STEP Support Programme

Mechanics STEP Questions: Solutions

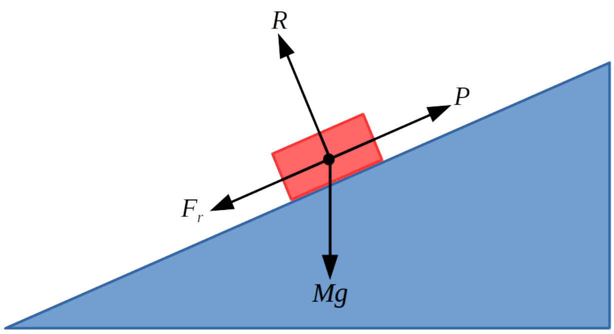
2012 S1 Q11

1 Preparation

(i) Since $0 \leqslant \theta < \frac{\pi}{2}$ we know that $\cos\theta$ and $\sin\theta$ are both positive. You can sketch a right-angled triangle with adjacent length 12 and opposite length 5. This gives $\cos\theta = \frac{12}{13}$ and $\sin\theta = \frac{5}{13}$.

Another approach is to use $\sec^2\theta = 1 + \tan^2\theta$ and $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$.

(ii) **(a)** You should have a diagram looking something like this:



P is the "Pulling force" and F_r is the friction. At the point of moving we have $F_r = \mu R = 0.7R$. Resolving forces parallel and perpendicular to the slope we have:

$$R = Mg\cos 30^\circ = 100\sqrt{3}$$
$$P = F_r + Mg\sin 30^\circ = 70\sqrt{3} + 100$$

(b) In this case the P and F_r forces reverse their direction. Here we have $P = 70\sqrt{3} - 100$. Note that is is smaller to the answer to part **(b)**, which is what we should expect!

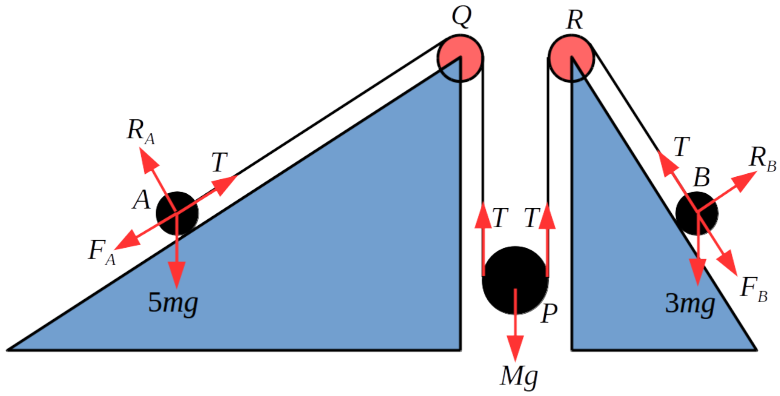
(iii) Since the string is light and in-extensible, and the pulley is smooth then the tension (T) is the same at both ends of the string. Since $m_1 > m_2$, m_1 will start moving downwards and m_2 will start moving upwards. They will move with the same acceleration, a (remember, the string is in-extensible). The equations are:

$$m_1 g - T = m_1 a$$
$$T - m_2 g = m_2 a \quad \Longrightarrow$$
$$a = \frac{m_1 - m_2}{m_1 + m_2} g$$

This answer seems sensible, the greater the difference between the two masses, then the greater the acceleration, and the greater the combined weight of the system then the slower the acceleration.

2 The Mechanics question

This is the diagram with the forces added:



The plane with A on is inclined at angle $\theta = \arctan \frac{7}{24}$, i.e. $\tan\theta = \frac{7}{24}$, and so we have $\sin\theta = \frac{7}{25}$ and $\cos\theta = \frac{24}{25}$.
The plane with B on is inclined at angle $\phi = \arctan \frac{4}{3}$, i.e. $\tan\phi = \frac{4}{3}$, and so we have $\sin\phi = \frac{4}{5}$ and $\cos\phi = \frac{3}{5}$.

(i) Resolving perpendicular to the slopes gives:

$$\begin{aligned} A: \quad & R_A = 5mg\cos\theta = 5mg \times \frac{24}{25} \\ B: \quad & R_B = 3mg\cos\phi = 3mg \times \frac{3}{5} \end{aligned}$$

Resolving up the slopes for A and B, and upwards for P, we have:

$$\begin{aligned} A: \quad & T = F_A + 5mg\sin\theta \\ & \quad = 5mg\mu \times \frac{24}{25} + 5mg \times \frac{7}{25} \\ B: \quad & T = F_B + 3mg\sin\phi \\ & \quad = 3mg\mu \times \frac{3}{5} + 3mg \times \frac{4}{5} \\ P: \quad & 2T = Mg \end{aligned}$$

Rearranging the first two equations gives:

$$5T = 24mg\mu + 7mg \quad (1)$$
$$5T = 9mg\mu + 12mg \quad (2)$$

Then (1) – (2) gives:

$$\begin{aligned} 15mg\mu &= 5mg \\ \mu &= \frac{1}{3} \end{aligned}$$

Substituting back we have:

$$\begin{aligned} 5T &= 24mg \times \frac{1}{3} + 7mg \\ 5T &= 15mg \\ T &= 3mg \end{aligned}$$

and since $Mg = 2T = 6mg$ we have $M = 6m$.

(ii) Since when $M = 6m$ we had A and B on the point of moving up the slopes, when $M = 9m$ they **will** be moving up the slope! A little bit of thought is needed here. If A goes up the slope by 5cm and B goes up by 3cm then P will go down by $\frac{1}{2}(5+3) = 4$cm. In a similar way we have $a_P = \frac{1}{2}(a_A + a_B)$.

The equations of motion are:

$$\begin{aligned}
A: \quad 5ma_A &= T - 5mg\mu \times \frac{24}{25} - 5mg \times \frac{7}{25} \\
5ma_A &= T - \frac{8}{5}mg - \frac{7}{5}mg \\
5ma_A &= T - 3mg && (3) \\
B: \quad 3ma_B &= T - 3mg\mu \times \frac{3}{5} - 3mg \times \frac{4}{5} \\
3ma_B &= T - \frac{3}{5}mg - \frac{12}{5}mg \\
3ma_B &= T - 3mg && (4) \\
P: \quad Ma_P &= Mg - 2T \\
9ma_P &= 9mg - 2T && (5)
\end{aligned}$$

(3) − (4) gives $5ma_A = 3ma_B$ i.e. $a_B = \frac{5}{3}a_A$. Using (3) and $a_P = \frac{1}{2}(a_A + a_B)$ in (5) gives:

$$\begin{aligned}
\tfrac{9}{2}m(a_A + a_B) &= 9mg - 2(5ma_A + 3mg) \\
9a_A + 9a_B &= 18g - 20a_A - 12g \\
9a_A + 9 \times \tfrac{5}{3}a_A &= 6g - 20a_A \\
44a_A &= 6g \\
a_A &= \frac{3g}{22}
\end{aligned}$$

We then have $a_B = \frac{5}{3} \times a_A = \dfrac{5g}{22}$ and $a_P = \frac{1}{2}(a_A + a_B) = \dfrac{2g}{11}$.

2010 S1 Q10

3 **Preparation** No questions to answer here!

4 **The Mechanics question** The velocity of the particle is given by:

$$\dot{\mathbf{r}} = \left(\mathrm{e}^t \cos t - \mathrm{e}^t \sin t\right)\mathbf{i} + \left(\mathrm{e}^t \sin t + \mathrm{e}^t \cos t\right)\mathbf{j}\,.$$

If the angle between the displacement and velocity is θ then we have:

$$\begin{aligned}\mathbf{r}\cdot\dot{\mathbf{r}} &= |\mathbf{r}| \times |\dot{\mathbf{r}}| \times \cos\theta \\ \mathrm{e}^{2t}\cos^2 t + \mathrm{e}^{2t}\sin^2 t &= \mathrm{e}^t \times \sqrt{2}\mathrm{e}^t \times \cos\theta \\ \cos\theta &= \frac{1}{\sqrt{2}} \\ \theta &= \frac{\pi}{4}\end{aligned}$$

The acceleration is given by:

$$\begin{aligned}\ddot{\mathbf{r}} &= \left(\mathrm{e}^t \cos t - \mathrm{e}^t \sin t - \mathrm{e}^t \sin t - \mathrm{e}^t \cos t\right)\mathbf{i} + \left(\mathrm{e}^t \sin t + \mathrm{e}^t \cos t + \mathrm{e}^t \cos t - \mathrm{e}^t \sin t\right)\mathbf{j} \\ &= \left(-2\mathrm{e}^t \sin t\right)\mathbf{i} + \left(2\mathrm{e}^t \cos t\right)\mathbf{j}\end{aligned}$$

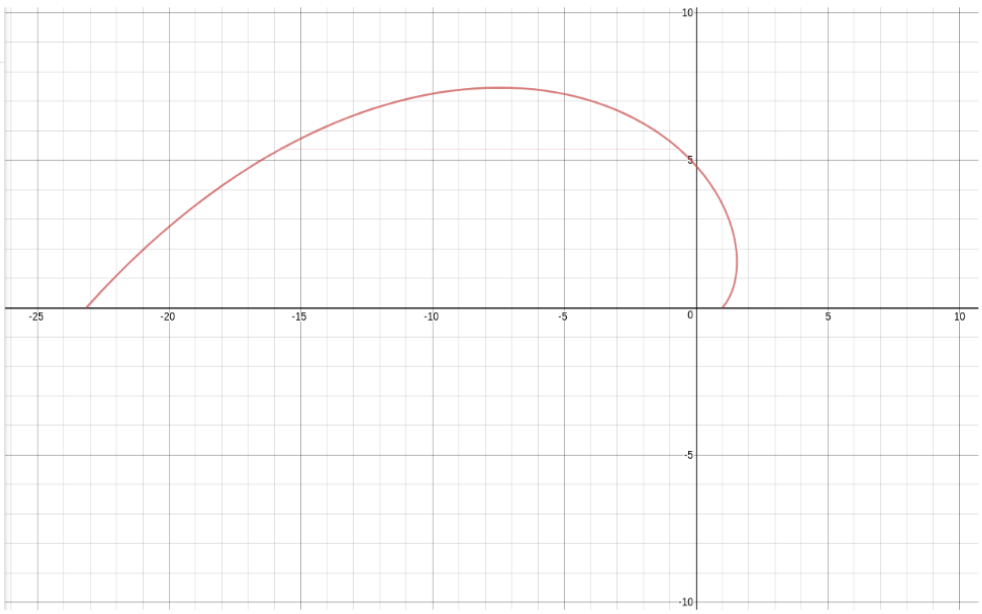
We then have:

$$\begin{aligned}\mathbf{r}\cdot\ddot{\mathbf{r}} &= -2\mathrm{e}^{2t}\cos t\sin t + 2\mathrm{e}^{2t}\sin t\cos t \\ &= 0\end{aligned}$$

and so $\mathbf{r}$ and $\ddot{\mathbf{r}}$ are perpendicular.

To sketch the path, I would start by plotting a few points (which is rather unusual for a STEP question!). We have:

$$\begin{array}{ll} t = 0 & \text{particle at } (1, 0) \\ t = \frac{\pi}{2} & \text{particle at } (0, \mathrm{e}^{\pi/2}) \\ t = \pi & \text{particle at } (\mathrm{e}^{\pi}, 0) \end{array}$$



Note that since the x component of the velocity is equal to $\mathrm{e}^t\left(\cos t - \sin t\right)$ then this is positive for $0 \leqslant t < \frac{\pi}{4}$, and so x is increasing in this range.

If Q is T seconds behind P, then the position of Q at time t is given by:

$$\mathbf{r}_Q = \left(\mathrm{e}^{t-T}\cos(t-T)\right)\mathbf{i} + \left(\mathrm{e}^{t-T}\sin(t-T)\right)\mathbf{j}$$

We then have:

$$\mathbf{r}_P - \mathbf{r}_Q = \left(\mathrm{e}^t\cos t\right)\mathbf{i} + \left(\mathrm{e}^t\sin t\right)\mathbf{j} - \left[\left(\mathrm{e}^{t-T}\cos(t-T)\right)\mathbf{i} + \left(\mathrm{e}^{t-T}\sin(t-T)\right)\mathbf{j}\right]$$

The distance squared between P and Q is:

$$\begin{aligned}
|\mathbf{r}_P - \mathbf{r}_Q|^2 &= \left(\mathrm{e}^t\cos t - \mathrm{e}^{t-T}\cos(t-T)\right)^2 + \left(\mathrm{e}^t\sin t - \mathrm{e}^{t-T}\sin(t-T)\right)^2 \\
&= \mathrm{e}^{2t}\left[\left(\cos t - \mathrm{e}^{-T}\cos(t-T)\right)^2 + \left(\sin t - \mathrm{e}^{-T}\sin(t-T)\right)^2\right] \\
&= \mathrm{e}^{2t}\Big[\cos^2 t - 2\mathrm{e}^{-T}\cos t\cos(t-T) + \mathrm{e}^{-2T}\cos^2(t-T) \\
&\qquad + \sin^2 t - 2\mathrm{e}^{-T}\sin t\sin(t-T) + \mathrm{e}^{-2T}\sin^2(t-T)\Big] \\
&= \mathrm{e}^{2t}\left[1 + \mathrm{e}^{-2T} - 2\mathrm{e}^{-T}\left(\cos t\cos(t-T) + \sin t\sin(t-T)\right)\right] \\
&= \mathrm{e}^{2t}\left[1 + \mathrm{e}^{-2T} - 2\mathrm{e}^{-T} \times \cos\left[t - (t-T)\right]\right] \\
&= \mathrm{e}^{2t}\left[1 + \mathrm{e}^{-2T} - 2\mathrm{e}^{-T}\cos T\right]
\end{aligned}$$

This means we have $|\mathbf{r}_P - \mathbf{r}_Q| = \mathrm{e}^t\sqrt{1 + \mathrm{e}^{-2T} - 2\mathrm{e}^{-T}\cos T}$, and since T is a constant the distance is proportional to e^t.

1993 S1 Q11

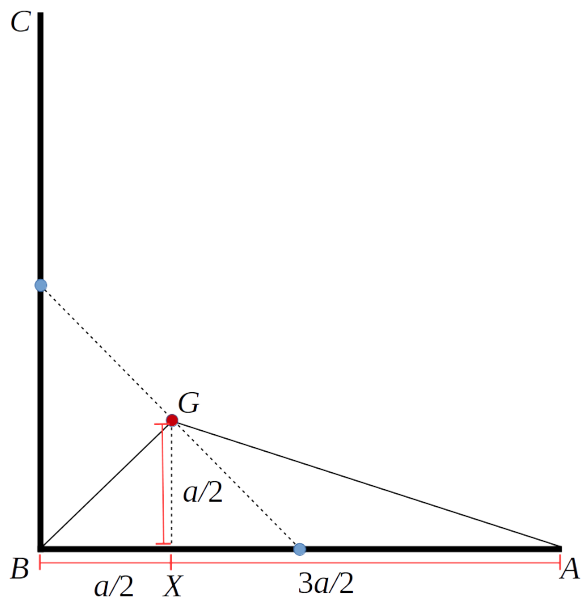
5 Preparation

(i) Since the rods are equal, the centre of mass will be the mean of the two individual centres of mass. The centre of mass is at:

$$\frac{(0,a)+(a,0)}{2} = \left(\frac{a}{2}, \frac{a}{2}\right)$$

It the masses of the two rods were different then you would need to find the weighted mean. There are some useful topics notes in the STEP II Mechanics module.

(ii) Your diagram should look something like this:

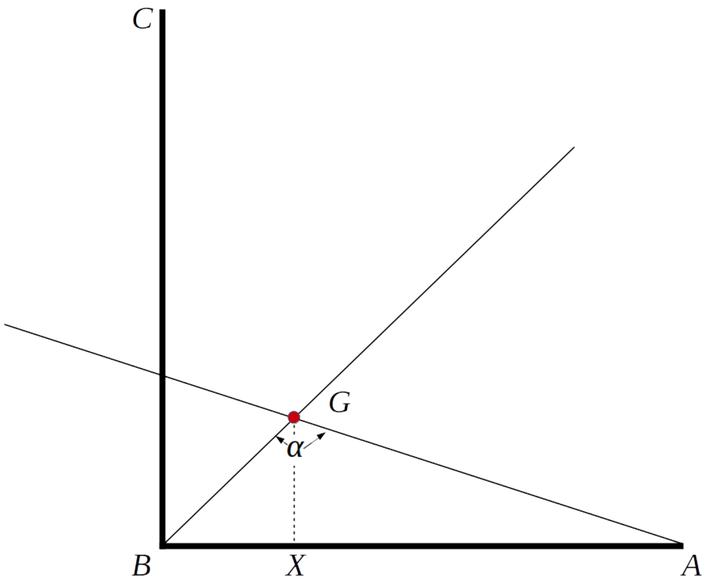


From this we have:

$$\tan BGX = \frac{\frac{a}{2}}{\frac{a}{2}} = 1$$

$$\tan AGX = \frac{\frac{3a}{2}}{\frac{a}{2}} = 3$$

(iii) Your diagram should look like:



You can see that $\alpha = \angle BGX + \angle AGX$.

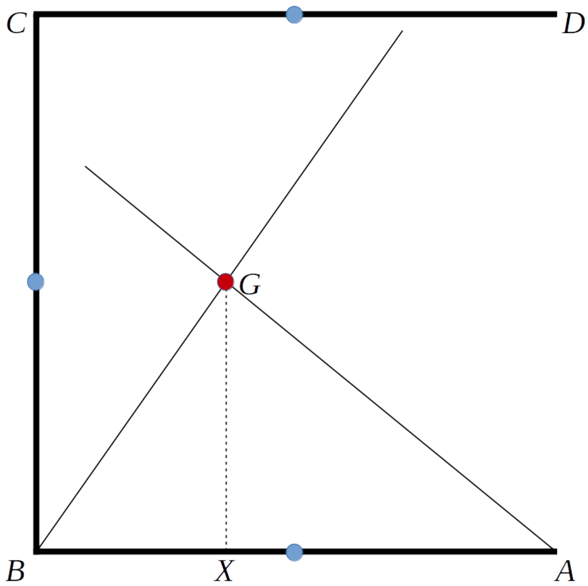
$$\begin{aligned} \tan\alpha &= \tan(BGX + AGX) \\ &= \frac{\tan BGX + \tan AGX}{1 - \tan BGX \times \tan AGX} \\ &= \frac{1+3}{1-3} \\ &= -2 \end{aligned}$$

6 The Mechanics question

First thing to do is to find the centre of mass. The wire is uniform and the 3 sides are equal length (length $2a$ say), so the centre of mass of each side is in the middle of each side. Let the point B be at $(0,0)$.

The centre of mass of whole wire is:

$$\frac{(a,0)+(0,a)+(a,2a)}{3}=\left(a,\tfrac{2}{3}a\right)$$



From this diagram we have:

$$\tan BGX = \frac{\frac{2}{3}a}{a}=\frac{2}{3}$$
$$\tan AGX = \frac{\frac{4}{3}a}{a}=\frac{4}{3}$$

Then if the angle between the two directions is θ we have:

$$\begin{aligned}\tan\theta &= \tan(BGX+AGX)\\ &= \frac{\tan BGX+\tan AGX}{1-\tan BGX\times\tan AGX}\\ &= \frac{\frac{2}{3}+\frac{4}{3}}{1-\frac{2}{3}\times\frac{4}{3}}\\ &= \frac{2}{1-\frac{8}{9}}\\ &= \frac{18}{9-8}\\ &= 18\end{aligned}$$

and so the angle between the directions is $\theta=\tan^{-1}18$.

2006 S2 Q11

7 Preparation

(i) Since the particle is moving on a smooth horizontal plane, gravity has no part to play.

$\ddot{x} = F/m$ and $\ddot{y} = 0$. Therefore the position of the particle at time t is given by $x = \frac{1}{2m}Ft^2 + u\cos\alpha\, t$ and $y = u\sin\alpha\, t$.

(a) If $\alpha = -\pi$ then $x = \frac{1}{2m}Ft^2 - ut$ and $y = 0$, i.e. the particle moves in the x direction only. It returns to the origin when $t = \frac{2mu}{F}$.

(b) If $\alpha = -\frac{3}{4}\pi$ then $x = \frac{1}{2m}Ft^2 - \frac{1}{\sqrt{2}}ut$ and $y = -\frac{1}{\sqrt{2}}ut$. It returns to the y-axis when $x = 0$, i.e. $t = \frac{\sqrt{2}mu}{F}$.

(c) If $\alpha = \frac{1}{2}\pi$ then $x = \frac{1}{2m}Ft^2$ and $y = ut$. It reaches the line $y = x$ when $\frac{1}{2m}Ft^2 = ut$ i.e. $t = \frac{2mu}{F}$.

(ii) Equating coefficients in $g\sin 2\theta + f\cos 2\theta = R\sin(2\theta+\beta) = R\sin 2\theta\cos\beta + R\cos 2\theta\sin\beta$ gives $R\cos\beta = g$ and $R\sin\beta = f$. We then have $R^2 = f^2 + g^2$ and $\tan\beta = \frac{f}{g}$. Therefore:

$$g\sin 2\theta + f\cos 2\theta = \sqrt{f^2+g^2}\,\sin\left(2\theta + \tan^{-1}\left(\frac{f}{g}\right)\right)$$

and the maximum value is $\sqrt{f^2+g^2}$.

8 The Mechanics question

Here gravity does play a part! If x is the horizontal distance northwards and y is the vertical distance then we have $\ddot{x} = -f$ and $\ddot{y} = -g$. Integrating and using the initial velocity gives:

$$\begin{aligned} x &= -\tfrac{1}{2}ft^2 + u\cos\theta\, t \\ y &= -\tfrac{1}{2}gt^2 + u\sin\theta\, t \end{aligned}$$

The projectile lands when $y = 0$, so when $t = T = \dfrac{2u\sin\theta}{g}$. Substituting this into x gives:

$$\begin{aligned} OA &= -\tfrac{1}{2}f \times \left(\frac{2u\sin\theta}{g}\right)^2 + u\cos\theta \times \frac{2u\sin\theta}{g} \\ &= \frac{2u^2\sin\theta}{g^2}\left(g\cos\theta - f\sin\theta\right) \end{aligned}$$

(i) If the wind starts to blow the projectile back towards O before it lands at A then we need $\dot{x} < 0$ for some $t < T$.

$$\begin{aligned} & \dot{x} < 0 \\ \Longrightarrow\quad & (u\cos\theta - ft) < 0 \quad \text{for some } t < T \\ \Longrightarrow\quad & u\cos\theta < ft \\ \Longrightarrow\quad & u\cos\theta < f \times \frac{2u\sin\theta}{g} \\ \Longrightarrow\quad & \tan\theta > \frac{g}{2f} \end{aligned}$$

Therefore if $\alpha = \tan^{-1}\left(\frac{g}{2f}\right)$ we have the projectile being blown back towards A as long as $\theta > \alpha$.

(ii) We have:

$$\begin{aligned} OB &= \frac{2u^2\sin 45^\circ}{g^2}\left(g\cos 45^\circ - f\sin 45^\circ\right) \\ &= \frac{u^2}{g^2}(g - f) \end{aligned}$$

and:

$$\begin{aligned} OA &= \frac{u^2}{g^2}\left(2g\cos\theta\sin\theta - 2f\sin^2\theta\right) \\ &= \frac{u^2}{g^2}\left(g\sin 2\theta + f\cos 2\theta - f\right) \\ &= \frac{u^2}{g^2}\left(\sqrt{f^2+g^2}\sin\left(2\theta + \beta\right) - f\right) \end{aligned}$$

where $\tan\beta = \frac{f}{g}$. The maximum value of OA is $\frac{u^2}{g^2}\left(\sqrt{f^2+g^2}-f\right)$ and so we have:

$$\begin{aligned}\frac{OB}{OA} &= \frac{\frac{u^2}{g^2}(g-f)}{\frac{u^2}{g^2}\left(\sqrt{f^2+g^2}-f\right)}\\ &= \frac{g-f}{\sqrt{f^2+g^2}-f}\end{aligned}$$

The horizontal and vertical distances for the second particle are given by:

$$\begin{aligned}x &= -\tfrac{1}{2}ft^2 + \frac{ut}{\sqrt{2}}\\ y &= -\tfrac{1}{2}gt^2 + \frac{ut}{\sqrt{2}}\end{aligned}$$

So if $f = g$ then the particle stays on the line $y = x$ throughout the motion. In particular, when it lands it lands at the origin (where it was fired from, i.e. $OB = 0$).

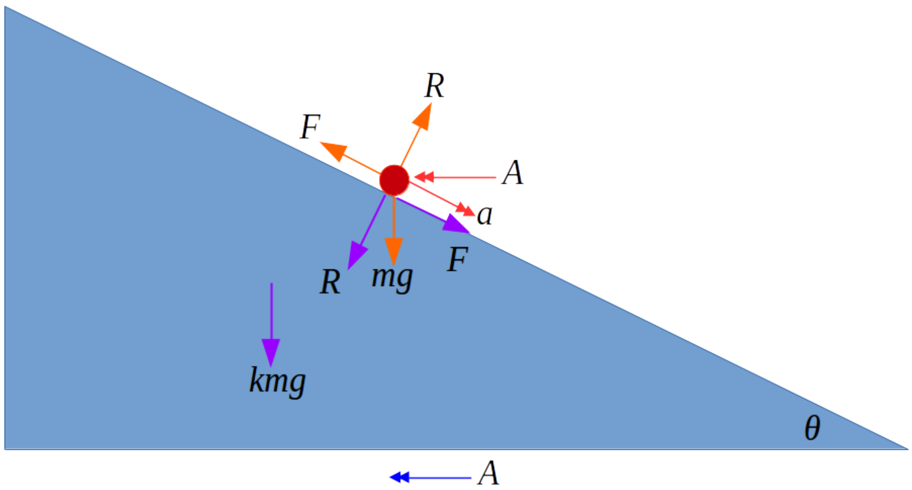
2008 S2 Q11

9 Preparation

(i) If $\tan\theta = \frac{3}{4}$ and θ is acute then drawing a right-angled triangle will show you that $\cos\theta = \frac{4}{5}$ and $\sin\theta = \frac{3}{5}$.

10 The Mechanics question

Your diagram might look like this:



Here the forces acting on the particle are orange and the forces acting on the wedge are purple. I have called the acceleration of the wedge A. You could instead draw two different diagrams, one showing the forces and accelerations for the particle and one showing the forces and accelerations for the wedge.

(i) Resolving horizontally (and taking "to the left" as the positive direction) gives:

$$\begin{aligned} \text{wedge}: \quad & kmA = R\sin\theta - F\cos\theta \\ \text{particle}: \quad & mA - ma\cos\theta = F\cos\theta - R\sin\theta \end{aligned}$$

Adding these gives:

$$kmA + mA - ma\cos\theta = 0 \qquad \Longrightarrow \qquad A = \frac{a\cos\theta}{k+1}\,.$$

If P is descending at an angle of $45°$, then the magnitude of the horizontal and vertical components of the acceleration of P must be the same. This means we have:

$$\begin{aligned} a\sin\theta &= a\cos\theta - A \\ \cancel{a}\sin\theta &= \cancel{a}\cos\theta - \frac{\cancel{a}\cos\theta}{k+1} \\ \frac{\sin\theta}{\cos\theta} &= 1 - \frac{1}{k+1} \\ \tan\theta &= \frac{k+1-1}{k+1} = \frac{k}{k+1} \end{aligned}$$

If $k = 3$ we have $\tan\theta = \frac{3}{4}$ and so $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$.

Resolving vertically for the particle gives:

$$\begin{aligned} ma\sin\theta &= mg - F\sin\theta - R\cos\theta \\ ma \times \tfrac{3}{5} &= mg - \mu R \times \tfrac{3}{5} - R \times \tfrac{4}{5} \\ 3ma &= 5mg - 3\mu R - 4R \end{aligned}$$

The horizontal equation for the wedge gives us:

$$\begin{aligned}
kmA &= R\sin\theta - F\cos\theta \\
km \times \frac{a\cos\theta}{k+1} &= R\sin\theta - \mu R\cos\theta \\
\tfrac{3}{4} \times ma \times \tfrac{4}{5} &= \tfrac{3}{5}R - \tfrac{4}{5}\mu R \\
3ma &= 3R - 4\mu R \\
R &= \frac{3ma}{3-4\mu}
\end{aligned}$$

Substituting into our vertical equation for the particle gives:

$$\begin{aligned}
3ma &= 5mg - (3\mu+4)R \\
3ma &= 5mg - \frac{3ma(3\mu+4)}{3-4\mu} \\
3a + \frac{3a(3\mu+4)}{3-4\mu} &= 5g \\
3a\left(\frac{3-4\mu+3\mu+4}{3-4\mu}\right) &= 5g \\
a &= \frac{5g(3-4\mu)}{3(7-\mu)}
\end{aligned}$$

(ii) Using the horizontal equation for the wedge, and taking $F = \mu R$ (i.e, assuming the wedge — and particle — are moving) we have:

$$\begin{aligned}
kmA &= R\sin\theta - F\cos\theta \\
&= R\sin\theta - \mu R\cos\theta \\
&= R\cos\theta\,(\tan\theta - \mu)
\end{aligned}$$

Hence if $\tan\theta \leqslant \mu$ this implies that the acceleration of the wedge will be zero, and the acceleration of the particle will be zero, i.e. it doesn't move.