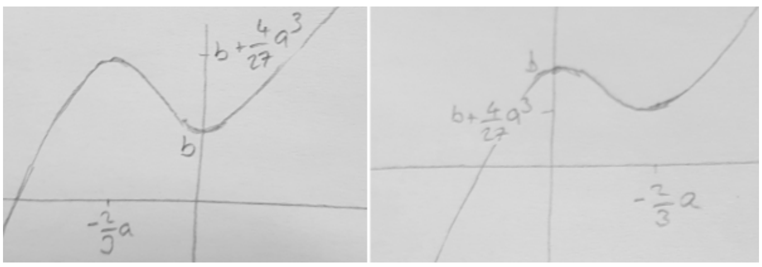


This sketch illustrates $b < 0$ and $b + \frac{4}{27}a^3 > 0$:



In both cases, the graphs show there are three distinct real roots. Therefore, if $27b^2 + 4a^3b < 0$, $x^3 + ax^2 + b = 0$ has three distinct real roots.

Similarly, if $27b^2 + 4a^3b > 0$ then b and $b + \frac{4}{27}a^3$ must be either both positive or both negative These sketches illustrate $b > 0$ and $b + \frac{4}{27}a^3 > 0$, the first showing $a > 0$ and the second showing $a < 0$:



These sketches illustrate $b < 0$ and $b + \frac{4}{27}a^3 < 0$:

In all these cases, the graphs show there is only one root. Therefore, if $27b^2 + 4a^3b > 0, x^3 + ax^2 + b$ has fewer than three distinct real roots.

You can start by considering the turning points, assuming that there are three roots. This is a dangerous approach (as you could end up only proving the converse statement), but can be made to work with careful use of "if and only if".

To do this, you might start by showing that:

the equation has three distinct real roots
if and only if
the turning points are on opposite sides of the y-axis

You will need to sketch graphs of cubics showing all the different cases to justify this. Once you have done this you can say:

the turning points are on opposite sides of the y-axis
if and only if
the y coordinates y_1, y_2 of the turning points have different signs
if and only if
$y_1 \times y_2 < 0$

and then use your y_1 and y_2 to complete the argument.

Warm down

4 Euclid did not have the SSS condition for congruence at this point in his book, so for this question we restricted ourselves to SAS.

If we do allow ourselves the SSS condition, then the easiest way to prove that an isosceles triangle has two angles is as in question **1(i)**.

Start by showing that $\triangle BCD$ is congruent to $\triangle BAE$ by using SAS You are given that $BA = BC$, and since the extensions AD and CE are the same length we also have $BE = BD$. The angle at B is a shared angle, so $\angle ABE = \angle CBD$.

Having shown $\triangle BCD$ is congruent to $\triangle BAE$ you now know that $\angle BCD = \angle BAE$, $DC = AE$ and $\angle ADC = \angle AEC$.

Now use SAS again with $\triangle ADC$ and $\triangle CEA$ to show that these two triangles are congruent. We now know that $\angle DAC = \angle ECA$, and since $\angle BAC = 180° - \angle DAC$ and $\angle BCA = 180° - \angle ECA$ we have $\angle BAC = \angle BCA$.

It may find it helpful to name the angles (for example: "Let $\angle BAE = \angle BCD = \beta$") and mark them on your diagram.