STEP Support Programme

STEP 3 Specification Pure Notes

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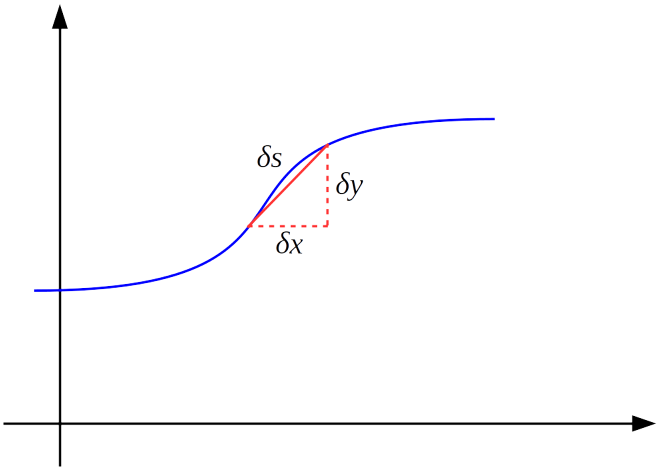
These notes are designed to help students in preparing for STEP 3. They cover the "bold and italic" sections of the STEP 3 specification which are not covered in the A-level single Mathematics specifications, or A-level Further Maths Common Core. Many of these topics will be covered in the optional A Level Further Mathematics modules.

There are more notes on the various sections of the specification in the STEP 3 modules.

Further Calculus

Consider a continuous curve $y = \mathrm{f}(x)$. When the x coordinate is changed by a small amount, δx, then the y coordinate is also changed by a small amount δy. If the change in x is small enough then the curve can be approximated by a straight line and the change in the length of the curve is approximately $\delta s \approx \sqrt{(\delta x)^2 + (\delta y)^2}$ (from Pythagoras' theorem).



Note that in the diagram above, the straight line is not a very good approximation to the curve. This is because in this case δx is not "small".

We can rearrange this to get $\delta s \approx \sqrt{1 + \left(\frac{\delta y}{\delta x}\right)^2} \times \delta x$. The length along the curve can then be found by considering what happens to a sum of these δs terms as $\delta x \to 0$ and so the length of the curve between the points with x coordinate a and b is:

$$\int_{x=a}^{x=b} \sqrt{\left[1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right]}\, \mathrm{d}x$$

Alternatively, if the curve is written in a parameterised form — so that we have $x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$ — then we can rearrange the expression for δs to get $\delta s = \sqrt{\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2} \times \delta t$. Taking the limit as before gives the length of the curve as:

$$\int_{t=p}^{t=q} \sqrt{\left[\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2\right]}\, \mathrm{d}t$$

Further Vectors

The vector product of the two vectors $\mathbf{a}$ and $\mathbf{b}$ is:

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta\, \hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, and θ is the angle between them. The magnitude of the vector product is equal to the area of the parallelogram with sides $\mathbf{a}$ and $\mathbf{b}$.

The direction of $\hat{\mathbf{n}}$ is given by the *right hand rule* — if $\mathbf{a}$ is the thumb and $\mathbf{b}$ is the index finger then the direction of $\hat{\mathbf{n}}$ is given by the direction of the second finger[1].

If you move your thumb to where the index finger was and the index finger to where the thumb was you should find that your second finger is now pointing in the opposite direction. This means that:

$$\mathbf{b} \times \mathbf{a} = -\mathbf{a} \times \mathbf{b}$$

The vector product is *anti-symmetric* — i.e. if you swap $\mathbf{a}$ and $\mathbf{b}$ the sign changes.

Note that $\mathbf{a} \times \mathbf{a} = 0$ as in this case we have $\theta = 0$.

If $\mathbf{a} \times \mathbf{b} = 0$ then either $\mathbf{a} = 0$, or $\mathbf{b} = 0$, or $\mathbf{a}$ and $\mathbf{b}$ are parallel (i.e. one is a multiple of the other).

In determinant form we have:

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{or} \quad \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & a_1 & b_1 \\ \mathbf{j} & a_2 & b_2 \\ \mathbf{k} & a_3 & b_3 \end{vmatrix}$$

where $\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.

This is the way I use to calculate the cross product — trying to remember it as $(a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$ feels like too much work for me!

Vector equation of a line

Let $\mathbf{r}$ be a general point on a line that passes through point A with position vector $\mathbf{a}$ and which is parallel to the vector $\mathbf{b}$. Then we can write the equation of the line as:

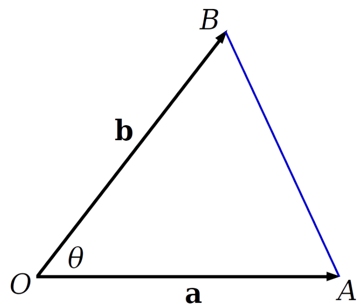
$$(\mathbf{r} - \mathbf{a}) \times \mathbf{b} = 0 \quad \text{or} \quad \mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}\,.$$

This is because the vector $\mathbf{r} - \mathbf{a}$ is parallel to $\mathbf{b}$, and the vector product of two parallel vectors is equal to 0.

[1] This assumes that you do not have hyper-mobile joints or any broken fingers.

Area of a triangle

Consider a triangle with vertices O, A and B.



Let the position vector of point A be $\mathbf{a}$ and the position vector of point B be $\mathbf{b}$. Also let the angle between $\mathbf{a}$ and $\mathbf{b}$ be θ. The area of the triangle is given by:

$$\tfrac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}|$$

(Remembering that $\hat{\mathbf{n}}$ is a unit vector, so $|\hat{\mathbf{n}}| = 1$).

If the triangle had vertices A, B and C, then the vectors of the lengths relative to C are $\overrightarrow{CA} = \mathbf{a}-\mathbf{c}$ and $\overrightarrow{CB} = \mathbf{b}-\mathbf{c}$. If the angle between $\overrightarrow{CA}$ and $\overrightarrow{CB}$ is θ then the area of the triangle is:

$$\tfrac{1}{2}|\overrightarrow{CA}||\overrightarrow{CB}|\sin\theta = \tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})|$$

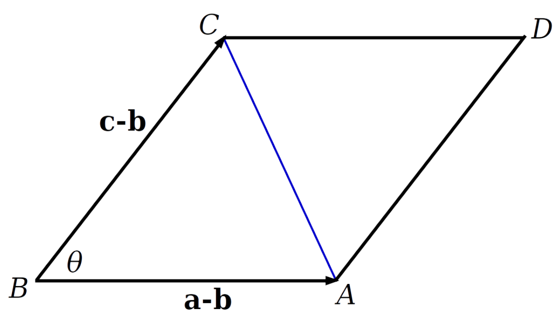
Rearranging this gives:

$$\begin{aligned}\tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})| &= \tfrac{1}{2}|(\mathbf{a}\times\mathbf{b})-(\mathbf{c}\times\mathbf{b})-(\mathbf{a}\times\mathbf{c})+(\mathbf{c}\times\mathbf{c})|\\ &= \tfrac{1}{2}|(\mathbf{a}\times\mathbf{b})+(\mathbf{b}\times\mathbf{c})+(\mathbf{c}\times\mathbf{a})|\end{aligned}$$

Note that $\mathbf{c}\times\mathbf{b} = -\mathbf{b}\times\mathbf{c}$ and that $\mathbf{c}\times\mathbf{c} = 0$.

Which is a nicely symmetrical result. You will not be expected to be able to quote this — however deriving this from $\tfrac{1}{2}|(\mathbf{a}-\mathbf{c})\times(\mathbf{b}-\mathbf{c})|$ would be fair game!

Area of a parallelogram Consider a parallelogram with vertices A, B, C and D.



The area of the parallelogram is twice the area of triangle ABC and so is equal to $|(\mathbf{a}-\mathbf{b})\times(\mathbf{c}-\mathbf{b})|$.

If the parallelogram had vertices at O, A and B then the area would be given by $|\mathbf{a}\times\mathbf{b}|$.

Hyperbolic Functions

For an introduction to hyperbolic functions see the warm-up of Assignment 21.

You can use Osborn's Rule to covert trigonometric identities written in terms of sine and cosine into a corresponding hyperbolic identity. Essentially you replace any $\sin^2\theta$ with $-\sinh^2\theta$, and $\sin A \sin B$ with $-\sinh A \sinh B$.

$$\begin{aligned}
&\tanh x = \frac{\sinh x}{\cosh x}\\
&\cosh^2 x - \sinh^2 x = 1\\
&1 - \tanh^2 x = \operatorname{sech}^2 x\\
&\coth^2 x - 1 = \operatorname{cosech}^2 x\\
&\cosh(A \pm B) = \cosh A \cosh B \pm \sinh A \sinh B\\
&\sinh(A \pm B) = \sinh A \cosh B \pm \sinh B \cosh A\\
&\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}\\
&\sinh 2x = 2 \sinh x \cosh x\\
&\cosh 2x = \cosh^2 x + \sinh^2 x\\
&\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x\\
&\tanh 2x = \frac{2\tanh x}{1 + \tanh^2 x}
\end{aligned}$$

Note that the identities for $\sinh 2x$, $\cosh 2x$, $\tanh 2x$ can be derived from the identities for $\sinh(A+B)$ etc., and also that the third and fourth identities can be derived from the first two. You can also derive $\tanh(A \pm B)$ from $\sinh(A \pm B)$ and $\cosh(A \pm B)$:

$$\begin{aligned}
\tanh(A+B) &= \frac{\sinh(A+B)}{\cosh(A+B)}\\
&= \frac{\sinh A \cosh B + \sinh B \cosh A}{\cosh A \cosh B + \sinh A \sinh B}\\
&= \frac{\frac{\sinh A}{\cosh A} + \frac{\sinh B}{\cosh B}}{1 + \frac{\sinh A}{\cosh A}\frac{\sinh B}{\cosh B}}\\
&= \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}
\end{aligned}$$

You can prove many of these from the definitions:

$$\begin{aligned}
\cosh x &= \tfrac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})\\
\sinh x &= \tfrac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})
\end{aligned}$$

See also the STEP 3 Hyperbolic Functions Topic Notes.

Differential Equations

See also the STEP 3 Differential Equations Topic Notes.

Substitutions

A "typical" Differential Equations STEP question involves using a substitution to transform a differential equation into one that can be more easily solved. This is often followed up with a related differential equation where the appropriate substitution needs to be found.

Example: 2008 STEP 2 Question 7

> **(i)** By writing $y = u(1+x^2)^{\frac{1}{2}}$, where u is a function of x, find the solution of the equation
>
> $$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = xy + \frac{x}{1+x^2}$$
>
> for which $y = 1$ when $x = 0$.

In this case differentiating the given $y = u(1+x^2)^{\frac{1}{2}}$ and then substituting for y and $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ results in the differential equation

$$\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}x} = xu\sqrt{1+x^2}$$

which can then be solved by separating the variables.

The next part of the question states:

> **(ii)** Find the solution of the equation
>
> $$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = x^2y + \frac{x^2}{1+x^3}$$
>
> for which $y = 1$ when $x = 0$.

which can solved by using a substitution which is similar to the one in part **(i)**.

Other methods

Sometimes STEP questions will ask you to solve a differential equation using a particular method.

Example: 2003 STEP 3 Question 8

(i) Show that the gradient at a point $(x\,,\,y)$ on the curve

$$(y+2x)^3\,(y-4x)=c\,,$$

where c is a constant, is given by

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{16x-y}{2y-5x}\,.$$

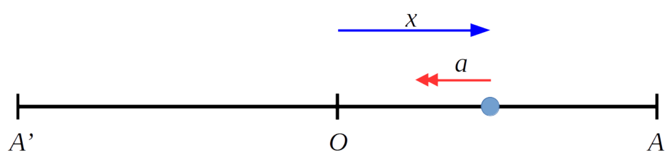
(ii) By considering the derivative with respect to x of $(y+ax)^n\,(y+bx)$, or otherwise, find the general solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{10x-4y}{3x-y}\,.$$

In part **(ii)** of this question you are asked to differentiate an expression and use this to solve the differential equation (and so use "anti-differentiation" to solve it). The first part practises the skills needed for this.

Simple Harmonic Motion

If a particle moves along a straight line so that its acceleration is proportional to its distance from a fixed point O on that line and the acceleration is always directed towards O, then the particle moves with *simple harmonic motion*. The equation can be written as $\ddot{x}=-kx$ where $k>0$ and the negative sign shows that acceleration is always working in the opposite direction to the displacement of x (so is always trying to return the particle to O).



In the diagram above the particle will move between A and A' repeatedly. It doesn't matter whether it is travelling away from O or towards O, the acceleration is always acting towards O. The distance OA is called the *amplitude* of the motion.

If a pendulum swings through a **small** angle then the motion can be approximated by a simple harmonic motion model. See STEP 3 Mechanics Specification Support for more on this.

In the Simple Harmonic equation $\ddot{x}=-\omega^2 x$, ω represents the *angular frequency* and we have $\omega=\frac{2\pi}{T}$, where T is the time take for the motion to make one complete oscillation. T is the *time period* of the motion.

The general solution to the SHM equation $\ddot{x}=-\omega^2 x$ is $x=A\sin(\omega t+\alpha)$.