STEP Support Programme

2024 STEP 2 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the Cambridge Assessment Admissions Testing website. These are the general comments for the STEP 2 2024 exam from the Examiner's report:

Many candidates produced good solutions to the questions, with the majority of candidates opting to focus on the pure questions of the paper. Candidates demonstrated very good ability, particularly in the area of manipulating algebra. Many candidates produced clear diagrams which in many cases meant that they were more successful in their attempts at their questions than those who did not do so. The paper also contained a number of places where the answer to be reached was given in the question. In such cases, candidates must be careful to ensure that they provide sufficient evidence of the method used to reach the result in order to gain full credit.

Please send any corrections, comments or suggestions to step@maths.cam.ac.uk.

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Question 1

1 In the equality

$$4 + 5 + 6 + 7 + 8 = 9 + 10 + 11,$$

the sum of the five consecutive integers from 4 upwards is equal to the sum of the next three consecutive integers.

Throughout this question, the variables n, k and c represent positive integers.

(i) Show that the sum of the $n + k$ consecutive integers from c upwards is equal to the sum of the next n consecutive integers if and only if

$$2n^2 + k = 2ck + k^2.$$

(ii) Find the set of possible values of n, and the corresponding values of c, in each of the cases

(a) $k = 1$

(b) $k = 2$.

(iii) Show that there are no solutions for c and n if $k = 4$.

(iv) Consider now the case where $c = 1$.

(a) Find two possible values of k and the corresponding values of n.

(b) Show, given a possible value N of n, and the corresponding value K of k, that

$$N' = 3N + 2K + 1$$

will also be a possible value of n, with

$$K' = 4N + 3K + 1$$

as the corresponding value of k.

(c) Find two further possible values of k and the corresponding values of n.

Examiner's report

There were a large number of attempts at this question, with many good answers seen and many attempting most parts of the question.

Many were able to show that the required formula in part **(i)** will be satisfied if the stated sum is satisfied, but many did not explain that the result applies in both directions with sufficient detail. Of those who failed to show the result in this part the main error was an incorrect choice of limits when expressing the sums either in sigma notation or as an arithmetic series.

Candidates generally demonstrated an understanding of what was required for part **(ii)**, but a significant number did not express their solution in a clear form, for example by finding just one case or listing the first few without specifying a general relationship. A number of candidates got confused between squares and square roots and having deduced that the square of n is equal to c then concluded that n must be a square number.

In part **(iii)** most candidates successfully identified the equation that needed to be satisfied, but were unable to explain clearly why there were no solutions.

Part **(iv)(a)** was generally completed well, including by candidates who had struggled in earlier parts of the question. Many good answers to part **(iv)(b)** were also seen, although a number of cases did not explicitly identify the relationship that must exist between the values of N and K.

Most candidates recognised that the result of part **(iv)(b)** could be used to generate the further solutions required in part **(iv)(c)**.

Solution

(i) Note that the first term in the first sum is c and the last term is equal to $c+n+k-1$. The first term in the second sum is $c+n+k$ and the last term is $c+n+k+n-1$. It can be helpful to write down an example with specific values to check that you have the correct last term for the first sum, it's very easy to erroneously write the last term as $c+n+k$ not realising that this would give $n+k+1$ terms in the first sum. I have used the $\frac{n}{2}(a+l)$ formula to find the sum of the sequences.

The sum of the $n+k$ consecutive integers from c upwards is equal to:

$$\frac{n+k}{2}\left[c+(c+n+k-1)\right] = \frac{n+k}{2}(2c+n+k-1)$$

The sum of the next n consecutive integers is:

$$\frac{n}{2}\left[(c+n+k)+(c+n+k+n-1)\right] = \frac{n}{2}(2c+3n+2k-1)$$

These two sums are equal if and only if

$$\begin{aligned}(n+k)(2c+n+k-1) &= n(2c+3n+2k-1)\\ 2nc+n^2+nk-n+2ck+nk+k^2-k &= 2nc+3n^2+2kn-n\\ n^2+2ck+k^2-k &= 3n^2\\ \iff 2ck+k^2 &= 2n^2+k\end{aligned}$$

(ii) **(a)** If $k = 1$ then the result in part **(i)** becomes:

$$2n^2 + 1 = 2c + 1$$

Which implies that $c = n^2$. Therefore n can be any positive integer and c is the corresponding square number.

(b) If $k = 2$ then the result in part **(i)** becomes:

$$2n^2 + 2 = 4c + 4 \implies n^2 = 2c + 1$$

This means that n^2 has to be odd, and hence n can be any odd positive integer greater than one, and the corresponding value of c is given by $c = \frac{1}{2}(n^2 - 1)$

(iii) If $k = 4$ we have:

$$2n^2 + 4 = 8c + 16 \implies n^2 = 4c + 6$$

This means that we need $n^2 = 4(c+1) + 2$, i.e. it is 2 more than a multiple of 4. We know that this is an even number, hence we must have $n = 2m$ for some integer m, but this gives $n^2 = 4m^2$, hence we can never find an n such that n^2 has the form $4(c+1) + 2$.

(iv) **(a)** When $c = 1$ we have:

$$2n^2 + k = 2k + k^2 \implies 2n^2 = k(k+1)$$

Trying the first few values of k we have:

$$k(k+1) = 2, 6, 12, 20, 30, 42, 56, 72$$

Of these, 2 and 72 are twice a square number so two possible values of (k, n) are $(1, 1)$ and $(8, 6)$.

You could instead have worked out some values of $2n^2$, but I think it's slightly harder to spot which of these are equal to $k(k+1)$.

(b) If (K, N) is a solution then we have $2N^2 = K(K+1)$.
Using the new values we have:

$$\begin{aligned} 2N'^2 &= 2(3N + 2K + 1)^2 \\ &= 2(9N^2 + 4K^2 + 1 + 12NK + 6N + 4K) \\ &= 18N^2 + 8K^2 + 2 + 24NK + 12N + 8K \end{aligned}$$

and

$$\begin{aligned} K'(K'+1) &= (4N + 3K + 1)(4N + 3K + 2) \\ &= 16N^2 + 9K^2 + 2 + 24KN + 12N + 9K \end{aligned}$$

Taking the difference between these we have:

$$\begin{aligned} 2N'^2 - K'(K'+1) &= (18N^2 + 8K^2 + 2 + 24NK + 12N + 8K) \\ &\quad - (16N^2 + 9K^2 + 2 + 24KN + 12N + 9K) \\ &= 2N^2 - K^2 - K \\ &= 2N^2 - K(K+1) \\ &= 0 \end{aligned}$$

Hence if (K, N) is a solution then so is (K', N').

(c) From part **(a)** we know that $n = 6$, $k = 8$ is a solution. Using this gives:

$$\begin{aligned} N' &= 3N + 2K + 1 \\ &= 3 \times 6 + 2 \times 8 + 1 \\ &= 35 \\ K' &= 4N + 3K + 1 \\ &= 4 \times 6 + 3 \times 8 + 1 \\ &= 49 \end{aligned}$$

and so $n = 35$ and $k = 49$ is another solution.
Repeating the process gives:

$$\begin{aligned} N' &= 3N + 2K + 1 \\ &= 3 \times 35 + 2 \times 49 + 1 \\ &= 204 \\ K' &= 4N + 3K + 1 \\ &= 4 \times 35 + 3 \times 49 + 1 \\ &= 288 \end{aligned}$$

Note that you still could attempt this part even if something went wrong with part **(iv) b**, and you can just apply the result even if you didn't manage to show it!

Question 2

2 *In this question, you need not consider issues of convergence.*

(i) Find the binomial series expansion of $(8+x^3)^{-1}$, valid for $|x|<2$.

Hence show that, for each integer $m \geqslant 0$,

$$\int_0^1 \frac{x^m}{8+x^3}\,\mathrm{d}x = \sum_{k=0}^{\infty}\left(\frac{(-1)^k}{2^{3(k+1)}}\cdot\frac{1}{3k+m+1}\right).$$

(ii) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}-\frac{2}{3k+2}+\frac{4}{3k+1}\right) = \int_0^1 \frac{1}{x+2}\,\mathrm{d}x\,,$$

and evaluate the integral.

(iii) Show that

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{72(2k+1)}{(3k+1)(3k+2)}\right) = \pi\sqrt{a} - \ln b\,,$$

where a and b are integers which you should determine.

Examiner's report

This question was attempted by approximately three-quarters of the candidates and many very good solutions were seen.

In part **(i)** candidates were generally able to find the binomial series expansion, but some did not give a clear enough statement of the general term. A number of candidates did not recognize that it would be possible to use the series expansion to establish the integration result that was required and instead attempted to use integration by parts or to produce a proof by induction.

Most candidates recognized the way in which part **(i)** could be applied to answer part **(ii)** and this part was generally answered well. A number of candidates forgot to evaluate the integral before moving on to the next part of the question.

In part **(iii)** a large number of candidates recognized that the use of partial fractions would allow them to apply part **(i)** again, but many did not then realise that there was a need for partial fractions to be applied a second time. Almost all candidates who reached the final integral were able to recognize it and either state the answer or use an appropriate substitution.

Solution

At the start of this question it tells you that "you need not consider issues of convergence". To do this question you need to swap the sum and the integral signs around, which is ok for finite sums and integrals with finite limits, but sometimes weird things happen when you throw ∞ into the mix. However we didn't want you to worry about that, hence the note at the start! This Youtube video discusses some of the possible issues, and how you can tell whether it is ok to swap or not.

(i) The binomial expansion of $(8+x^3)^{-1}$ is given by:

$$\begin{aligned}\left[8\left(1+\frac{x^3}{8}\right)\right]^{-1} &= \frac{1}{8}\left(1+\frac{x^3}{8}\right)^{-1} \\ &= \frac{1}{8}\left[1+(-1)\left(\frac{x^3}{8}\right)+\frac{(-1)(-2)}{2!}\left(\frac{x^3}{8}\right)^2+\right. \\ &\qquad \left.\frac{(-1)(-2)(-3)}{3!}\left(\frac{x^3}{8}\right)^3+\cdots+(-1)^n\left(\frac{x^3}{8}\right)^n+\cdots\right] \\ &= \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k}\end{aligned}$$

At first I just left the sum as an expanded series of terms, but then when I looked at the next part I realised that it might be useful to write the sum using sigma notation, and to use 2^3 rather than 8.

Using this in the given integral we have:

$$\begin{aligned}\int_0^1 \frac{x^m}{8+x^3}\,\mathrm{d}x &= \int_0^1 x^m \times \frac{1}{8}\sum_{k=0}^{\infty}(-1)^k\left(\frac{x}{2}\right)^{3k}\,\mathrm{d}x \\ &= \frac{1}{8}\sum_{k=0}^{\infty}\int_0^1 (-1)^k\frac{x^{3k+m}}{2^{3k}}\,\mathrm{d}x \\ &= \frac{1}{8}\sum_{k=0}^{\infty}\left[(-1)^k\frac{x^{3k+m+1}}{2^{3k}(3k+m+1)}\right]_0^1 \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{1^{3k+m+1}}{8\times 2^{3k}(3k+m+1)} \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}(3k+m+1)}\end{aligned}$$

(ii) Using the result from part **(i)** with $m = 2, 1$ and 0 gives:

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{3(k+1)}}\left(\frac{1}{3k+3}-\frac{2}{3k+2}+\frac{4}{3k+1}\right) \\ &= \int_0^1 \frac{x^2}{8+x^3}\,\mathrm{d}x - 2\int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x + 4\int_0^1 \frac{1}{8+x^3}\,\mathrm{d}x \\ &= \int_0^1 \frac{x^2-2x+4}{8+x^3}\,\mathrm{d}x\end{aligned}$$

Since $(-2)^3 + 8 = 0$ we know that $x + 2$ must be a factor of $x^3 + 8$. Factorising gives $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$ and so we have:

$$\begin{aligned}\int_0^1 \frac{x^2 - 2x + 4}{8 + x^3}\,\mathrm{d}x &= \int_0^1 \frac{1}{x+2}\,\mathrm{d}x \\ &= \big[\ln(x+2)\big]_0^1 \\ &= \ln 3 - \ln 2\end{aligned}$$

(iii) Using partial fractions gives:

$$\begin{aligned}\frac{72(2k+1)}{(3k+1)(3k+2)} &= \frac{A}{3k+1} + \frac{B}{3k+2} \\ \implies 144k + 72 &= A(3k+2) + B(3k+1) \\ k = -\frac{1}{3} &\implies A = 24 \\ k = -\frac{2}{3} &\implies B = 24\end{aligned}$$

And so we have:

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{3(k+1)}} \left(\frac{72(2k+1)}{(3k+1)(3k+2)} \right) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{2^{3(k+1)}} \left(\frac{24}{3k+1} + \frac{24}{3k+2} \right) \\ &= 24 \int_0^1 \frac{1}{8+x^3}\,\mathrm{d}x + 24 \int_0^1 \frac{x}{8+x^3}\,\mathrm{d}x \\ &= \int_0^1 \frac{24x+24}{8+x^3}\,\mathrm{d}x\end{aligned}$$

Using partial fractions again gives:

$$\begin{aligned}\frac{24x+24}{8+x^3} &= \frac{A}{x+2} + \frac{Bx+C}{x^2-2x+4} \\ \implies 24x + 24 &= A(x^2 - 2x + 4) + (Bx + C)(x + 2) \\ x^2 &\implies 0 = A + B \\ x &\implies 24 = -2A + 2B + C \\ \text{const} &\implies 24 = 4A + 2C\end{aligned}$$

Substituting for B and C into the x equation gives:

$$24 = -4A + (12 - 2A) \implies A = -2$$

and we also have $B = 2$ and $C = 16$. This gives:

$$\int_0^1 \frac{24x+24}{8+x^3}\,\mathrm{d}x = \int_0^1 \frac{2x+16}{x^2-2x+4} - \frac{2}{x+2}\,\mathrm{d}x$$

We can split up the first term into two terms which are integrable:

$$\begin{aligned}\int_0^1 \frac{2x+16}{x^2-2x+4} - \frac{2}{x+2}\,\mathrm{d}x &= \int_0^1 \frac{2x-2}{x^2-2x+4} + \frac{18}{x^2-2x+4} - \frac{2}{x+2}\,\mathrm{d}x \\ &= \int_0^1 \frac{2x-2}{x^2-2x+4} + \frac{18}{(x-1)^2+3} - \frac{2}{x+2}\,\mathrm{d}x\end{aligned}$$

The first term now has the form $\dfrac{f'(x)}{f(x)}$ and the second can be found using a $\tan^{-1}$ substitution, which you can either do by deriving the result or by just quoting it. We have:

$$\begin{aligned}
\int_0^1 \frac{2x-2}{x^2-2x+4}\,dx &= \big[\ln|x^2-2x+4|\big]_0^1 \\
&= \ln 3 - \ln 4 \\
\int_0^1 \frac{18}{(x-1)^2+3}\,dx &= \left[\frac{18}{\sqrt{3}}\tan^{-1}\left(\frac{x-1}{\sqrt{3}}\right)\right]_0^1 \\
&= 0 - 6\sqrt{3}\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = 6\sqrt{3}\times\frac{\pi}{6} \\
\int_0^1 \frac{2}{x+2}\,dx &= 2\big[\ln|x+2|\big]_0^1 \\
&= 2(\ln 3 - \ln 2)
\end{aligned}$$

and so we have:

$$\begin{aligned}
\int_0^1 \frac{24x+24}{8+x^3}\,dx &= (\ln 3 - \ln 4) + \pi\sqrt{3} - 2(\ln 3 - \ln 2) \\
&= \pi\sqrt{3} - \ln 3
\end{aligned}$$

Question 3

3 The unit circle is the circle with radius 1 and centre the origin, O.

N and P are distinct points on the unit circle. N has coordinates $(-1, 0)$, and P has coordinates $(\cos\theta, \sin\theta)$, where $-\pi < \theta < \pi$. The line NP intersects the y-axis at Q, which has coordinates $(0, q)$.

(i) Show that $q = \tan\frac{1}{2}\theta$.

(ii) In this part, $q \neq 1$.

(a) Let $\mathrm{f}_1(q) = \dfrac{1+q}{1-q}$. Show that $\mathrm{f}_1(q) = \tan\frac{1}{2}\left(\theta + \frac{1}{2}\pi\right)$.

(b) Let Q_1 be the point with coordinates $(0, \mathrm{f}_1(q))$ and P_1 be the point of intersection (other than N) of the line NQ_1 and the unit circle. Describe geometrically the relationship between P and P_1.

(iii) **(a)** P_2 is the image of P under an anti-clockwise rotation about O through angle $\frac{1}{3}\pi$. The line NP_2 intersects the y-axis at the point Q_2 with coordinates $(0, \mathrm{f}_2(q))$. Find $\mathrm{f}_2(q)$ in terms of q, for $q \neq \sqrt{3}$.

(b) In this part, $q \neq -1$. Let $\mathrm{f}_3(q) = \dfrac{1-q}{1+q}$, let Q_3 be the point with coordinates $(0, \mathrm{f}_3(q))$ and let P_3 be the point of intersection (other than N) of the line NQ_3 and the unit circle. Describe geometrically the relationship between P and P_3.

(c) In this part, $0 < q < 1$. Let $\mathrm{f}_4(q) = \mathrm{f}_2^{-1}\Big(\mathrm{f}_3\big(\mathrm{f}_2(q)\big)\Big)$, let Q_4 be the point with coordinates $(0, \mathrm{f}_4(q))$ and let P_4 be the point of intersection (other than N) of the line NQ_4 and the unit circle. Describe geometrically the relationship between P and P_4.

Examiner's report

This question was attempted by approximately three-quarters of the candidates, but only a few were able to achieve a fully complete solution to the question. This question was one where a diagram was very helpful and approaches that were supported by geometrical understanding were generally more successful than attempts that relied solely on algebra.

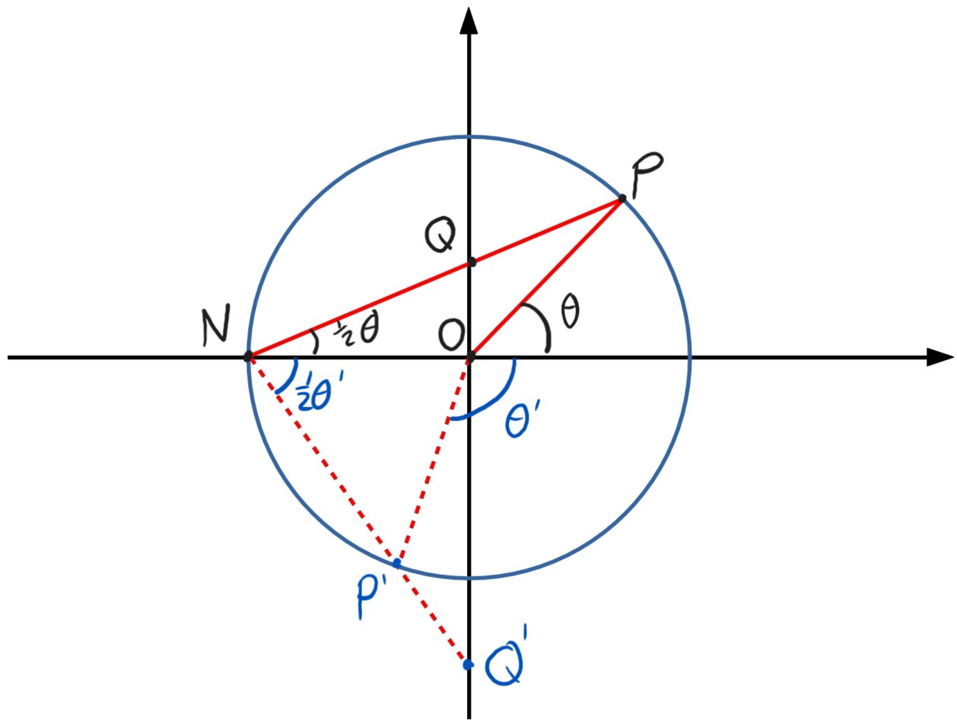
Part **(i)** was very well done, but candidates who used geometrical arguments generally did not address cases not covered by their diagram, usually this was the case where the value of θ was negative.

Part **(ii)** was also done well, but some candidates failed to give enough working to support their answer in **(a)**, which is very important in questions where the answer is given. Similarly, in **(b)** a number of candidates did not show clearly how they interpreted their algebraic work to reach a geometrical description.

Part **(iii)** was found to be difficult by a large number of candidates. Part **(a)** was generally done well, although some care with the algebra and exact trigonometric values was needed. Many were then unable to identify a relationship between f_3 and the previously seen functions and did not reach a correct geometrical description in words. A small number did well on part **(c)** and were able to interpret the inverse of function f_2 geometrically, but very few reached a fully simplified geometrical transformation.

Solution

(i) From the "angle at the circumference is half the angle at the centre" theorem we have $\angle QNO = \angle PNO = \frac{1}{2}\theta$, and this holds for all values $-\pi < \theta < \pi$. The diagram below shows two examples.



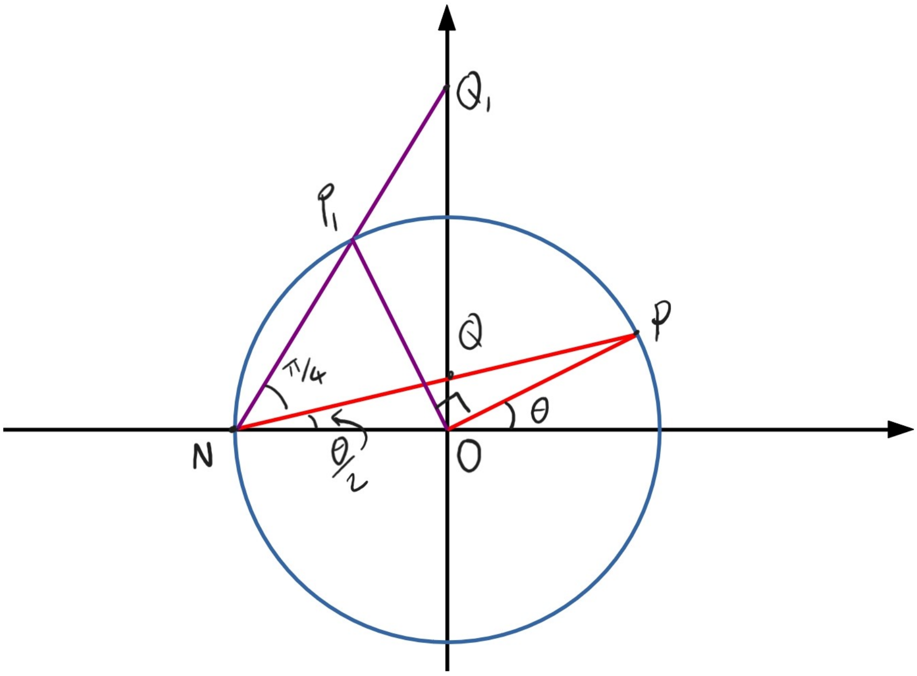
If θ is positive then we have $OQ = ON \tan\left(\frac{1}{2}\theta\right) = \tan\left(\frac{1}{2}\theta\right)$ (as we have $ON = 1$).

If θ is negative then we have $OQ = -\tan\left(\frac{1}{2}|\theta|\right) = \tan\left(\frac{1}{2}\theta\right)$ as $-|\theta| = \theta$ when θ is negative.

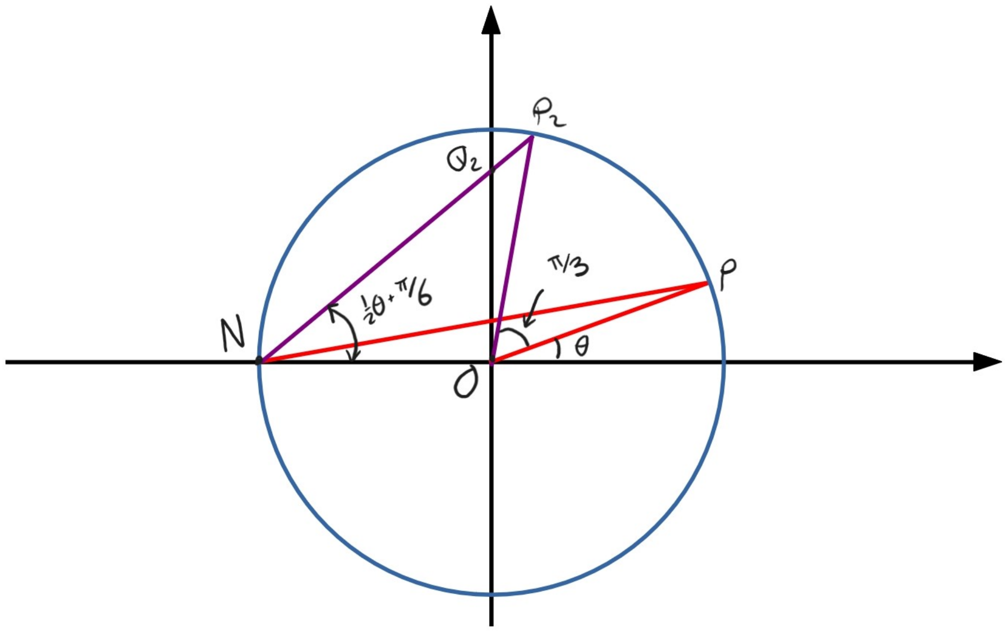
(ii) **(a)** We have:

$$\begin{aligned} f_1(q) &= \frac{1+q}{1-q} \\ &= \frac{1+\tan\left(\frac{1}{2}\theta\right)}{1-\tan\left(\frac{1}{2}\theta\right)} \\ &= \frac{\tan\left(\frac{1}{4}\pi\right)+\tan\left(\frac{1}{2}\theta\right)}{1-\tan\left(\frac{1}{4}\pi\right)\tan\left(\frac{1}{2}\theta\right)} \\ &= \tan\left(\frac{1}{4}\pi+\frac{1}{2}\theta\right) \\ &= \tan\left(\frac{1}{2}\left(\theta+\frac{1}{2}\pi\right)\right) \end{aligned}$$

(b) We have $\angle ONQ_1 = \frac{1}{2}\theta + \frac{1}{4}\pi$. This means that the angle OP_1 makes with the positive x axis is $\theta + \frac{1}{2}\pi$, i.e. P_1 is a rotation of P by $\frac{1}{2}\pi$ radians anticlockwise about the origin.



(iii) (a) The diagram below shows P and P_2.



We have:

$$\begin{aligned} OQ_2 &= \tan\left(\tfrac{1}{2}\theta + \tfrac{1}{6}\pi\right) \\ &= \frac{\tan\left(\frac{1}{2}\theta\right) + \tan\left(\frac{1}{6}\pi\right)}{1 - \tan\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{6}\pi\right)} \\ &= \frac{q + \frac{1}{\sqrt{3}}}{1 - \frac{q}{\sqrt{3}}} \\ &= \frac{q\sqrt{3}+1}{\sqrt{3}-q} \end{aligned}$$

Note that in the question we are told that $q \neq \sqrt{3}$. This can be used as a check that your answer is reasonable!

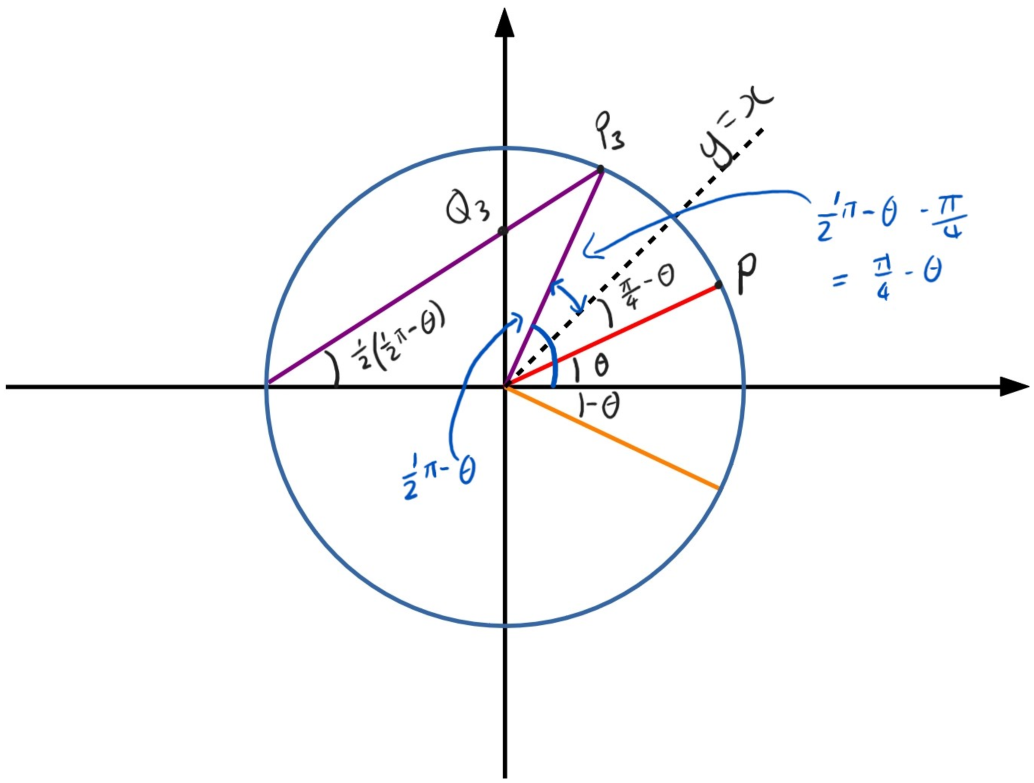
(b) We have:

$$\begin{aligned} f_3(q) &= \frac{1-q}{1+q} \\ &= \frac{1 - \tan\left(\frac{1}{2}\theta\right)}{1 + \tan\left(\frac{1}{2}\theta\right)} \\ &= \frac{\tan\left(\frac{1}{4}\pi\right) - \tan\left(\frac{1}{2}\theta\right)}{1 + \tan\left(\frac{1}{2}\theta\right)\tan\left(\frac{1}{4}\pi\right)} \\ &= \tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\theta\right) \\ &= \tan\left(\tfrac{1}{2}\left(\tfrac{1}{2}\pi - \theta\right)\right) \end{aligned}$$

Alternatively you could use $f_3(q) = f_1(-q)$, which is a little quicker.

We know the angle OP_3 makes with the positive x axis is equal to $\frac{1}{2}\pi - \theta$, and so P_3 has coordinates $\left(\cos\left(\frac{1}{2}\pi - \theta\right), \sin\left(\frac{1}{2}\pi - \theta\right)\right) = (\sin\theta, \cos\theta)$. Hence P_3 is the reflection of P in the line $y = x$.

Alternatively you could draw a sketch to show why P_3 is the reflection of P in the line $y = x$, something like the diagram below:

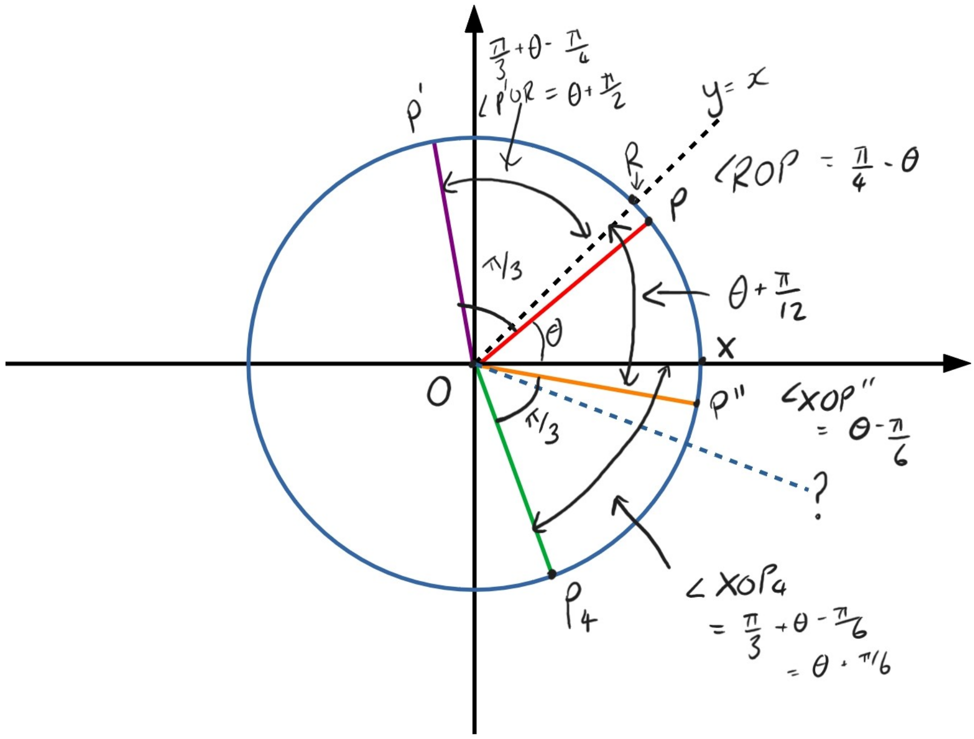


Solution continues on next page!

(c) In this part we need to combine the transformations from the previous parts. We have:

$$\begin{aligned} f_2 &: \text{Rotation anti-clockwise about } O \text{ through angle } \tfrac{1}{3}\pi \\ f_3 &: \text{Reflection in the line } y = x \\ f_2^{-1} &: \text{Rotation clockwise about } O \text{ through angle } \tfrac{1}{3}\pi \end{aligned}$$

I know that these transformations when combined are the same as a reflection, but it's not immediately clear which line you need to reflect in. To help me work this out I drew another diagram (this one is a bit messy!). All of the angles below are the magnitudes of the angles (rather than trying to show their directions as well).



From this I can see that $\angle POP_4 = 2\theta + \frac{1}{6}\pi$. Hence P_4 is a reflection of P in the line which makes an angle of $-\frac{1}{12}\pi$ with the positive x-axis.

Alternatively you could find the value of θ which would mean that $P = P_4$, which would locate the line of reflection. This would be the starting point which, after being rotated by $\frac{1}{3}\pi$ anti-clockwise, then lies on $y = x$ so doesn't move before being rotated clockwise again. This means that the angle below the positive x axis must be $\frac{1}{3}\pi - \frac{1}{4}\pi = \frac{1}{12}\pi$.
This question wanted you ti find a single transformation that takes P to P_4.

Question 4

4 In this question, if O, C and D are non-collinear points in three dimensional space, we will call the non-zero vector $\mathbf{v}$ a *bisecting vector* for angle COD if $\mathbf{v}$ lies in the plane COD, the angle between $\mathbf{v}$ and $\overrightarrow{OC}$ is equal to the angle between $\mathbf{v}$ and $\overrightarrow{OD}$, and both angles are less than $90°$.

(i) Let O, X and Y be non-collinear points in three-dimensional space, and define $\mathbf{x} = \overrightarrow{OX}$ and $\mathbf{y} = \overrightarrow{OY}$.

Let $\mathbf{b} = |\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}$.

(a) Show that $\mathbf{b}$ is a bisecting vector for angle XOY.

Explain, using a diagram, why any other bisecting vector for angle XOY is a positive multiple of $\mathbf{b}$.

(b) Find the value of λ such that the point B, defined by $\overrightarrow{OB} = \lambda\mathbf{b}$, lies on the line XY. Find also the ratio in which the point B divides XY.

(c) Show, in the case when OB is perpendicular to XY, that the triangle XOY is isosceles.

(ii) Let O, P, Q and R be points in three-dimensional space, no three of which are collinear. A bisecting vector is chosen for each of the angles POQ, QOR and ROP. Show that the three angles between them are either all acute, all obtuse or all right angles.

Examiner's report

About one third of candidates attempted this question. Many of the attempts struggled to explain the reasoning sufficiently clearly, often missing one or two important details.

In part **(i)** most candidates were able to prove that the cosines of the two angles were equal, but did not justify details such as checking that everything lies in the same plane. The most problematic part of this part of the question for candidates was **(i)(b)**. The most successful approach was to use the fact that XB and BY lie on the same line and then use algebra to calculate the value of λ and the ratio. A small number of candidates were able to achieve full marks with geometric arguments, but most such attempts did not give sufficient justification to be fully convincing. Candidates often produced successful attempts at **(i)(c)**, although again some failed to justify elements of the solution fully enough in some parts.

Part **(ii)** was found to be harder than the rest of the question. The majority who attempted this part correctly identified the dot product to be considered, but most did not recognize the symmetry within the three expressions.

Solution

Note that "non-collinear" means that the three points do not lie on a straight line. The stem of the question here is basically saying "the bisecting vector of an angle is what you think it is", but is being mathematically precise about it.

(i) **(a)** We have:

$$\begin{aligned}\mathbf{b}\cdot\mathbf{x} &= |\mathbf{x}|\big(\mathbf{x}\cdot\mathbf{y}\big) + |\mathbf{y}|\big(\mathbf{x}\cdot\mathbf{x}\big)\\ &= |\mathbf{x}|\big(\mathbf{x}\cdot\mathbf{y}\big) + |\mathbf{y}||\mathbf{x}|^2\end{aligned}$$

If we let the angle between $\mathbf{b}$ and $\mathbf{x}$ be α then we have:

$$\begin{aligned}\mathbf{b}\cdot\mathbf{x} &= |\mathbf{b}||\mathbf{x}|\cos\alpha\\ \implies \cos\alpha &= \frac{|\mathbf{x}|\big(\mathbf{x}\cdot\mathbf{y}\big) + |\mathbf{y}||\mathbf{x}|^2}{|\mathbf{b}||\mathbf{x}|}\\ &= \frac{\big(\mathbf{x}\cdot\mathbf{y}\big) + |\mathbf{y}||\mathbf{x}|}{|\mathbf{b}|}\end{aligned}$$

Similarly, if the angle between $\mathbf{b}$ and $\mathbf{y}$ is β then we have:

$$\begin{aligned}\cos\beta &= \frac{|\mathbf{x}||\mathbf{y}|^2 + |\mathbf{y}|(\mathbf{x}\cdot\mathbf{y})}{|\mathbf{b}||\mathbf{y}|}\\ &= \frac{|\mathbf{x}||\mathbf{y}| + (\mathbf{x}\cdot\mathbf{y})}{|\mathbf{b}||\mathbf{y}|}\end{aligned}$$

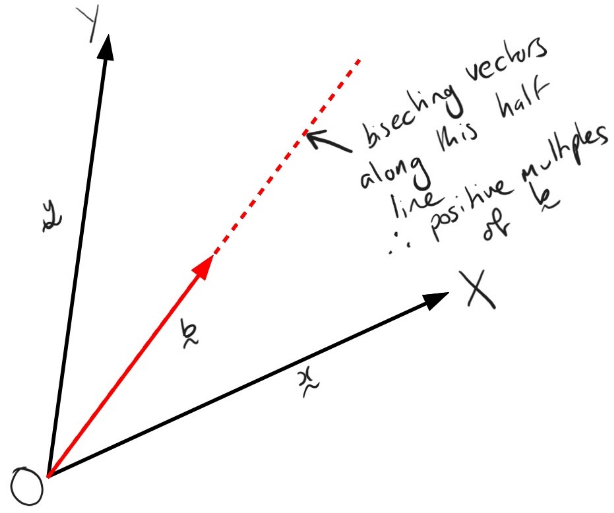
We therefore have $\cos\alpha = \cos\beta$, and since $\cos\theta$ is a one-to-one function for $0 \leqslant \theta \leqslant \pi$ we have $\alpha = \beta$. We also need to show that these angles are acute in order to say that $\mathbf{b}$ is a bisecting vector.

We have:

$$\begin{aligned}|\mathbf{x}||\mathbf{y}| + \mathbf{x}\cdot\mathbf{y} &= |\mathbf{x}||\mathbf{y}| + |\mathbf{x}||\mathbf{y}|\cos\theta\\ &= |\mathbf{x}||\mathbf{y}|(1+\cos\theta)\\ &> 0\end{aligned}$$

Where θ is the angle between $\mathbf{x}$ and $\mathbf{y}$. Note that since O, X and Y are non-collinear we have $0 < \theta < \pi$ and so $-1 < \cos\theta < 1$.

We then have $\cos\alpha = \dfrac{\big(\mathbf{x}\cdot\mathbf{y}\big) + |\mathbf{y}||\mathbf{x}|}{|\mathbf{b}|} > 0$ and so we have $0 < \alpha < \frac{\pi}{2}$ and hence α and β are acute.



(b) If B lies on the line XY then XB must be a multiple of XY which gives:

$$\begin{aligned}\lambda(|\mathbf{x}|\mathbf{y} + |\mathbf{y}|\mathbf{x}) - \mathbf{x} &= \mu(\mathbf{y} - \mathbf{x}) \\ (\lambda|\mathbf{y}| - 1 + \mu)\mathbf{x} &= (\mu - \lambda|\mathbf{x}|)\mathbf{y}\end{aligned}$$

Since $\mathbf{x}$ and $\mathbf{y}$ are not parallel (as O, X and Y are non-collinear) then the coefficients on each side must be equal to 0. This gives:

$$\begin{aligned}\lambda|\mathbf{y}| - 1 + \mu &= 0 \\ \mu - \lambda|\mathbf{x}| &= 0\end{aligned}$$

Eliminating μ gives:

$$\begin{aligned}\lambda|\mathbf{y}| - 1 + \lambda|\mathbf{x}| &= 0 \\ \implies \quad \lambda &= \frac{1}{|\mathbf{x}| + |\mathbf{y}|}\end{aligned}$$

We also have $\mu = \lambda|\mathbf{x}| = \dfrac{|\mathbf{x}|}{|\mathbf{x}| + |\mathbf{y}|}$, and so $\lambda\mathbf{b} = \dfrac{|\mathbf{x}|}{|\mathbf{x}| + |\mathbf{y}|}(\mathbf{y} - \mathbf{x})$. This means that B divides XY in the ratio:

$$\begin{gathered}\frac{|\mathbf{x}|}{|\mathbf{x}| + |\mathbf{y}|} : 1 - \frac{|\mathbf{x}|}{|\mathbf{x}| + |\mathbf{y}|} \\ \frac{|\mathbf{x}|}{|\mathbf{x}| + |\mathbf{y}|} : \frac{|\mathbf{y}|}{|\mathbf{x}| + |\mathbf{y}|} \\ |\mathbf{x}| : |\mathbf{y}|\end{gathered}$$

(c) If OB is perpendicular to XY then we have:

$$\begin{aligned}\mathbf{b}\cdot(\mathbf{y}-\mathbf{x}) &= 0\\ (|\mathbf{x}|\mathbf{y}+|\mathbf{y}|\mathbf{x})\cdot(\mathbf{y}-\mathbf{x}) &= 0\\ |\mathbf{x}||\mathbf{y}|^2+|\mathbf{y}|\mathbf{x}\cdot\mathbf{y}-|\mathbf{x}|\mathbf{x}\cdot\mathbf{y}-|\mathbf{y}||\mathbf{x}|^2 &= 0\\ (|\mathbf{y}|-|\mathbf{x}|)(|\mathbf{x}||\mathbf{y}|+\mathbf{x}\cdot\mathbf{y}) &= 0\end{aligned}$$

and since O, X and Y are non-collinear we have $|\mathbf{x}||\mathbf{y}|+\mathbf{x}\cdot\mathbf{y}\neq 0$, which means that we have $|\mathbf{x}|=|\mathbf{y}|$ and so the triangle is isosceles.

(ii) Let $\mathbf{p}=\overrightarrow{OP}$ etc.

The bisecting vector of POQ is $|\mathbf{p}|\mathbf{q}+|\mathbf{q}|\mathbf{p}$ and the bisecting vector of QOR is $|\mathbf{q}|\mathbf{r}+|\mathbf{r}|\mathbf{q}$. The angle between these vectors is given by:

$$\begin{aligned}\cos\theta &= \frac{\big(|\mathbf{p}|\mathbf{q}+|\mathbf{q}|\mathbf{p}\big)\cdot\big(|\mathbf{q}|\mathbf{r}+|\mathbf{r}|\mathbf{q}\big)}{\big||\mathbf{p}|\mathbf{q}+|\mathbf{q}|\mathbf{p}\big|\times\big||\mathbf{q}|\mathbf{r}+|\mathbf{r}|\mathbf{q}\big|}\\ &= \frac{|\mathbf{p}||\mathbf{q}|\mathbf{q}\cdot\mathbf{r}+|\mathbf{p}||\mathbf{r}|\mathbf{q}\cdot\mathbf{q}+|\mathbf{q}||\mathbf{q}|\mathbf{p}\cdot\mathbf{r}+|\mathbf{q}||\mathbf{r}|\mathbf{p}\cdot\mathbf{q}}{\big||\mathbf{p}|\mathbf{q}+|\mathbf{q}|\mathbf{p}\big|\times\big||\mathbf{q}|\mathbf{r}+|\mathbf{r}|\mathbf{q}\big|}\\ &= \frac{|\mathbf{q}|\Big(|\mathbf{p}|\mathbf{q}\cdot\mathbf{r}+|\mathbf{p}||\mathbf{r}||\mathbf{q}|+|\mathbf{q}|\mathbf{p}\cdot\mathbf{r}+|\mathbf{r}|\mathbf{p}\cdot\mathbf{q}\Big)}{\big||\mathbf{p}|\mathbf{q}+|\mathbf{q}|\mathbf{p}\big|\times\big||\mathbf{q}|\mathbf{r}+|\mathbf{r}|\mathbf{q}\big|}\end{aligned}$$

Note that $|\mathbf{p}||\mathbf{r}||\mathbf{q}|+|\mathbf{p}|\mathbf{q}\cdot\mathbf{r}+|\mathbf{q}|\mathbf{p}\cdot\mathbf{r}+|\mathbf{r}|\mathbf{p}\cdot\mathbf{q}$ (i.e. the bit in big brackets in the numerator) is symmetrical in $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, that is if you swap any two of them you get the same result. This means that the corresponding results for the cosine of the angles between the other bisecting vectors will also have the same factor in them. All the other factors in $\cos\theta$ are strictly positive (and this will hold for $\cos\phi$ and $\cos\psi$ as well, where ϕ and ψ are the angles between the other pairs of bisecting vectors). This means that the signs of $\cos\theta$, $\cos\phi$ and $\cos\psi$ must all be the same and we have:

$$\begin{aligned}\cos\theta,\cos\phi,\cos\psi>0 &\implies \text{all three angles are acute}\\ \cos\theta,\cos\phi,\cos\psi<0 &\implies \text{all three angles are obtuse}\\ \cos\theta=\cos\phi=\cos\psi=0 &\implies \text{all three angles are right angles}\end{aligned}$$

Question 5

5 **(i)** The functions f_1 and F_1, each with domain $\mathbb{Z}$, are defined by

$$\mathrm{f}_1(n) = n^2 + 6n + 11,$$

$$\mathrm{F}_1(n) = n^2 + 2.$$

Show that F_1 has the same range as f_1.

(ii) The function g_1, with domain $\mathbb{Z}$, is defined by

$$\mathrm{g}_1(n) = n^2 - 2n + 5.$$

Show that the ranges of f_1 and g_1 have empty intersection.

(iii) The functions f_2 and g_2, each with domain $\mathbb{Z}$, are defined by

$$\mathrm{f}_2(n) = n^2 - 2n - 6,$$

$$\mathrm{g}_2(n) = n^2 - 4n + 2.$$

Find any integers that lie in the intersection of the ranges of the two functions.

(iv) Show that $p^2 + pq + q^2 \geqslant 0$ for all real p and q.

The functions f_3 and g_3, each with domain $\mathbb{Z}$, are defined by

$$\mathrm{f}_3(n) = n^3 - 3n^2 + 7n,$$

$$\mathrm{g}_3(n) = n^3 + 4n - 6.$$

Find any integers that lie in the intersection of the ranges of the two functions.

Examiner's report

This was the second most popular question, but many candidates did not recognise the significance of the domain being used for the functions in this question. This meant that many applied techniques relevant to functions where the domain is the set of real numbers and therefore reached incorrect answers.

The first requirement of part **(iv)** was not affected by this misunderstanding and most candidates were able to prove the required result successfully, usually by completing the square.

A number of good solutions were seen, however. Those who completed the square and used the difference of two squares were often successful in parts **(i)**, **(ii)** and **(iii)**. Some of these were also able to make progress on the end of part **(iv)**.

Solution

It is quite easy to skip over the fact that these functions are only defined on the integers! The functions are described in terms on n to help remind you of this, but even so it is an easy mistake to make (which I made myself on the first attempt).

(i) Completing the square gives:

$$f_1(n) = (n+3)^2 + 2$$

Which has the same form as $F_1(N) = N^2+2$ with $n+3 = N$. This means that both functions have as their range the square numbers plus 2.

(ii) We have:

$$\begin{aligned} f_1(n) &= (n+3)^2 + 2 \\ g_1(n) &= (n-1)^2 + 4 \end{aligned}$$

If there is going to be an integer in common in both the ranges of f_1 and g_1 then we need:

$$(s+3)^2 + 2 = (t-1)^2 + 4 \implies (s+3)^2 = (t-1)^2 + 2$$

But there are no square numbers which differ by 2 (the smallest gap between two different square number is 3), and so you cannot find integer values s and t which satisfy this. Hence there are no values which appear in both the range of f_1 and in the range of g_1.

(iii) In this case we have:

$$\begin{aligned} f_2(n) &= (n-1)^2 - 7 \\ g_2(n) &= (n-2)^2 - 2 \end{aligned}$$

So if we are going to find s and t such that $f_2(s) = g_2(t)$ then we need:

$$\begin{aligned} (s-1)^2 &= (t-2)^2 + 5 \\ (s-1)^2 - (t-2)^2 &= 5 \\ \big[(s-1)+(t-2)\big]\big[(s-1)-(t-2)\big] &= 5 \\ \big[s+t-3\big]\big[s-t+1\big] &= 5 \end{aligned}$$

Since s and t are integers there are 4 possible cases depending on how 5 is written as a product of two integers.

Case 1

$$\begin{aligned} &s+t-3 = 5 \\ &s-t+1 = 1 \\ &\quad\implies s = 4, t = 4 \\ &\quad\text{gives } f_2(4) = g_2(4) = 2 \end{aligned}$$

Case 2

$$\begin{aligned} &s+t-3 = 1 \\ &s-t+1 = 5 \\ &\quad\implies s = 4, t = 0 \\ &\quad\text{gives } f_2(4) = g_2(0) = 2 \end{aligned}$$

Case 3

$$\begin{aligned} s+t-3 &= -5 \\ s-t+1 &= -1 \\ &\implies s=-2, t=0 \\ &\text{gives } \mathrm{f}_2(-2)=\mathrm{g}_2(0)=2 \end{aligned}$$

Case 4

$$\begin{aligned} s+t-3 &= -1 \\ s-t+1 &= -5 \\ &\implies s=-2, t=4 \\ &\text{gives } \mathrm{f}_2(-2)=\mathrm{g}_2(4)=2 \end{aligned}$$

So in all these cases the only value which appears in both the range of $\mathrm{f}_2(n)$ and $\mathrm{g}_2(n)$ is 2.

(iv) We can write:

$$p^2+pq+q^2=(p+q)^2-pq$$

and so if $pq \leqslant 0$ we have $p^2+pq+q^2 \geqslant 0$.

Also:

$$p^2+pq+q^2=(p-q)^2+3pq$$

so if $pq \geqslant 0$ we have $p^2+pq+q^2 \geqslant 0$. So since pq is either greater than equal to 0 or less than or equal to zero we must have $p^2+pq+q^2 \geqslant 0$.

Alternatively you can double p^2+pq+q^2 and use:

$$2(p^2+pq+q^2)=(p+q)^2+p^2+q^2 \geqslant 0$$

which is perhaps a little neater!

This first request feels a bit disconnected from the next part, but I expect it will be useful in some way or another! In the previous parts we were completing the square, so here it might be an idea to "complete the cube".

We have

$$\begin{aligned} \mathrm{f}_3(n) &= n^3-3n^3+7n \\ &= (n-1)^3+4n+1 \end{aligned}$$

If we equate $\mathrm{f}_3(s)$ and $\mathrm{g}_3(t)$ we have:

$$\begin{aligned} (s-1)^3+4s+1 &= t^3+4t-6 \\ (s-1)^3-t^3 &= 4t-4s-7 \\ \left[(s-1)-t\right]\left[(s-1)^2+(s-1)t+t^2\right] &= 4t-4s-7 \end{aligned}$$

Where the last line uses the difference of two cubes formula $A^3-B^3=(A-B)(A^2+AB+B^2)$ - which looks like the first result from this part!

Rearranging gives:

$$\left[s-t-1\right]\left[(s-1)^2+(s-1)t+t^2\right]+4(s-t-1)=-11$$
$$\left[s-t-1\right]\left[(s-1)^2+(s-1)t+t^2+4\right]=-11$$

From earlier we know that $(s-1)^2+(s-1)t+t^2 \geqslant 0$, and so $(s-1)^2+(s-1)t+t^2+4 \geqslant 4$. Hence the only possible factorisation of -11 is -1×11, which gives:

$$s-t-1=-1 \implies s=t$$
$$(s-1)^2+(s-1)t+t^2+4=11$$
$$(t-1)^2+(t-1)t+t^2=7$$
$$3t^2-3t-6=0$$
$$3(t+1)(t-2)=0$$

and so we have $s=t=-1$ or $s=t=2$ which gives $\mathrm{f}_3(-1)=\mathrm{g}_3(-1)=-11$ and $\mathrm{f}_3(2)=\mathrm{g}_3(2)=10$ as the only two common values in the ranges of $\mathrm{f}_3(n)$ and $\mathrm{g}_3(n)$.

Note that you would get the same answers if you equated $\mathrm{f}_3(n)$ and $\mathrm{g}_3(n)$ but this is not a valid method as you don't know that the inputs into the functions must be the same in order to get the same outputs.

Note also that the command "Find any" means find (with justification) all of the integers that lie in the intersection and explain why there are no others.

Question 6

6 *In this question, you need not consider issues of convergence.*

(i) The sequence T_n, for $n = 0, 1, 2, \ldots$, is defined by $T_0 = 1$ and, for $n \geqslant 1$, by

$$T_n = \frac{2n-1}{2n} T_{n-1}.$$

Prove by induction that

$$T_n = \frac{1}{2^{2n}} \binom{2n}{n},$$

for $n = 0, 1, 2, \ldots$.

[Note that $\binom{0}{0} = 1$.]

(ii) Show that in the binomial series for $(1-x)^{-\frac{1}{2}}$,

$$(1-x)^{-\frac{1}{2}} = \sum_{r=0}^{\infty} a_r x^r,$$

successive coefficients are related by

$$a_r = \frac{2r-1}{2r} a_{r-1}$$

for $r = 1, 2, \ldots$.

Hence prove that $a_r = T_r$ for all $r = 0, 1, 2, \ldots$.

(iii) Let b_r be the coefficient of x^r in the binomial series for $(1-x)^{-\frac{3}{2}}$, so that

$$(1-x)^{-\frac{3}{2}} = \sum_{r=0}^{\infty} b_r x^r.$$

By considering $\dfrac{b_r}{a_r}$, find an expression involving a binomial coefficient for b_r, for $r = 0, 1, 2, \ldots$.

(iv) By considering the product of the binomial series for $(1-x)^{-\frac{1}{2}}$ and $(1-x)^{-1}$, prove that

$$\frac{(2n+1)}{2^{2n}} \binom{2n}{n} = \sum_{r=0}^{n} \frac{1}{2^{2r}} \binom{2r}{r},$$

for $n = 1, 2, \ldots$.

Examiner's report

A large number of attempts at this question were seen and many candidates were able to produce good solutions.

Part **(i)** was answered well, although a number of candidates did not select the correct base case. Several solutions did not give sufficient detail in the proof to gain full credit however. Since the required form is known, it is important that steps in the solution are shown clearly.

In part **(ii)** many candidates started by calculating some of the terms and then attempted to spot a pattern, or match the terms to the desired result. To gain full credit a solution that showed the form of the general term of the binomial series expansion was required. In a small number of cases combinatorial coefficients with non-integer arguments were used, but no explanation was given of the meaning of this notation.

In part **(iii)** a number of candidates again did not write down the binomial coefficients and used pattern spotting. In general, candidates who had successfully answered part **(ii)** were also successful in part **(iii)**. Some good solutions to part **(iv)** were seen, but often there was insufficient detail in the solutions to gain full credit.

Solution

(i) When $n = 0$ we are given $T_0 = 1$ and substituting this into the required result gives:

$$\frac{1}{2^{2\times 0}}\binom{2\times 0}{0} = \frac{1}{2^0}\binom{0}{0} = 1$$

and so the result is correct when $n = 0$.

Assume the result is true when $n = k$, so we have $T_k = \dfrac{1}{2^{2k}}\dbinom{2k}{k}$. Consider $n = k+1$:

$$\begin{aligned}
T_{k+1} &= \frac{2(k+1)-1}{2(k+1)}T_k \\
&= \frac{2(k+1)-1}{2(k+1)} \times \frac{1}{2^{2k}}\binom{2k}{k} \\
&= \frac{2k+1}{2(k+1)} \times \frac{1}{2^{2k}} \times \frac{(2k)!}{k!k!} \\
&= \frac{2k+2}{2k+2} \times \frac{2k+1}{2(k+1)} \times \frac{1}{2^{2k}} \times \frac{(2k)!}{k!k!} \\
&= \frac{1}{2\times 2\times 2^{2k}} \times \frac{(2k+2)!}{(k+1)!(k+1)!} \\
&= \frac{1}{2^{2k+2}} \times \frac{(2(k+1))!}{(k+1)!(k+1)!} \\
&= \frac{1}{2^{2(k+1)}}\binom{2(k+1)}{k+1}
\end{aligned}$$

which is the required result.

(ii) We have:

$$(1-x)^{-\frac{1}{2}} = 1 + \left(-\tfrac{1}{2}\right)(-x) + \tfrac{1}{2!}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)(-x)^2 + \tfrac{1}{3!}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)(-x)^3 + \cdots + \tfrac{(-1)^n}{n!}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\cdots\left(-\tfrac{2n-1}{2}\right)x^n + \cdots$$

Therefore we have:

$$a_r = a_{r-1} \times \frac{-1}{r}\left(-\frac{2r-1}{2}\right)$$
$$a_r = \frac{2r-1}{2r}a_{r-1}$$

This is the same relationship as seen in part **(i)**. We have $a_0 = 1 = T_0$ and so $a_r = T_r$ for all $r = 0, 1, 2, ...$

(iii) We have:

$$\begin{aligned} b_r &= \frac{(-1)^r}{r!}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2r+1}{2}\right) \\ &= \frac{1}{r!}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\cdots\left(\frac{2r+1}{2}\right) \\ &= (2r+1)a_r \end{aligned}$$

and so we have:

$$b_r = \frac{2r+1}{2^{2r}}\binom{2r}{r}$$

(iv) We have $(1-x)^{-\frac{1}{2}} \times (1-x)^{-1} = (1-x)^{-\frac{3}{2}}$. We also have $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$. Using the expansions we have:

$$(a_0 + a_1x + a_2x^2 + \cdots + a_rx^r + \cdots)(1 + x + x^2 + \cdots) = (b_0 + b_1x + b_2x^2 + \cdots + b_rx^r + \cdots)$$

Considering the coefficient of x^n on both sides then:

$$b_n = a_0 + a_1 + a_2 + \cdots + a_n$$
$$\frac{2n+1}{2^{2n}}\binom{2n}{n} = \sum_{r=0}^{n} \frac{1}{2^{2r}}\binom{2r}{r}$$

Question 7

7 **(i)** Sketch the curve C_1 with equation

$$\big(y^2 + (x-1)^2 - 1\big)\big(y^2 + (x+1)^2 - 1\big) = 0.$$

(ii) Consider the curve C_2 with equation

$$\big(y^2 + (x-1)^2 - 1\big)\big(y^2 + (x+1)^2 - 1\big) = \tfrac{1}{16}.$$

(a) Show that the line $y = k$ meets the curve C_2 at points for which

$$x^4 + 2(k^2-2)x^2 + \big(k^4 - \tfrac{1}{16}\big) = 0.$$

Hence determine the number of intersections between curve C_2 and the line $y = k$ for each positive value of k.

(b) Determine whether the points on curve C_2 with the greatest possible y-coordinate are further from, or closer to, the y-axis than those on curve C_1.

(c) Show that it is not possible for both $y^2 + (x-1)^2 - 1$ and $y^2 + (x+1)^2 - 1$ to be negative, and deduce that curve C_2 lies entirely outside curve C_1.

(d) Sketch the curves C_1 and C_2 on the same axes.

Examiner's report

While there were many attempts at this question, most did not achieve very high marks.

The sketch of the graph in part **(i)** needed to be clear that it was two circles each with a radius of 1, for example by stating that points on the curve must satisfy the equation of one of the circles, or by marking the centres of the circles and having the correct radius in each case.

Most candidates were able to reach the required result at the start of part **(ii)(a)**. Candidates often missed important features of the analysis in the remainder of this part, usually believing that the conclusions derived from considering the discriminant of the quadratic in x^2 was sufficient to analyse the number of roots in each case.

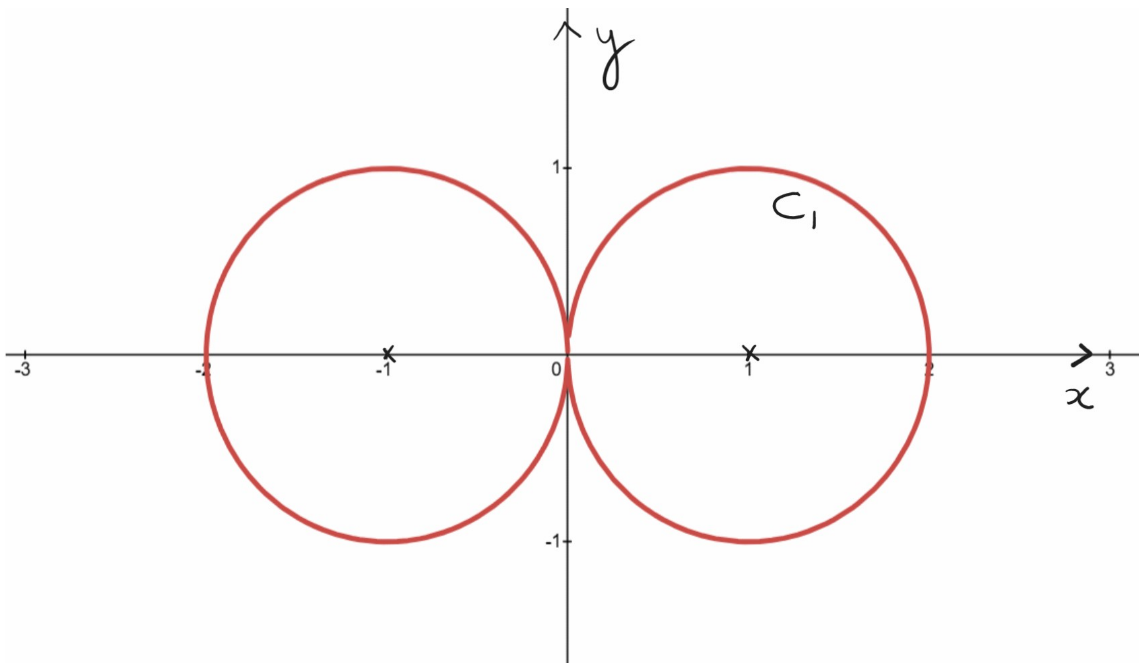
In part **(ii)(b)** candidates were able to use their results from **(ii)(a)** to find the maximum value for y, but many struggled to obtain the correct value for x or incorrectly considered distances from the x-axis rather than from the y-axis.

In **(ii)(c)** candidates often struggled to explain their reasoning clearly enough to give a fully convincing answer. In general candidates were comfortable in using algebra to prove that the two brackets cannot be negative at the same time and many were able to explain why this would mean that they must both be positive, but found it difficult to explain the significance of this in terms of the graph.

In part **(ii)(d)** candidates often produced a graph that contained some of the features that had been deduced earlier in the question. Many solutions did not show intersections with axes or co-ordinates of maximum and minimum points.

Solution

(i) We either have $y^2 + (x-1)^2 - 1 = 0 \implies y^2 + (x-1)^2 = 1$ or $y^2 + (x+1)^2 - 1 = 0 \implies y^2 + (x+1)^2 = 1$. Hence the curve is two circles with radius 1, one with centre $(-1, 0)$ and once with centre $(0, 1)$.



(ii) Before we start the question parts, here are a couple of thoughts. We know that C_2 cannot intersect with C_1 (as both equations cannot hold at the same time). If we take $x = 0$ the equation becomes $y^4 = \frac{1}{16}$, i.e. $y = \pm\frac{1}{2}$. This is enough to justify that C_2 lies "outside" C_1, but in part **(c)** we are told to deduce this following a different method, so we must use the prescribed method when we get to this point.

(a) Substituting $y = k$ gives:

$$\begin{aligned}
\left(k^2 + (x-1)^2 - 1\right)\left(k^2 + (x+1)^2 - 1\right) &= \tfrac{1}{16} \\
k^4 + k^2(x+1)^2 - k^2 + (x-1)^2k^2 - k^2 + (x-1)^2(x+1)^2 - (x-1)^2 - (x+1)^2 + 1 &= \tfrac{1}{16} \\
k^4 + k^2\left(x^2 + 2x + 1 + x^2 - 2x + 1 - 2\right) + (x^2-1)^2 - \left(x^2 - 2x + 1 + x^2 + 2x + 1\right) + 1 &= \tfrac{1}{16} \\
k^4 + 2k^2x^2 + x^4 - 2x^2 + 1 - 2x^2 - 2 + 1 &= \tfrac{1}{16} \\
x^4 + x^2\left(2k^2 - 4\right) + k^4 &= \tfrac{1}{16} \\
x^4 + 2(k^2-2)x^2 + \left(k^4 - \tfrac{1}{16}\right) &= 0
\end{aligned}$$

The number of solutions of this quartic equation will tell use how many intersections between C_2 and $y = k$ there are.
Solving for x^2 we have:

$$\begin{aligned} x^2 &= \frac{2(2-k^2) \pm \sqrt{4(k^2-2)^2 - 4\left(k^4 - \frac{1}{16}\right)}}{2} \\ &= (2-k^2) \pm \sqrt{(k^2-2)^2 - \left(k^4 - \frac{1}{16}\right)} \\ &= (2-k^2) \pm \sqrt{k^4 - 4k^2 + 4 - k^4 + \frac{1}{16}} \\ &= (2-k^2) \pm \sqrt{\frac{65}{16} - 4k^2} \end{aligned}$$

Remember that we need x^2 to be positive!

If $k^2 > \frac{65}{64}$ then there will be no real values of x^2, and so no solutions.

If $k^2 = \frac{65}{64}$ then we have $x^2 = 2 - \frac{65}{64} = \frac{63}{64}$ and so there are exactly two roots for x.

If $k^2 < \frac{65}{64}$ then the bit under the square root is positive. The larger root for x^2 will be positive so there will be at least two roots for x^2, but the smaller root might be zero, or negative.
If the smaller root is equal to zero then we have:

$$\begin{aligned} (2-k^2) - \sqrt{\tfrac{65}{16} - 4k^2} &= 0 \\ (2-k^2) &= \sqrt{\tfrac{65}{16} - 4k^2} \\ 4 - 4k^2 + k^4 &= \tfrac{65}{16} - 4k^2 \\ k^4 &= \tfrac{65}{16} - 4 \\ k^4 &= \tfrac{1}{16} \\ k^2 &= \tfrac{1}{4} \end{aligned}$$

If the smaller root is zero then we get one root from $x^2 = 0$, plus the other two from the positive value of x^2 making exactly three roots.
Overall we have:

$$\text{Number of roots} = \begin{cases} 2 \text{ for} & 0 \leqslant k^2 < \frac{1}{4} \\ 3 \text{ for} & k^2 = \frac{1}{4} \\ 4 \text{ for} & \frac{1}{4} \leqslant k^2 < \frac{65}{64} \\ 2 \text{ for} & k^2 = \frac{65}{64} \\ \text{none for} & k^2 > \frac{65}{64} \end{cases}$$

(b) The greatest y-coordinate happens when $k = \frac{\sqrt{65}}{8}$. At this point we have $x^2 = \frac{63}{64}$, and so the distance of from these points from the y-axis is $\frac{\sqrt{63}}{8} < 1$.

For curve C_1 the greatest y-coordinates are equal to 1 and occur when $x = \pm 1$, so the greatest y-coordinates are closer to the y-axis for C_2.

(c) If $y^2 + (x-1)^2 - 1$ is negative then we have:

$$y^2 + (x-1)^2 = 1 - c$$

for some $c > 0$. This means that a point (x, y) on this curve lies on a circle centre $(1, 0)$ and the circle has radius $\sqrt{1-c} < 1$. Therefore all of the x-coordinates of points on this curve are positive.

Similarly if $y^2 + (x+1)^2 - 1$ is negative then a point on this curve lies on a circle centre $(-1, 0)$ with radius $\sqrt{1-c'} < 1$. Therefore all of the x-coordinates of points on this curve are negative.

An x coordinate cannot be both positive and negative, so it is not possible for both $y^2 + (x-1)^2 - 1$ and $y^2 + (x+1)^2 - 1$ to be negative.

Since we have $(y^2 + (x-1)^2 - 1)(y^2 + (x-1)^2 - 1) = \frac{1}{16} > 0$ both brackets must be positive (as they cannot be negative). Hence the distance of a point on the curve from $(1, 0)$ has to be greater than one, and the distance of the point from $(-1, 0)$ has to be greater than one. Therefore C_2 lies outside C_1.

(d) Trying to sketch this graph smoothly is quite tricky (especially using a graphics tablet), but as long as the main features are clear a little wobblyness does not matter. If you sketch the graphs on Desmos you will see that the two curves are actually very close together and you will need to exaggerate the distance between them to show the key features.

The only points we have not yet looked at are the points where it crosses the x-axis. These are where $y = 0$, which gives:

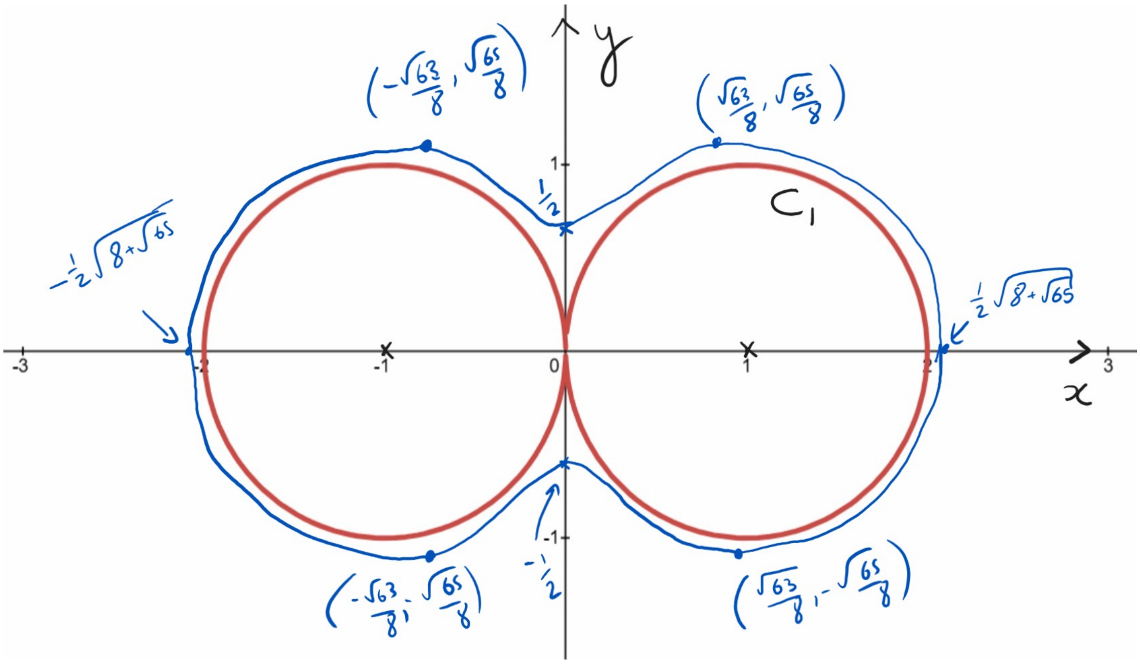
$$\begin{aligned}\big((x-1)^2 - 1\big)\big((x+1)^2 - 1\big) &= \tfrac{1}{16} \\ \big(x^2 - 2x\big)\big(x^2 + 2x\big) &= \tfrac{1}{16} \\ x^4 - 4x^2 &= \tfrac{1}{16} \\ 16x^4 - 64x^2 - 1 &= 0\end{aligned}$$

Solving for x^2 gives:

$$\begin{aligned}x^2 &= \frac{64 \pm \sqrt{64^2 + 64}}{32} \\ &= \frac{64 \pm 8\sqrt{64+1}}{32} \\ &= \frac{8 \pm \sqrt{65}}{4}\end{aligned}$$

Since x^2 is positive we must have $x^2 = \dfrac{8 + \sqrt{65}}{4}$, and so we have $x = \pm\frac{1}{2}\sqrt{8 + \sqrt{65}}$.

Putting this altogether the graph looks something like:



Question 8

8 In this question, the following theorem may be used without proof.

Let u_1, u_2, ... be a sequence of real numbers. If the sequence is

- *bounded above, so $u_n \leqslant b$ for all n, where b is some fixed number*
- *and increasing, so $u_n \leqslant u_{n+1}$ for all n*

then there is a number $L \leqslant b$ such that $u_n \to L$ as $n \to \infty$.

For positive real numbers x and y, define $\mathrm{a}(x,y) = \frac{1}{2}(x+y)$ and $\mathrm{g}(x,y) = \sqrt{xy}$. Let x_0 and y_0 be two positive real numbers with $y_0 < x_0$ and define, for $n \geqslant 0$

$$x_{n+1} = \mathrm{a}(x_n, y_n)\,,$$
$$y_{n+1} = \mathrm{g}(x_n, y_n)\,.$$

(i) By considering $\left(\sqrt{x_n} - \sqrt{y_n}\right)^2$, show that $y_{n+1} < x_{n+1}$, for $n \geqslant 0$. Show further that, for $n \geqslant 0$

- $x_{n+1} < x_n$
- $y_n < y_{n+1}$.

Deduce that there is a value M such that $y_n \to M$ as $n \to \infty$.

Show that $0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$ and hence that $x_n - y_n \to 0$ as $n \to \infty$.

Explain why x_n also tends to M as $n \to \infty$.

(ii) Let

$$\mathrm{I}(p,q) = \int_0^{\infty} \frac{1}{\sqrt{\left(p^2+x^2\right)\left(q^2+x^2\right)}}\,\mathrm{d}x,$$

where p and q are positive real numbers with $q < p$.

Show, using the substitution $t = \frac{1}{2}\left(x - \dfrac{pq}{x}\right)$ in the integral

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)\left(pq+t^2\right)}}\,\mathrm{d}t,$$

that

$$\mathrm{I}(p,q) = \mathrm{I}\big(\mathrm{a}(p,q), \mathrm{g}(p,q)\big).$$

Hence evaluate $\mathrm{I}(x_0, y_0)$ in terms of M.

Examiner's report

This question was attempted by approximately half of the candidates.

In part **(i)** candidates were often able to produce the necessary algebra, but some did not justify the strictness of the inequality or failed to use a full inductive structure for the proof. There was a roughly even split between candidates who identified that x_0 could be used as a bound in order to apply the given result and those who incorrectly used the fact that $y_n < x_n$. When considering the behaviour of $(x_n - y_n)$ some candidates incorrectly asserted that the fact that the sequence is bounded below by 0 is sufficient to show that the sequence tends to 0. Almost all candidates were able to show that this result implies that the two sequences tend to the same limit.

In part **(ii)** most candidates were able to apply the substitution, but some did not justify the new values of the limits or comment on the evenness of the integrand. The evaluation of the final integral was generally done well.

Solution

There is quite a lot of stuff in the stem of this question, but these are all definitions. They seem quite wordy as they are being precise. Basically the first bit is saying that if a sequence is increasing but bounded above then it must tend to a limit (which might or might not be equal to the known upper bound). We also have $\mathrm{a}(x, y)$ is the arithmetic mean of x and y, and $\mathrm{g}(x, y)$ is the geometric mean of x and y.

(i) We have:

$$\begin{aligned}(\sqrt{x_n} - \sqrt{y_n})^2 &= x_n + y_n - 2\sqrt{x_n y_n} \\ &= 2\mathrm{a}(x_n, y_n) - 2\mathrm{g}(x_n, y_n) \\ &= 2x_{n+1} - 2y_{n+1}\end{aligned}$$

Since $(\sqrt{x_n} - \sqrt{y_n})^2$ we have $x_{n+1} \geqslant y_{n+1}$. However we need the strict inequality. We are told that $y_0 < x_0$. If we have $y_k < x_k$ then $(\sqrt{x_k} - \sqrt{y_k})^2 = 2(x_{k+1} - y_{k+1}) \implies x_{k+1} > y_{k+1}$ and so we have $y_{n+1} < x_{n+1}$ for $n \geqslant 0$.

We have:

$$\begin{aligned}x_{n+1} &= \frac{1}{2}(x_n + y_n) \\ &< \frac{1}{2}(x_n + x_n) \\ \implies x_{n+1} &< x_n\end{aligned}$$

and

$$\begin{aligned}y_{n+1} &= \sqrt{x_n y_n} \\ &> \sqrt{y_n^2} \\ \implies y_{n+1} &> y_n\end{aligned}$$

We then have $y_n < x_n < x_0$, so the sequence y_n is bounded above by a fixed number, and we have $y_{n+1} > y_n$ so the sequence is increasing. Therefore, using the theorem from the stem of the question, there exists an $M \leqslant x_0$ such that $y_n \to M$ as $n \to \infty$.

We already have $x_{n+1} > y_{n+1}$, so we have $x_{n+1} - y_{n+1} > 0$. Using the first result from this part we also have:

$$\begin{aligned} x_{n+1} - y_{n+1} &= \tfrac{1}{2}(\sqrt{x_n} - \sqrt{y_n})^2 \\ &= \tfrac{1}{2}\big(x_n - 2\sqrt{x_n y_n} + y_n\big) \\ &< \tfrac{1}{2}\big(x_n + y_n - 2\sqrt{y_n y_n}\big) \qquad \text{as } y_n < x_n \\ &= \tfrac{1}{2}(x_n - y_n) \end{aligned}$$

Therefore we have $0 < x_{n+1} - y_{n+1} < \frac{1}{2}(x_n - y_n)$ and so the gap between values of x_k and y_k is halved each time k increases by one. So we have $x_n - y_n \to 0$ as $n \to \infty$.

Hence we must have $x_n \to M$ as $n \to \infty$.

(ii) We have:

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{\big(\frac{1}{4}(p+q)^2 + t^2\big)\big(pq + t^2\big)}}\, \mathrm{d}t = 2\int_{0}^{\infty} \frac{1}{\sqrt{\big(\frac{1}{4}(p+q)^2 + t^2\big)\big(pq + t^2\big)}}\, \mathrm{d}t$$

as the first integral is of an even function (so it has symmetry about the y axis). The integral on the RHS is the same as $2I\big(\mathrm{a}(p,q), \mathrm{g}(p,q)\big)$, as we have $\big(\mathrm{a}(p,q)\big)^2 = \frac{1}{4}(p+q)^2$ and $\big(\mathrm{g}(p,q)\big)^2 = pq$.

If we have $t = \frac{1}{2}\left(x - \frac{pq}{x}\right)$ then we have $\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}\left(1 + \frac{pq}{x^2}\right)$. As $x \to 0$ we have $t \to -\infty$ and as $x \to \infty$ we have $t \to \infty$.

It's probably best to try and simplify some of the terms in the integral rather than trying to do everything in one go. We have:

$$\begin{aligned} \frac{1}{4}(p+q)^2 + t^2 &= \frac{1}{4}(p+q)^2 + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2 \\ &= \frac{1}{4}\left(p^2 + \cancel{2pq} + q^2 + x^2 - \cancel{2pq} + \frac{(pq)^2}{x^2}\right) \\ &= \frac{1}{4x^2}\left(x^2p^2 + x^2q^2 + x^4 + (pq)^2\right) \\ &= \frac{1}{4x^2}\left(x^2 + p^2\right)\left(x^2 + q^2\right) \end{aligned}$$

Also:

$$\begin{aligned} pq + t^2 &= pq + \frac{1}{4}\left(x - \frac{pq}{x}\right)^2 \\ &= \frac{1}{4}\left(4pq + x^2 - 2pq + \frac{(pq)^2}{x^2}\right) \\ &= \frac{1}{4x^2}\left(x^4 + 2pqx^2 + (pq)^2\right) \\ &= \frac{1}{4x^2}\left(x^2 + pq\right)^2 \end{aligned}$$

Substituting all this gives:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{\left(\frac{1}{4}(p+q)^2+t^2\right)\left(pq+t^2\right)}}\,\mathrm{d}t \\
=&\int_{0}^{\infty} \frac{1}{\sqrt{\frac{1}{4x^2}\left(x^2+p^2\right)\left(x^2+q^2\right)\times\frac{1}{4x^2}\left(x^2+pq\right)^2}} \times \tfrac{1}{2}\left(1+\frac{pq}{x^2}\right)\mathrm{d}x \\
=&\int_{0}^{\infty} \frac{4x^2}{\sqrt{\left(x^2+p^2\right)\left(x^2+q^2\right)\times\left(x^2+pq\right)^2}} \times \tfrac{1}{2}\left(1+\frac{pq}{x^2}\right)\mathrm{d}x \\
=&2\int_{0}^{\infty} \frac{1}{\left(x^2+pq\right)\sqrt{\left(x^2+p^2\right)\left(x^2+q^2\right)}} \times \left(x^2+pq\right)\mathrm{d}x \\
=&2\int_{0}^{\infty} \frac{1}{\sqrt{\left(x^2+p^2\right)\left(x^2+q^2\right)}}\,\mathrm{d}x \\
=&2I(p,q)
\end{aligned}
$$

Hence we have $I\big(\mathrm{a}(p,q),\mathrm{g}(p,q)\big)=I(p,q)$.

We then have $I(x_0,y_0)=I(x_1,y_1)=I(x_2,y_2)=\cdots=I(M,M)$.

$$
\begin{aligned}
\int_0^{\infty}\frac{1}{\sqrt{(M^2+x^2)(M^2+x^2)}}\,\mathrm{d}x &= \int_0^{\infty}\frac{1}{(M^2+x^2)}\,\mathrm{d}x \\
&= \left[\frac{1}{M}\tan^{-1}\left(\frac{x}{M}\right)\right]_0^{\infty} \\
&= \frac{\pi}{2M}
\end{aligned}
$$

Question 9

9 A long straight trench, with rectangular cross section, has been dug in otherwise horizontal ground. The width of the trench is d and its depth $2d$. A particle is projected at speed v, where $v^2 = \lambda dg$, at an angle α to the horizontal, from a point on the ground a distance d from the nearer edge of the trench. The vertical plane in which it moves is perpendicular to the trench.

(i) The particle lands on the base of the trench without first touching either of its sides.

(a) By considering the vertical displacement of the particle when its horizontal displacement is d, show that $(\tan\alpha - \lambda)^2 < \lambda^2 - 1$ and deduce that $\lambda > 1$.

(b) Show also that $(2\tan\alpha - \lambda)^2 > \lambda^2 + 4(\lambda - 1)$ and deduce that $\alpha > 45°$.

(ii) Show that, provided $\lambda > 1$, α can always be chosen so that the particle lands on the base of the trench without first touching either of its sides.

Examiner's report

About one-third of the candidates attempted this question. In some cases the diagrams that were drawn showed that candidates had not taken care to fully understand the description of the situation.

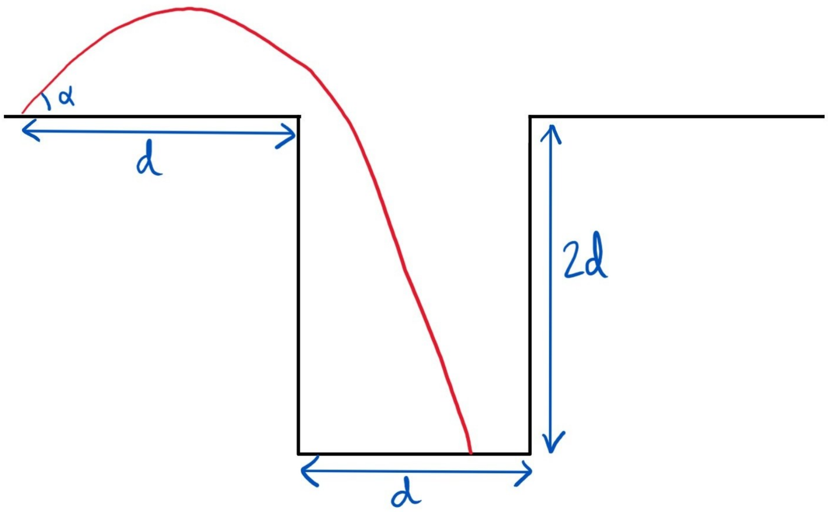
Part **(i)(a)** was completed well by many candidates, but some did not recognise the need to use a trigonometric identity to obtain a form which is a function of $\tan\alpha$ which is a common technique in questions of this form.

Part **(i)(b)** was mostly done well, although most did not score fully on the final deduction, often not applying the condition that $\lambda > 1$ correctly and reaching two cases, one of which they could not rule out.

Very few candidates produced a good solution to part **(ii)**, but those who managed to identify the correct starting point generally produced good solutions.

Solution

First thing - draw a diagram! I had to re-read the stem a few times to ensure I had the right picture.



(i) **(a)** Let the initial position of the particle be at $(0, 0)$. This horizontal displacement of the particle is given by $x = vt\cos\alpha$ and the vertical displacement is given by $y = vt\sin\alpha - \frac{1}{2}gt^2$. The time when the horizontal displacement is equal to d is given by:

$$d = vt\cos\alpha$$
$$\implies t = \frac{d}{v\cos\alpha}$$

Substituting this into the expression for the vertical displacement gives:

$$\begin{aligned} y &= vt\sin\alpha - \tfrac{1}{2}gt^2 \\ &= \frac{dv\sin\alpha}{v\cos\alpha} - \tfrac{1}{2}g\left(\frac{d}{v\cos\alpha}\right)^2 \\ &= d\tan\alpha - \frac{gd^2}{2v^2}\sec^2\alpha \\ &= d\tan\alpha - \frac{gd^2}{2v^2}(1+\tan^2\alpha) \end{aligned}$$

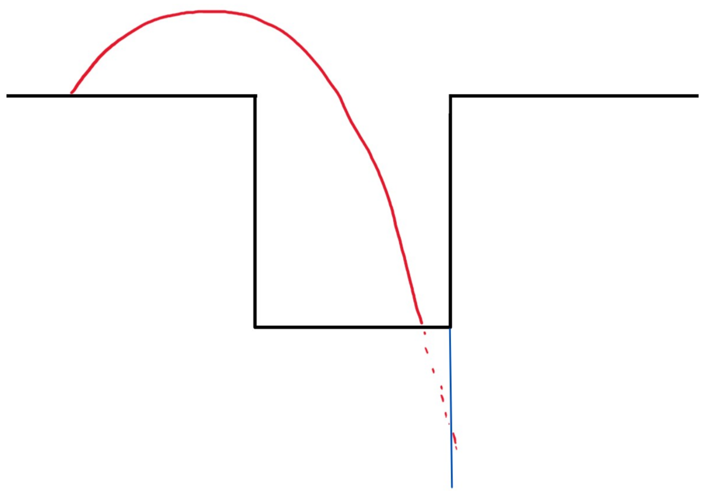
If the particle is to not hit the front edge of the trench then when $x = d$ we need $y > 0$ (i.e. it is still above ground level!). Remember that we are given $v^2 = \lambda dg$, and so $\dfrac{gd^2}{2v^2} = \dfrac{d}{2\lambda}$. This then gives:

$$\begin{aligned} d\tan\alpha - \frac{d}{2\lambda}(1+\tan^2\alpha) &> 0 \\ 2\lambda\tan\alpha - 1 - \tan^2\alpha &> 0 \\ \tan^2\alpha - 2\lambda\tan\alpha + 1 &< 0 \\ \big(\tan\alpha - \lambda\big)^2 - \lambda^2 + 1 &< 0 \\ \big(\tan\alpha - \lambda\big)^2 &< \lambda^2 - 1 \end{aligned}$$

We have $\lambda^2 - 1 > \big(\tan\alpha - \lambda\big)^2 \geqslant 0$ and so $\lambda^2 - 1 > 0 \implies \lambda^2 > 1$. From the definition of λ we know that λ must be positive and so we have $\lambda > 1$.

(b) You could tackle this question by solving when $y = -2d$, but this does mean that you have to solve a quadratic equation and the algebra gets a little more involved.

Instead imagine that the particle can pass through the floor of the trench. If the particle is going to miss the far side of the trench then when $x = 2d$ we need $y < -2d$.



When $x = 2d$ we have $t = \dfrac{2d}{v\cos\alpha}$. Substituting this into y gives:

$$\begin{aligned} y &= vt\sin\alpha - \tfrac{1}{2}gt^2 \\ &= \frac{2dv\sin\alpha}{v\cos\alpha} - \tfrac{1}{2}g\left(\frac{2d}{v\cos\alpha}\right)^2 \\ &= 2d\tan\alpha - \frac{2gd^2}{v^2}\sec^2\alpha \\ &= 2d\tan\alpha - \frac{2gd^2}{v^2}(1+\tan^2\alpha) \\ &= 2d\tan\alpha - \frac{2d}{\lambda}(1+\tan^2\alpha) \end{aligned}$$

Using $y < -2d$ gives:

$$\begin{aligned} 2d\tan\alpha - \frac{2d}{\lambda}(1+\tan^2\alpha) &< -2d \\ \lambda\tan\alpha - 1 - \tan^2\alpha &< -\lambda \\ \tan^2\alpha - \lambda\tan\alpha - \lambda + 1 &> 0 \end{aligned}$$

Completing the square last time was quite useful, so lets try that again:

$$\begin{aligned} \left(\tan\alpha - \tfrac{1}{2}\lambda\right)^2 - \tfrac{1}{4}\lambda^2 - \lambda + 1 &> 0 \\ 4\left(\tan\alpha - \tfrac{1}{2}\lambda\right)^2 - \lambda^2 - 4\lambda + 4 &> 0 \\ (2\tan\alpha - \lambda)^2 &> \lambda^2 + 4\lambda - 4 \\ (2\tan\alpha - \lambda)^2 &> \lambda^2 + 4(\lambda - 1) \end{aligned}$$

We know that $\lambda > 1$ and so $4(\lambda - 1) > 0$ and so we have:

$$\begin{aligned}(2\tan\alpha - \lambda)^2 &> \lambda^2\\ 4\tan^2\alpha - 4\lambda\tan\alpha + \lambda^2 &> \lambda^2\\ 4\tan^2\alpha - 4\lambda\tan\alpha &> 0\\ 4\tan\alpha(\tan\alpha - \lambda) &> 0\end{aligned}$$

We know that $0^\circ < \alpha < 90^\circ$ and so $\tan\alpha > 0$. This means that we must have:

$$\begin{aligned}\tan\alpha - \lambda &> 0\\ \tan\alpha &> \lambda > 1\end{aligned}$$

Therefore we have $\tan\alpha > 1$ and so $\alpha > 45^\circ$.

(ii) The condition for $y > 0$ when $x = d$ was $\tan^2\alpha - 2\lambda\tan\alpha + 1 < 0$ and the condition for $y < -2d$ was $\tan^2\alpha - \lambda\tan\alpha - \lambda + 1 > 0$.

Considering $\tan^2\alpha - 2\lambda\tan\alpha + 1 < 0$ we need:

$$\frac{2\lambda - \sqrt{4\lambda^2 - 4}}{2} < \tan\alpha < \frac{2\lambda + \sqrt{4\lambda^2 - 4}}{2}$$

Considering $\tan^2\alpha - \lambda\tan\alpha - \lambda + 1 > 0$ we need:

$$\tan\alpha > \frac{\lambda + \sqrt{\lambda^2 + 4(\lambda - 1)}}{2} \quad \text{or} \quad \tan\alpha < \frac{\lambda - \sqrt{\lambda^2 + 4(\lambda - 1)}}{2}$$

If we can show that:

$$\frac{\lambda + \sqrt{\lambda^2 + 4(\lambda - 1)}}{2} < \frac{2\lambda + \sqrt{4\lambda^2 - 4}}{2}$$

then the two ranges will overlap and we can pick a value of $\tan\alpha$ that satisfies both conditions.

We want:

$$\begin{aligned}\frac{\lambda + \sqrt{\lambda^2 + 4(\lambda - 1)}}{2} &< \frac{2\lambda + \sqrt{4\lambda^2 - 4}}{2}\\ \sqrt{\lambda^2 + 4(\lambda - 1)} &< \lambda + \sqrt{4\lambda^2 - 4}\\ \lambda^2 + 4(\lambda - 1) &< \left(\lambda + \sqrt{4\lambda^2 - 4}\right)^2 \quad \text{OK as both sides are positive}\\ \lambda^2 + 4(\lambda - 1) &< \lambda^2 + 2\lambda\sqrt{4\lambda^2 - 4} + 4\lambda^2 - 4\\ 4\lambda &< 4\lambda\sqrt{\lambda^2 - 1} + 4\lambda^2\\ 1 &< \sqrt{\lambda^2 - 1} + \lambda\end{aligned}$$

Since we know that $\lambda > 1$ and that $\sqrt{\lambda^2 - 1} > 0$, this condition is true and hence the ranges overlap and we can choose a value of α so that $\tan\alpha$ satisfies both conditions.

Similarly, we could instead have shown that

$$\frac{\lambda - \sqrt{\lambda^2 + 4(\lambda - 1)}}{2} > \frac{2\lambda - \sqrt{4\lambda^2 - 4}}{2}$$

which would mean the ranges overlap at the bottom end. However we only need to show that they overlap somewhere so you don't have to consider both "ends".

Question 10

10 A triangular prism lies on a horizontal plane. One of the rectangular faces of the prism is vertical; the second is horizontal and in contact with the plane; the third, oblique rectangular face makes an angle α with the horizontal. The two triangular faces of the prism are right angled triangles and are vertical. The prism has mass M and it can move without friction across the plane.

A particle of mass m lies on the oblique surface of the prism. The contact between the particle and the plane is rough, with coefficient of friction μ.

(i) Show that if $\mu < \tan\alpha$, then the system cannot be in equilibrium.

Let $\mu = \tan\lambda$, with $0 < \lambda < \alpha < \frac{1}{4}\pi$.

A force P is exerted on the vertical rectangular face of the prism, perpendicular to that face and directed towards the interior of the prism. The particle and prism accelerate, but the particle remains in the same position relative to the prism.

(ii) Show that the magnitude, F, of the frictional force between the particle and the prism is

$$F = \frac{m}{M+m}\left|(M+m)g\sin\alpha - P\cos\alpha\right|.$$

Find a similar expression for the magnitude, N, of the normal reaction between the particle and the prism.

(iii) Hence show that the force P must satisfy

$$(M+m)g\tan(\alpha-\lambda) \leqslant P \leqslant (M+m)g\tan(\alpha+\lambda).$$

Examiner's report

This question received the fewest attempts, although a good proportion of the attempts that were made were successful. Many candidates drew clear diagrams in their attempts at this question.

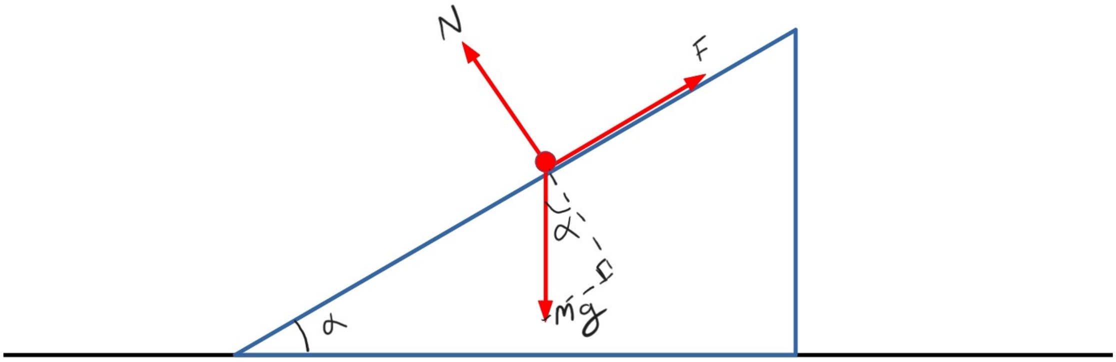
Part **(i)** was done fairly well, with most candidates resolving forces successfully. However, many candidates were not able to justify sufficiently well the situation in equilibrium as opposed to limiting equilibrium.

In part **(ii)** many candidates struggled to know how to deal with the force that acts on the prism and thought instead that it would be acting on the particle.

Those candidates who attempted part **(iii)** generally did well and many realised how to get both sides of the required inequality and were able to follow through the required manipulation.

Solution

(i) The forces on the particle are shown below:



If the particle is in equilibrium then resolving parallel and perpendicular to the slope we have:

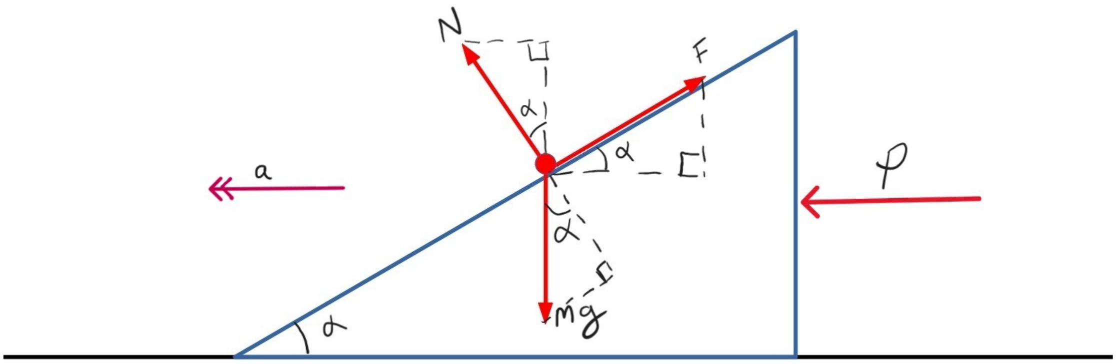
$$N = mg\cos\alpha$$
$$F = mg\sin\alpha$$

For equilibrium we need $F \leqslant \mu N$, and so the particle can be in equilibrium as long as:

$$mg\sin\alpha \leqslant \mu mg\cos\alpha \implies \tan\alpha \leqslant \mu$$

and the system cannot be in equilibrium if $\tan\alpha > \mu$.

(ii) Now we have an extra force acting on the prism. There is no frictional force between the prism and the plane.



Resolving horizontally for the whole system we have:

$$P = (M + m)a \tag{1}$$

Resolving horizontally and vertically for the particle we have:

$$ma = N\sin\alpha - F\cos\alpha \tag{2}$$
$$mg = N\cos\alpha + F\sin\alpha \tag{3}$$

Rearranging (2) gives:

$$F\cos\alpha = N\sin\alpha - ma$$

We want to eliminate N and a; we can use (3) and (1) to do this:

$$\begin{aligned}
F\cos^2\alpha &= N\cos\alpha\sin\alpha - ma\cos\alpha \\
F\cos^2\alpha &= \big(mg - F\sin\alpha\big)\sin\alpha - m\cos\alpha\times\frac{P}{M+m} \\
F\cos^2\alpha + F\sin^2\alpha &= mg\sin\alpha - \frac{Pm\cos\alpha}{M+m} \\
F &= m\left[g\sin\alpha - \frac{P\cos\alpha}{M+m}\right] \\
F &= \frac{m}{M+m}\big[(M+m)g\sin\alpha - P\cos\alpha\big]
\end{aligned}$$

Using (3) we have:

$$\begin{aligned}
N\cos\alpha &= mg - F\sin\alpha \\
&= mg - \frac{m}{M+m}\big[(M+m)g\sin\alpha - P\cos\alpha\big]\sin\alpha \\
&= mg - mg\sin^2\alpha + \frac{Pm\cos\alpha\sin\alpha}{M+m} \\
&= mg\cos^2\alpha + \frac{Pm\cos\alpha\sin\alpha}{M+m} \\
\implies N &= mg\cos\alpha + \frac{Pm\sin\alpha}{M+m} \\
&= \frac{m}{M+m}\big[(M+m)g\cos\alpha + P\sin\alpha\big]
\end{aligned}$$

(iii) If the particle is going to stay in the same place relative to the prism then we need to have:

$$-\mu N \leqslant F \leqslant \mu N$$

Using the expressions for F and N found earlier, and $\mu = \tan\lambda$, this gives:

$$\begin{aligned}
-\tan\lambda\frac{m}{M+m}\big[(M+m)g\cos\alpha + P\sin\alpha\big] \leqslant \frac{m}{M+m}\big[(M+m)g\sin\alpha - P\cos\alpha\big] \\
\leqslant \tan\lambda\frac{m}{M+m}\big[(M+m)g\cos\alpha + P\sin\alpha\big]
\end{aligned}$$

Dividing throughout by $\tan\alpha$ and $\dfrac{m}{M+m}$ gives:

$$\begin{aligned}
-\tan\lambda\big[(M+m)g + P\tan\alpha\big] \leqslant \big[(M+m)g\tan\alpha - P\big] \leqslant \tan\lambda\big[(M+m)g + P\tan\alpha\big] \\
P(1-\tan\lambda\tan\alpha) - \tan\lambda(M+m)g \leqslant (M+m)g\tan\alpha \leqslant P(1+\tan\lambda\tan\alpha) + \tan\lambda(M+m)g
\end{aligned}$$

Considering the first inequality we have:

$$\begin{aligned}
P(1-\tan\lambda\tan\alpha) - \tan\lambda(M+m)g &\leqslant (M+m)g\tan\alpha \\
P(1-\tan\lambda\tan\alpha) &\leqslant (M+m)g(\tan\alpha + \tan\lambda)
\end{aligned}$$

We know that $0 < \lambda < \alpha < \frac{1}{4}\pi$, and so $\tan\lambda\tan\alpha < 1$ and we can divide by $1 - \tan\lambda\tan\alpha$ whilst preserving the inequality sign:

$$\begin{aligned} P &\leqslant (M+m)g\frac{\tan\alpha + \tan\lambda}{1 - \tan\lambda\tan\alpha} \\ \implies P &\leqslant (M+m)g\tan(\alpha+\lambda) \end{aligned}$$

Considering the second inequality we have:

$$\begin{aligned} (M+m)g\tan\alpha &\leqslant P(1+\tan\lambda\tan\alpha) + \tan\lambda(M+m)g \\ (M+m)g(\tan\alpha - \tan\lambda) &\leqslant P(1+\tan\lambda\tan\alpha) \end{aligned}$$

We know that $0 < \lambda < \alpha < \frac{1}{4}\pi$, and so $0 < \tan\lambda\tan\alpha$ and we can divide by $1 + \tan\lambda\tan\alpha$ whilst preserving the inequality sign:

$$\begin{aligned} (M+m)g\frac{\tan\alpha - \tan\lambda}{1 + \tan\lambda\tan\alpha} &\leqslant P \\ \implies (M+m)g\tan(\alpha-\lambda) &\leqslant P \end{aligned}$$

Hence we have:

$$(M+m)g\tan(\alpha-\lambda) \leqslant P \leqslant (M+m)g\tan(\alpha+\lambda)$$

as required.

Question 11

11 **(i)** Sketch a graph of $y = x^{\frac{1}{x}}$ for $x > 0$, showing the location of any turning points.

Find the maximum value of $n^{\frac{1}{n}}$, where n is a positive integer.

N people are to have their blood tested for the presence or absence of an enzyme. Each person, independently of the others, has a probability p of having the enzyme present in a sample of their blood, where $0 < p < 1$. The blood test always correctly determines whether the enzyme is present or absent in a sample.

The following method is used.

- The people to be tested are split into r groups of size k, with $k > 1$ and $rk = N$.
- In every group, a sample from each person in that group is mixed into one large sample, which is then tested.
- If the enzyme is not present in the combined sample from a group, no further testing of the people in that group is needed.
- If the enzyme is present in the combined sample from a group, a second sample from each person in that group is tested separately.

(ii) Find, in terms of N, k and p, the expected number of tests.

(iii) Given that N is a multiple of 3, find the largest value of p for which it is possible to find an integer value of k such that $k > 1$ and the expected number of tests is at most N.

Show that this value of p is greater than $\frac{1}{4}$.

(iv) Show that, if pk is sufficiently small, the expected number of tests is approximately $N\left(\frac{1}{k} + pk\right)$.

In the case where $p = 0.01$, show that choosing $k = 10$ gives an expected number of tests which is only about 20% of N.

Examiner's report

This question received one of the lowest numbers of responses and many of the responses did not achieve many marks.

In part **(i)** candidates were generally able to complete the differentiation correctly and identify the location of the stationary point of the curve. Most were also able to identify the correct behaviour of the course as $x \to 0$, but several incorrectly believed that the function also approached 0 as $x \to \infty$. Some candidates were not able to justify why the maximum value when taking the function over integers must occur when $n = 2$ or $n = 3$. Many candidates were however able to explain clearly that the value is greater when $n = 3$ compared to $n = 2$.

Part **(ii)** proved to be difficult for many candidates, with many incorrectly calculating the probability that a combined test is found to be negative or omitting the first test when counting the number of tests required if a group did test positive.

Those candidates who had successfully solved part **(ii)** were able to produce good solutions to part **(iii)** as well. Part **(iv)** was also answered well, although several candidates did not justify the exclusion of higher order terms in the expansion of $(1-p)^k$.

Solution

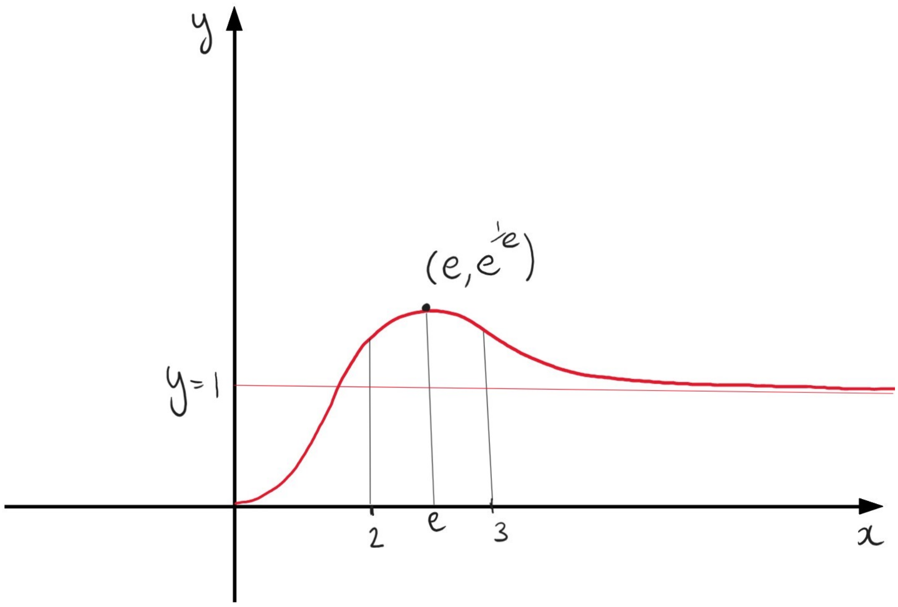
(i) We have:

$$\begin{aligned} y &= x^{\frac{1}{x}} \\ \ln y &= \frac{1}{x}\ln x \\ \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{x^2} - \frac{1}{x^2}\ln x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= y\left(\frac{1-\ln x}{x^2}\right) \end{aligned}$$

So the turning point of the graph is at $x = e, y = e^{\frac{1}{e}}$.

Let $x = \frac{1}{t}$ so $x^{\frac{1}{x}} = \left(\frac{1}{t}\right)^t = \frac{1}{t^t}$. Then as $x \to 0$ we have $t \to \infty$ and $\frac{1}{t^t} \to 0$. Hence $x^{\frac{1}{x}} \to 0$ as $x \to 0$.

We cannot use the same trick for $x \to \infty$, but we can note that $\frac{1}{x}\ln x \to 0$ as $x \to \infty$, and so $e^{\left(\frac{1}{x}\ln x\right)} = x^{\frac{1}{x}} \to e^0 = 1$ as $x \to \infty$.



This graph is a little exaggerated compared to the one that Desmos draws, but this one does make the maximum clearer!

We can see that the maximum lies between $x = 2$ and $x = 3$, so we only need to consider $2^{\frac{1}{2}}$ and $3^{\frac{1}{3}}$ to see which maximises $n^{\frac{1}{n}}$.

We have:

$$\begin{aligned} 8 &< 9 \\ 2^3 &< 3^2 \\ \left(2^3\right)^{\frac{1}{6}} &< \left(3^2\right)^{\frac{1}{6}} \\ 2^{\frac{1}{2}} &< 3^{\frac{1}{3}} \end{aligned}$$

and so the maximum value of $n^{\frac{1}{n}}$ is $3^{\frac{1}{3}}$.

(ii) The probability that no-one in a group has the enzyme is $(1-p)^k$. So the number of tests needed for one of the groups is 1 test with probability $(1-p)^k$ and $k+1$ tests with probability $1-(1-p)^k$. Therefore the expected number of tests for one group is given by:

$$1 \times (1-p)^k + (k+1) \times \left[1 - (1-p)^k\right] = k + 1 - k(1-p)^k$$

The number of expected tests for each group will be the same, so the total number of expected tests for all r groups is:

$$r\left[k + 1 - k(1-p)^k\right] = N\left[1 + \frac{1}{k} - (1-p)^k\right]$$

Note that the question asked for the expected number of tests in terms of N, k and p, so we needed to substitute $r = \dfrac{N}{k}$.

(iii) If the expected number of tests is at most N then we have:

$$N\left[1+\frac{1}{k}-(1-p)^k\right] \leqslant N$$
$$1+\frac{1}{k}-(1-p)^k \leqslant 1$$
$$\frac{1}{k} \leqslant (1-p)^k$$
$$k \geqslant \frac{1}{(1-p)^k}$$
$$k^{\frac{1}{k}} \geqslant \frac{1}{1-p}$$

From part **(i)** we know that the maximum value of $k^{\frac{1}{k}}$ occurs when $k = 3$. Therefore the maximum value of p is where:

$$\frac{1}{1-p} = 3^{\frac{1}{3}}$$
$$1-p = \frac{1}{3^{\frac{1}{3}}}$$
$$p = 1-\frac{1}{3^{\frac{1}{3}}}$$

To show that $p > \frac{1}{4}$, start by considering $p - \frac{1}{4}$:

$$p-\frac{1}{4} = \frac{3}{4}-\frac{1}{3^{\frac{1}{3}}}$$

We have $\left(\frac{3}{4}\right)^3 = \frac{27}{64} > \frac{1}{3}$ and so $\frac{3}{4} > \frac{1}{3^{\frac{1}{3}}}$. Therefore $p - \frac{1}{4} > 0$ and so $p > \frac{1}{4}$.

(iv) If we consider the expansion of $(1-p)^k$ we have:

$$(1-p)^k = 1 - pk + \frac{k(k-1)}{2!}p^2 + \cdots + (-1)^n\frac{k(k-1)\cdots(k-n+1)}{n!}p^n + \cdots$$

Note that:

$$\frac{k(k-1)\cdots(k-n+1)}{n!}p^n < \frac{k\times k\times\cdots\times k}{n!}p^n = \frac{(pk)^n}{n!}$$

and so if pk is "sufficiently small" we can ignore the higher terms and say $(1-p)^k \approx 1 - pk$. This gives an expected number of tests as:

$$N\left[1+\frac{1}{k}-(1-p)^k\right] \approx N\left[1+\frac{1}{k}-1+pk\right]$$
$$= N\left[\frac{1}{k}+pk\right]$$

Substituting $p = 0.01$ and $k = 10$ gives:

$$N\left[\frac{1}{k}+pk\right] = N\left[\frac{1}{10}+\frac{1}{100}\times 10\right]$$
$$= N\left[\frac{2}{10}\right]$$

and so the expected number of tests is about 20% of N.

Question 12

12 *In this question, you may use without proof the results*

$$\sum_{i=1}^{n} i^2 = \tfrac{1}{6}n(n+1)(2n+1) \quad \textit{and} \quad \sum_{i=1}^{n} i^3 = \tfrac{1}{4}n^2(n+1)^2.$$

Throughout the question, n and k are integers with $n \geqslant 3$ and $k \geqslant 2$.

(i) In a game, k players, including Ada, are each given a random whole number from 1 to n (that is, for each player, each of these numbers is equally likely and assigned independently of all the others). A player wins the game if they are given a smaller number than all the other players, so there may be no winner in this game.

Find an expression, in terms of n, k and a, for the probability that Ada is given number a, where $1 \leqslant a \leqslant n-1$, and all the other players are given larger numbers. Hence show that, if $k = 4$, the probability that there is a winner in this game is

$$\frac{(n-1)^2}{n^2}.$$

(ii) In a second game, k players, including Ada and Bob, are each given a random whole number from 1 to n. A player wins the game if they are given a smaller number than all the other players or if they are given a larger number than all the other players, so it is possible for there to be zero, one or two winners in this game.

Find an expression, in terms of n, k and d, for the probability that Ada is given number a and Bob is given number $a+d+1$, where $1 \leqslant d \leqslant n-2$ and $1 \leqslant a \leqslant n-d-1$, and all the other players are given numbers greater than a and less than $a+d+1$. Hence show that, if $k = 4$, the probability that there are two winners in this game is

$$\frac{(n-2)(n-1)^2}{n^3}.$$

If $k = 4$, what is the minimum value of n for which there are more likely to be exactly two winners than exactly one winner in this game?

Examiner's report

This question received one of the lowest numbers of responses and many of the responses did not achieve many marks.

In both parts of the question candidates often failed to calculate the probabilities that were needed to start the calculations, although many candidates were able to calculate the relevant combinatorial factors successfully and showed accurate algebraic manipulation when using the formulae provided in the question.

Those candidates who reached the final part of the question were able to identify a suitable approach for comparing the probability of there being one winner with the probability of there being two winners, although this was not executed correctly in most cases.

Solution

(i) The probability that a number is greater than a is given by $\dfrac{n-a}{n}$. The probability that Ada picks number a is $\dfrac{1}{n}$. The probability that Ada picks a and all the $k-1$ other plays pick a number greater than a is:

$$\frac{1}{n} \times \left(\frac{n-a}{n}\right)^{k-1}$$

If $k = 4$ then the probability that a specific player wins is equal to the probability that Ada gets $a = 1, 2, \cdots, n-1$ and the other three players all get higher numbers, so is given by:

$$\sum_{a=1}^{n-1} \frac{1}{n} \times \left(\frac{n-a}{n}\right)^3 = \frac{1}{n^4}\sum_{a=1}^{n-1}(n-a)^3$$

Using a substitution of $i = n - a$ this is equal to:

$$\frac{1}{n^4}\sum_{i=1}^{n-1} i^3 = \frac{1}{n^4} \times \frac{1}{4}(n-1)^2 n^2 = \frac{(n-1)^2}{4n^2}$$

However this is just the probability that Ada wins, but it could be Bob/Charlie/Doris who wins. This means that the probability that there is a winner is four times this, i.e. it is $\dfrac{(n-1)^2}{n^2}$.

Instead of changing the variable in the sum you could expand $(n-a)^3$ and evaluate the terms separately but this is a little more work.

(ii) The probability that Ada gets a is $\dfrac{1}{n}$ and that Bob gets $a+d+1$ is also $\dfrac{1}{n}$. There are d numbers between a and $a+d+1$ so the probability that one player gets a number between a and $a+d+1$ is $\dfrac{d}{n}$. Therefore the probability that Ada gets a and Bob gets $a+d+1$ and all the others have number between these is:

$$\frac{1}{n} \times \frac{1}{n} \times \left(\frac{d}{n}\right)^{k-2} = \frac{d^{k-2}}{n^k}$$

The size of the gap between Ada's number and Bob's number must be at least 1 and is at most $n-2$, so we have $1 \leqslant d \leqslant n-2$.

The smallest possible value of a is 1. For a fixed d (so a fixed gap between Ada's and Bob's numbers) the maximum number that Ada can have is equal to $n-1-d$. Therefore we have $1 \leqslant a \leqslant n-1-d$.

Therefore the probability that Ada has a number lower than everyone else's, and Bob has a number higher than everyone else's, is:

$$\sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \frac{d^{k-2}}{n^4}$$

When $k=4$ we have:

$$\begin{aligned}
\sum_{d=1}^{n-2} \sum_{a=1}^{n-d-1} \frac{d^2}{n^4} &= \frac{1}{n^4} \sum_{d=1}^{n-2} d^2 \sum_{a=1}^{n-d-1} 1 \\
&= \frac{1}{n^4} \sum_{d=1}^{n-2} d^2(n-d-1) \\
&= \frac{1}{n^4} \left(\sum_{d=1}^{n-2} (n-1)d^2 - \sum_{d=1}^{n-2} d^3 \right) \\
&= \frac{1}{n^4} \left((n-1) \times \frac{1}{6}(n-2)(n-1)\big[2(n-2)+1\big] - \frac{1}{4}(n-2)^2(n-1)^2 \right) \\
&= \frac{1}{24n^4}(n-2)(n-1)\Big(4(n-1)(2n-3) - 6(n-2)(n-1)\Big) \\
&= \frac{1}{24n^4}(n-2)(n-1)\Big([8n^2-20n+12] - [6n^2-18n+12]\Big) \\
&= \frac{1}{24n^4}(n-2)(n-1)\Big(2n^2-2n\Big) \\
&= \frac{1}{12n^3}(n-2)(n-1)^2
\end{aligned}$$

Note that factorisation is your friend! It is almost always a very good idea to take out common factors rather than expand the brackets.

This is the probability that Ada has a number smaller than everyone else and that Bob has a number larger than everyone else. However it would be the same probability if Charlie had a number smaller than everyone else and Doris had a number larger than everyone else.

The number of choices for the person who has the smallest number is 4, and then the number of options for the largest number is 3 so we need to multiply our single probability by $4 \times 3 = 12$ to get:

$$\text{Probability of two winners } = \frac{1}{n^3}(n-2)(n-1)^2$$

The probability that there is at least one winner is the probability that there is a player with a number smaller than everyone else, or there is a player with a number higher than everyone else, or both are true. This is equal to twice the probability from part **(i)** subtract the probability from part **(ii)** as otherwise you double count the probability that a player has a number lower than everyone else and that another player has a score higher than everyone else.

We have:

$$\text{P(There is at least one winner)} = \frac{2(n-1)^2}{n^2} - \frac{(n-2)(n-1)^2}{n^3}$$

$$\begin{aligned}\text{P(There is exactly one winner)} &= \text{P(There is at least one winner)} - \text{P(There are exactly two winners)} \\ &= \frac{2(n-1)^2}{n^2} - \frac{2(n-2)(n-1)^2}{n^3}\end{aligned}$$

For P(There are exactly two winners) > P(There is exactly one winner) then we need:

$$\begin{aligned}\frac{(n-2)(n-1)^2}{n^3} &> \frac{2(n-1)^2}{n^2} - \frac{2(n-2)(n-1)^2}{n^3} \\ \frac{3(n-2)(n-1)^2}{n^3} &> \frac{2(n-1)^2}{n^2} \\ 3(n-2) &> 2n \quad \text{as we have } n \geqslant 3 \\ n &> 6\end{aligned}$$

and so the smallest value of n is 7.