STEP Support Programme

2019 STEP 3

General comments

There are often many ways to approach a STEP question. Your methods may be different to the ones shown here but correct maths done correctly (and explained fully, especially in the case of a "show that") always gets the marks.

The full examiners report and mark-schemes for this paper can be found on the Cambridge Assessment Admissions Testing website.

Several of the questions on this paper involved graph sketching, and the examiner's report mentioned that in some cases candidates did not show all of the "salient points". You may find the topics notes in the STEP 2 Curve Sketching module helpful. As well as finding stationary points (which might also be called points parallel to the x-axis), you can find points parallel to the y axis by setting $\dfrac{\mathrm{d}x}{\mathrm{d}y} = 0$.

Question 1

1 The coordinates of a particle at time t are x and y. For $t \geqslant 0$, they satisfy the pair of coupled differential equations

$$\dot{x} = -x - ky$$
$$\dot{y} = x - y$$

where k is a constant. When $t = 0$, $x = 1$ and $y = 0$.

(i) Let $k = 1$. Find x and y in terms of t and sketch y as a function of t.

Sketch the path of the particle in the x-y plane, giving the coordinates of the point at which y is greatest and the coordinates of the point at which x is least.

(ii) Instead, let $k = 0$. Find x and y in terms of t and sketch the path of the particle in the x-y plane.

Examiner's report

This was the third most popular question attracting 83% of candidates, though it was only the 6^{th} most successful with an average mark just under 8/20, and no candidate achieved full marks on it. Small mistakes when starting the question often resulted in very different differential equations and hence solutions which lost lots of marks. Some candidates simply did not know how to solve a differential equation and they appeared to waste a lot of time. Drawing the spiral in part **(i)**, lots of candidates guessed that the extrema occurred on the axes losing substantial marks. Candidates should have spent a few minutes more sketching their graphs as these sketches were the main feature of the question and a lot of candidates forgot to mark salient points. Attempting the final sketch, only a handful considered the behaviour for large t.

Solution

When dealing with a pair of coupled equations the first thing to try and do is to eliminate one of the variables. We can rearrange the first equation to get $y = \frac{1}{k}(-x - \dot{x})$ which can be differentiated to get $\dot{y} = \frac{1}{k}(-\dot{x} - \ddot{x})$. These can both be substituted into the second equation in order to eliminate y. In a similar way you could eliminate x.

For some notes on solving second order differential equations see the STEP 3 Differential Equations Topic Notes. You might also find the STEP 2 Curve Sketching module topic notes helpful.

Part (i) - 12 marks

In this part $k = 1$. We have $y = -x - \dot{x}$ and therefore $\dot{y} = -\dot{x} - \ddot{x}$. Substituting this into the second equation gives:

$$\begin{aligned} \dot{y} &= x - y \\ -\dot{x} - \ddot{x} &= x - (-x - \dot{x}) \\ 0 &= \ddot{x} + 2\dot{x} + 2x \end{aligned}$$

Be careful with your double negatives! A small mistake here can lead to the wrong equation.

The *Auxiliary equation* in this case is $\lambda^2 + 2\lambda + 2 = 0$ which has solutions $\lambda = -1 \pm i$ which means that the general solution is:

$$\begin{aligned} x &= A\mathrm{e}^{(-1+\mathrm{i})t} + B\mathrm{e}^{(-1-\mathrm{i})t} \\ &= \mathrm{e}^{-t}\left(A\mathrm{e}^{\mathrm{i}t} + B\mathrm{e}^{-\mathrm{i}t}\right) \\ &= \mathrm{e}^{-t}\left(A\cos t + A\sin t + B\cos t - B\sin t\right) \\ &= \mathrm{e}^{-t}\left(C\cos t + D\sin t\right) \end{aligned}$$

You don't need to show all the working above, you can go straight to the last line.

We know that when $t = 0, x = 1$ which gives us $C = 1$. To find y we first need to find $\dot{x}$

$$\begin{aligned} \dot{x} &= -\mathrm{e}^{-t}(C\cos t + D\sin t) + \mathrm{e}^{-t}(-C\sin t + D\cos t) \\ &= \mathrm{e}^{-t}\big((D - C)\cos t - (C + D)\sin t\big) \end{aligned}$$

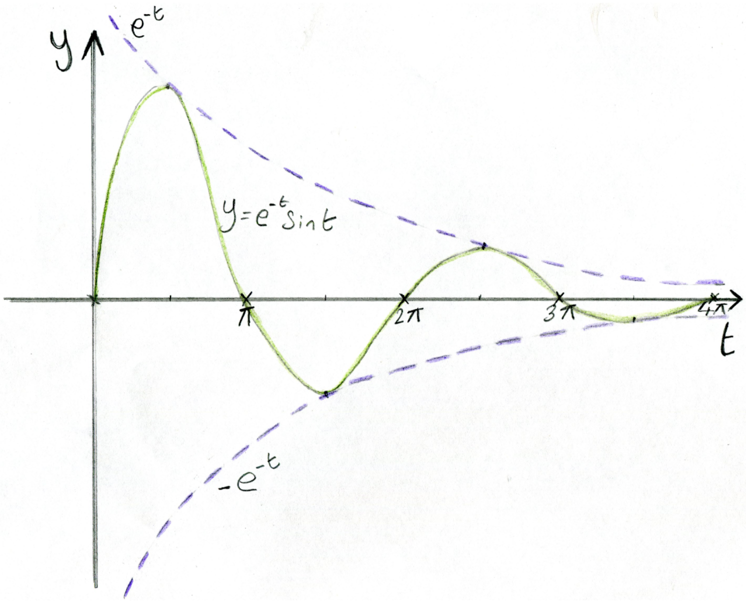
This gives:

$$\begin{aligned} y &= -x - \dot{x} \\ &= -\mathrm{e}^{-t}\left(C\cos t + D\sin t\right) + \mathrm{e}^{-t}\big((C - D)\cos t + (C + D)\sin t\big) \\ &= \mathrm{e}^{-t}\big(-D\cos t + C\sin t\big) \end{aligned}$$

When $t = 0, y = 0$ so $D = 0$. x and y are therefore:

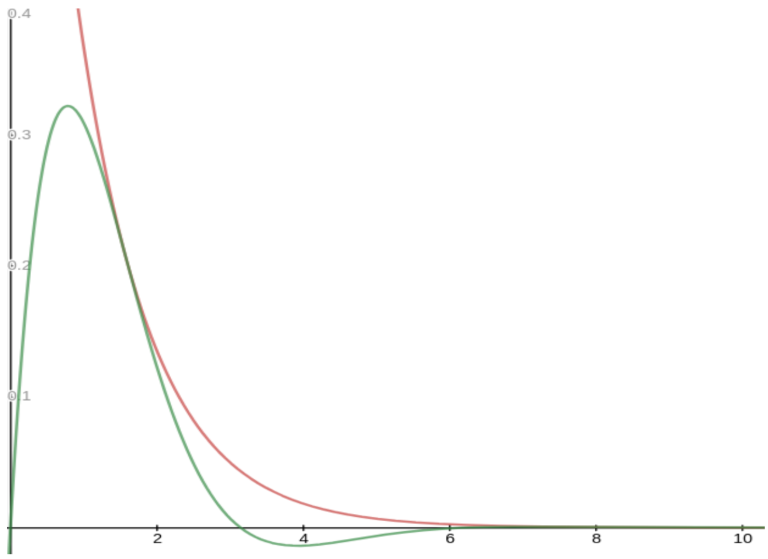
$$\begin{aligned} x &= \mathrm{e}^{-t}\cos t \\ y &= \mathrm{e}^{-t}\sin t \end{aligned}$$

The next request is to sketch y as a function of t. The easiest way to approach this is to sketch e^{-t} and $-\mathrm{e}^{-t}$ (possibly dotted so they don't get confused with y), and then you know that when $t = 0, \pi, 2\pi, \cdots$ we have $y = 0$, when $t = \frac{1}{2}\pi, \frac{5}{2}\pi \cdots$ then y lies on e^{-t} and when $t = \frac{3}{2}\pi, \frac{7}{2}\pi$ then y will lie on $-\mathrm{e}^{-t}$. This will give you enough information to complete the sketch.



When I did this I marked on π, 2π etc. and tried to make my graph go through these points. It is probably easier to draw the graph and then label π etc. at the points where it intersects the t axis. There is probably enough detail here, but the points where the curve meets $\pm e^{-t}$ could have been marked more clearly. My sketch makes these points look like maxima and minima, but in fact they cannot be stationary points. The curve we are sketching is bounded by $y = \pm e^{-t}$, so where it meets the boundary curves it "touches" the boundary rather than crosses, and so at these points $y = e^{-t} \sin t$ is tangential to the boundary curve $y = e^{-t}$ or $y = -e^{-t}$.

If you use a graph plotter you can see that the curve actually looks more like this:



and here you can clearly see that the maximum occurs before the curve hits $y = e^{-t}$. This graph might be technically more accurate than the first one, but the first one is more likely to get you the marks (and if you did produce something like the second one there may be a suspicion that you used IT to help you).

We know that $x^2 + y^2 = (\mathrm{e}^{-t})^2 = \mathrm{e}^{-2t}$, so you can imagine x and y being on a circle where the radius is decreasing.

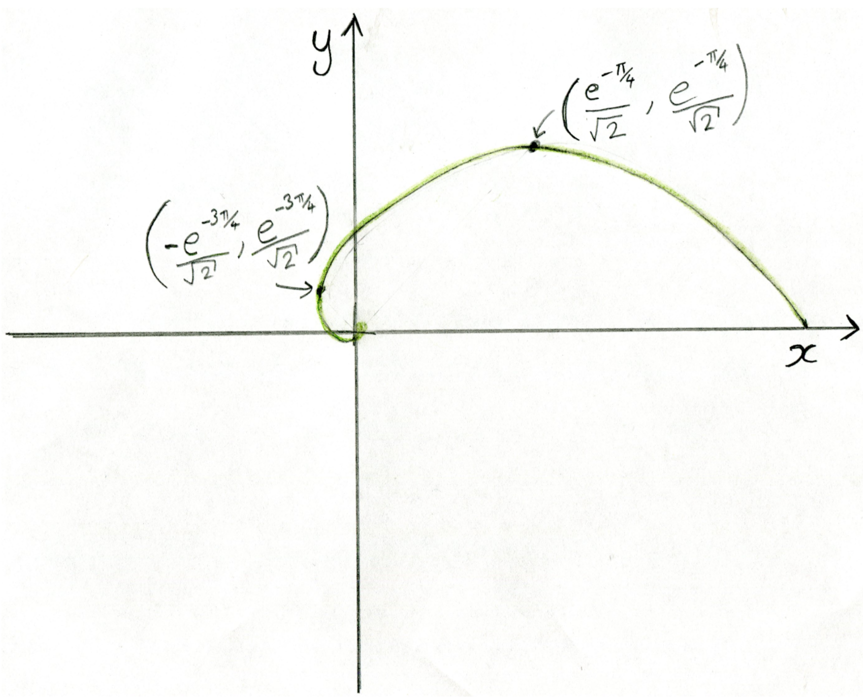
To find the greatest value of y it is tempting to look at your sketch and maybe conclude that y is greatest when $t = \frac{1}{2}\pi$, but this isn't in fact the maximum of y (see argument above!). y is greatest when $\dot{y} = 0$ i.e. when:

$$\begin{aligned}
x &= y && \text{from the second equation given in the stem}\\
\mathrm{e}^{-t}\sin t &= \mathrm{e}^{-t}\cos t &&\\
\sin t &= \cos t && \mathrm{e}^{-1} > 0 \text{ so it is fine to divide by this}\\
\tan t &= 1 && \text{The maximum value of } y \text{ must occur before } t = \tfrac{1}{2}\pi\\
t &= \frac{\pi}{4} &&
\end{aligned}$$

I have probably gone into more detail that necessary in showing in each case why I can divide by things. You could also have found $\dot{y}$ by differentiating your equation for y.

The coordinates of the maximum value of y are therefore $\left(\dfrac{\mathrm{e}^{-\frac{\pi}{4}}}{\sqrt{2}}, \dfrac{\mathrm{e}^{-\frac{\pi}{4}}}{\sqrt{2}}\right)$.

x is at a minimum when $\dot{x} = 0 \implies x = -y$ which happens when $\tan t = -1$, so when $t = \frac{3}{4}\pi$. The coordinates of the minimum of x are $\left(-\dfrac{\mathrm{e}^{-\frac{3\pi}{4}}}{\sqrt{2}}, \dfrac{\mathrm{e}^{-\frac{3\pi}{4}}}{\sqrt{2}}\right)$.



When drawing a spiral it is tempting to put the stationary points and the points where the curve is vertical on the y and x axes (this certainly feels like a more "natural" spiral to draw). However the question was careful to ask you to find the points where y is greatest and x is least, which is a big hint for how this spiral should look.

Part (ii) - 8 marks

When $k = 0$ the equations become:

$$\dot{x} = -x$$
$$\dot{y} = x - y$$

The first equation has solution $x = Ae^{-t}$, and since $x = 1$ when $t = 0$ we have $A = 1$. The second equation is therefore:

$$\dot{y} + y = e^{-t}$$

This has an integrating factor of e^t, so multiplying throughout by this gives:

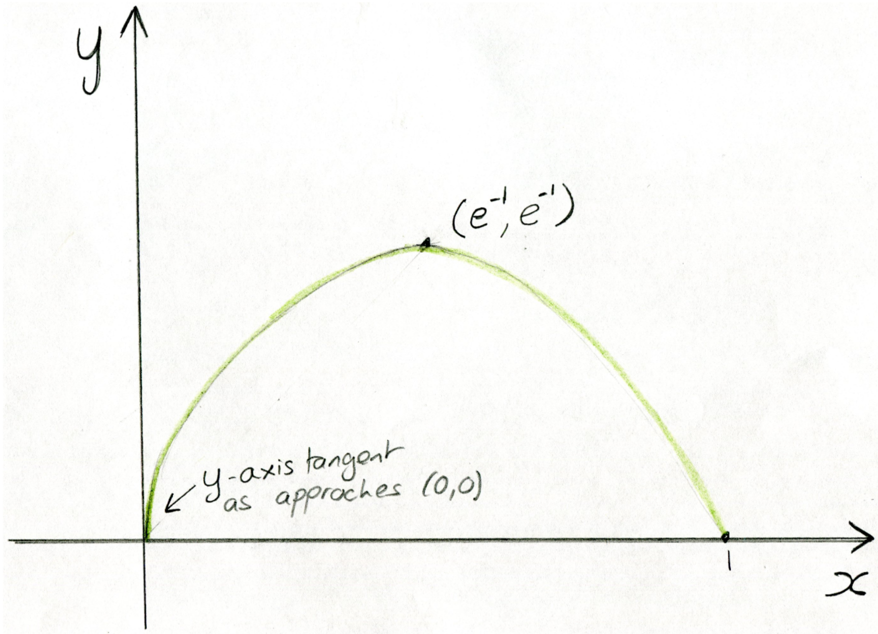
$$\begin{aligned} e^t\dot{y} + e^t y &= 1 \\ \frac{d}{dt}\left(e^t y\right) &= 1 \\ e^t y &= t + c \\ t = 0, y = 0 &\implies c = 0 \\ y &= te^{-t} \end{aligned}$$

To sketch the graph, note that $0 < x \leqslant 1$ and as $t \to \infty$, $y \to 0$. When $t = 0$ the particle is at $(1, 0)$.

To see if there are any stationary points, we can find the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{dt} \div \frac{dx}{dt} \\ &= \frac{e^{-t} - te^{-t}}{-e^{-t}} \\ &= t - 1 \qquad \text{as } e^{-1} > 0 \end{aligned}$$

Therefore there is a stationary point when $t = 1$, so it will be at the point (e^{-1}, e^{-1}). As $t \to \infty$, $\frac{dy}{dx} \to \infty$ so as the curve approaches $(0, 0)$ it will become vertical.



Note that you can write things onto your graph to make it clear what you were intending to sketch, which maybe useful if you are worried about your artistic abilities. You can even write things like "graph should tend to this line and not cross it, unlike my drawing where my line wobbled over".

Question 2

2 The definition of the derivative f' of a (differentiable) function f is

$$\mathrm{f}'(x) = \lim_{h \to 0} \frac{\mathrm{f}(x+h) - \mathrm{f}(x)}{h}\,. \qquad (*)$$

(i) The function f has derivative f' and satisfies

$$\mathrm{f}(x+y) = \mathrm{f}(x)\mathrm{f}(y)$$

for all x and y, and $\mathrm{f}'(0) = k$ where $k \neq 0$. Show that $\mathrm{f}(0) = 1$.

Using $(*)$, show that $\mathrm{f}'(x) = k\mathrm{f}(x)$ and find $\mathrm{f}(x)$ in terms of x and k.

(ii) The function g has derivative g' and satisfies

$$\mathrm{g}(x+y) = \frac{\mathrm{g}(x) + \mathrm{g}(y)}{1 + \mathrm{g}(x)\mathrm{g}(y)}$$

for all x and y, $|\mathrm{g}(x)| < 1$ for all x, and $\mathrm{g}'(0) = k$ where $k \neq 0$.

Find $\mathrm{g}'(x)$ in terms of $\mathrm{g}(x)$ and k, and hence find $\mathrm{g}(x)$ in terms of x and k.

Examiner's report

The most popular question, it was also the most successful with an average score of over $^{11}/_{20}$ and many fully correct solutions. Most candidates that noticed that the equation for $\mathrm{f}(x+y)$ implied $\mathrm{f}(x) = 0$ for all x, or $\mathrm{f}(0) = 1$ correctly eliminated the former, but quite a few did not realise that it was a possibility to consider. Nearly every candidate successfully found $\mathrm{f}'(x) = \mathrm{f}(x) \lim_{h \to 0} \left(\frac{\mathrm{f}(h) - 1}{h} \right)$ and most also proceeded correctly from there to find the required differential equation. Finding $\mathrm{f}(x)$ was generally successful although some did not check the boundary conditions. In part **(ii)**, there were fewer issues demonstrating that $\mathrm{g}(0) = 0$ than there had been with $\mathrm{f}(0) = 1$ in part **(i)**. The simplification in order to find the limit to obtain $\mathrm{g}'(x)$ was usually successful. Solution of that differential equation was often well done, either using partial fractions or as a hyperbolic function, although some mistakenly identified the solution as a tan function.

Solution

Part (i) - 9 marks

Take $y = 0$, then we have:

$$\begin{aligned} f(x+0) &= f(x)f(0) \\ f(x) &= f(x)f(0) \\ f(x)\big(f(0)-1\big) &= 0 \\ \implies f(x) &= 0 \text{ for all } x, \text{ or } f(0) = 1 \\ \text{therefore } f(0) &= 1 \text{ since we are told that } f'(0) \neq 0 \\ &\qquad \text{and so } f(x) \text{ cannot be 0 for all } x \end{aligned}$$

The first thing I tried was setting $x = y = 0$ which was not helpful in that it gave $f(0) = 0$ or $f(0) = 1$.

Using $(*)$ we have:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h\to 0} \frac{f(h) - 1}{h} \qquad \text{since } f(x) \text{ is independent of } h \\ &= f(x)f'(0) \qquad \text{using } (*) \text{ with } x = 0 \\ &= kf(x) \end{aligned}$$

Rearranging gives:

$$\begin{aligned} \frac{f'(x)}{f(x)} &= k \\ \ln|f(x)| &= kx + c \\ |f(x)| &= Ae^{kx} \end{aligned}$$

The modulus signs mean that $f(x) = \pm Ae^{kx}$, but since A could be positive or negative then we can ignore the $\pm$ sign and just write $f(x) = Ae^{kx}$. If we wanted to be formal about it we could say $|f(x)| = Ae^{kx} \implies f(x) = Be^{kx}$, where $B = \pm A$. There is no need to be this formal though, you can just quietly lose the modulus signs here!

Using $f(0) = 1$ means that we have $f(x) = e^{kx}$.

Part (ii) - 11 marks

Let's start by doing the same sort of things that we did for part **(i)**.

Take $y = 0$, which gives:

$$\begin{aligned} g(x) &= \frac{g(x) + g(0)}{1 + g(x)g(0)} \\ \cancel{g(x)} + [g(x)]^2 g(0) &= \cancel{g(x)} + g(0) \\ \big([g(x)]^2 - 1\big)g(0) &= 0 \end{aligned}$$

We are told in the question that $|g(x)| < 1$ for all x, and so we know that $[g(x)]^2 - 1 \neq 0$. Hence $g(0) = 0$.

Using $(*)$ we have:

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{\dfrac{g(x)+g(h)}{1+g(x)g(h)} - g(x)}{h} \\
&= \lim_{h\to 0} \frac{g(x)+g(h) - g(x)[1+g(x)g(h)]}{h(1+g(x)g(h))} \\
&= \lim_{h\to 0} \frac{g(h)(1-[g(x)]^2)}{h(1+g(x)g(h))} \\
&= \left(1-[g(x)]^2\right) \lim_{h\to 0} \frac{g(h)}{h(1+g(x)g(h))} \\
&= \left(1-[g(x)]^2\right) \lim_{h\to 0} \frac{g(h)/h}{(1+g(x)g(h))} \\
&= \left(1-[g(x)]^2\right) \lim_{h\to 0} \frac{g(h)}{h} \qquad \text{as } g(h) \to 0 \text{ as } h \to 0 \\
&= \left(1-[g(x)]^2\right) \times g'(0) \\
&= k\left(1-[g(x)]^2\right)
\end{aligned}$$

We therefore have $\dfrac{g'(x)}{1-(g(x))^2} = k$. Integrating with respect to x gives:

$$\begin{aligned}
\int \frac{g'(x)}{1-(g(x))^2}\,dx &= kx + c \\
\int \frac{g'(x)}{1-(\tanh t)^2} \times \frac{\operatorname{sech}^2 t}{g'(x)}\,dt &= kx + c \\
&\qquad \left(\text{using a substitution of } g(x) = \tanh t\right) \\
\int 1\,dt &= kx + c \\
\tanh^{-1}\left(g(x)\right) &= kx + c \\
g(x) &= \tanh(kx + c)
\end{aligned}$$

Using $g(0) = 0$ (found earlier) we have $\tanh(c) = 0$ and so $c = 0$. This means we have $g(x) = \tanh(kx)$.

Alternative integration method

Instead we could have integrated using partial fractions. We have:

$$\frac{1}{1-\left(g(x)\right)^2} = \frac{1/2}{1-g(x)} + \frac{1/2}{1+g(x)}$$

and so:

$$\begin{aligned}
\int \frac{g'(x)}{1-(g(x))^2}\,dx &= kx + c \\
\tfrac{1}{2}\int \left(\frac{g'(x)}{1-g(x)} + \frac{g'(x)}{1+g(x)}\right) dx &= kx + c \\
\tfrac{1}{2}\Big[-\ln|1-g(x)| + \ln|1+g(x)| \Big] &= kx + c
\end{aligned}$$

We know that $|g(x)| < 1$ for all x, so we can write this as:

$$\begin{aligned}
\frac{1}{2}\ln\left(\frac{1+g(x)}{1-g(x)}\right) &= kx + c \\
\ln\left(\frac{1+g(x)}{1-g(x)}\right) &= 2kx + 2c
\end{aligned}$$

Using $g(0) = 0$ gives $c = 0$, and so:

$$\begin{aligned}
\frac{1+g(x)}{1-g(x)} &= e^{2kx} \\
1 + g(x) &= e^{2kx} - g(x)e^{2kx} \\
g(x)\big(1 + e^{2kx}\big) &= e^{2kx} - 1 \\
g(x) &= \frac{e^{2kx} - 1}{e^{2kx} + 1}
\end{aligned}$$

This is equivalent to $g(x) = \tanh(kx)$ since:

$$\begin{aligned}
\tanh(kx) &= \frac{\sinh kx}{\cosh kx} \\
&= \frac{e^{kx} - e^{-kx}}{e^{kx} + e^{-kx}} \\
&= \frac{e^{2kx} - 1}{e^{2kx} + 1}
\end{aligned}$$

Question 3

3 The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(i) You are given that the transformation represented by $\mathbf{A}$ has a line L_1 of invariant points (so that each point on L_1 is transformed to itself). Let (x, y) be a point on L_1. Show that $\big((a-1)(d-1) - bc\big)xy = 0$.

Show further that $(a-1)(d-1) = bc$.

What can be said about $\mathbf{A}$ if L_1 does not pass through the origin?

(ii) By considering the cases $b \neq 0$ and $b = 0$ separately, show that if $(a-1)(d-1) = bc$ then the transformation represented by $\mathbf{A}$ has a line of invariant points. You should identify the line in the different cases that arise.

(iii) You are given instead that the transformation represented by $\mathbf{A}$ has an invariant line L_2 (so that each point on L_2 is transformed to a point on L_2) and that L_2 does not pass through the origin. If L_2 has the form $y = mx + k$, show that $(a-1)(d-1) = bc$.

Examiner's report

This was the 6th most popular question, but the least successful of those six, indeed one of the four least successful in the whole paper scoring just under a quarter of the marks. Barely a handful of attempts scored full marks. Candidates frequently overlooked the last result in **(i)**, or merely wrote $\mathbf{A} = \mathbf{I}$, presumably assuming that was hardly worth any marks. Those who got to part **(iii)** typically did well on it, even if they had done poorly on parts **(i)** and **(ii)**. Candidates often omitted or struggled to deal with cases. Common pitfalls were unnecessary division by zero without considering if the denominator were zero, thinking that the question stated that all invariant points lie on a line leads to claims that $\mathbf{A}$ is non-linear or doesn't exist, as mentioned failure to justify that $\mathbf{A} = \mathbf{I}$, and use of $\det(\mathbf{A} - \mathbf{I}) = 0$ with no justification.

Solution

There is a big difference between a *line of invariant points* and an*invariant line*. In the first case every point is mapped back to itself,whereas in the second every point on a line is mapped to another point on that line. A line of invariant points is an invariant line, but the opposite isn't true — not every invariant line is a line of invariant points.

In the examiners report it says that candidates who attempted part **(iii)** tended to do well on it, even if they had not done well on the previous parts. It is always worth having a go at the last parts of a question even if you don't think you have completed the previous parts fully correctly.

Part (i) - 9 marks

Since L_1 is a line of invariant points we have:

$$\mathbf{A}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

and so we have $(a-1)x+by=0$ and $cx+(d-1)y=0$. From the first of these we have $y=-\frac{1}{b}(a-1)x$, and substituting this into the second equation gives:

$$cx - \tfrac{1}{b}(a-1)(d-1)x = 0$$
$$(a-1)(d-1)x - cbx = 0$$
$$\Big[(a-1)(d-1) - bc\Big]x = 0 \qquad (*)$$
$$\Big[(a-1)(d-1) - bc\Big]xy = 0$$

Where the last line was achieved by multiplying by y.

From the line marked $(*)$, we know that either $x=0$, or $(a-1)(d-1)-bc=0$. If $x=0$ then for all y we must have $by=0$ and $dy=y$ which means that $b=0, d=1$ and hence in this case $(a-1)(d-1)-bc=0$ again. Therefore we have $(a-1)(d-1)=bc$.

If L_1 does not pass through the origin, then we know that it must have the form $y=mx+k$[1], or $x=k$, or $y=k$ where in each case $k \neq 0$.

If $x=k$ then we have:

$$\begin{pmatrix} ak+by \\ ck+dy \end{pmatrix} = \begin{pmatrix} k \\ y \end{pmatrix}$$
$$ak+by=k$$
$$ck+dy=y$$

Since this is true for all y then setting $y=0$ gives $a=1$ (as $k \neq 0$) and $c=0$, and setting $y=1$ gives $b=0$ and $d=1$. This means that $\mathbf{A}$ is the identity matrix.

If $y=k$ then we have:

$$ax+bk=x$$
$$cx+dk=k$$

[1] Usually I would use $y=mx+c$, but c is already used in the matrix $\mathbf{A}$ so it is better to use a different letter here!

This is true for all x, and by setting $x = 0$ and $x = 1$ we again get $\mathbf{A}$ to be the identity matrix.

If $y = mx + k$ we have:

$$\begin{aligned} ax + b(mx + k) &= x \\ cx + d(mx + k) &= mx + k \end{aligned}$$

This needs to be true for all x. Setting $x = 0$ gives $bk = 0$, and so we have $b = 0$ (as $k \neq 0$). From the first equation above this means that we have $a = 1$. In the second equation setting $x = 0$ gives $dk = k$ and so $d = 1$, which implies $c = 0$.

So in every case, the only way that we can have a line of invariant points which does not pass through the origin is if we have $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Some candidates just wrote down $\mathbf{A} = \mathbf{I}$ with no working in the real exam. Perhaps they had been taught this as a fact in their A-levels. If a STEP question wants you to just write down a statement it will usually say "state", or "write down", otherwise you should probably expect to have to show some justification for your answer.

Part (ii) - 5 marks

Here the question is asking you to show that $(a-1)(d-1) = bc \implies$ there is a line of invariant points (rather than the opposite way around as in the start of part **(i)**).

Starting with the $b = 0$ case, this means that we have $(a-1)(d-1) = 0$ and so either $a = 1$ or $d = 1$. Starting with $a = 1$ we have an invariant line iff:

$$\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{aligned} x &= x \\ cx + dy &= y \end{aligned}$$

and so we have an invariant line $(1-d)y = cx$ (points on this line satisfy the second equation and so are invariant points).

Similarly, if $d = 1$ we need:

$$\begin{pmatrix} a & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{aligned} ax &= x \\ cx + y &= y \end{aligned}$$

The first equation implies that $a = 1$ or $x = 0$. If we take $a = 1$ we end up with $\mathbf{A} = \mathbf{I}$ again, however the line $x = 0$ is an invariant line of this more general matrix.

If $b \neq 0$ we need

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{aligned} ax + by &= x \\ cx + dy &= y \end{aligned}$$

If $b \neq 0$ then the point (x, y) is an invariant point if and only if:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

So we need $(a-1)x + by = 0$ and $cx + (d-1)y = 0$. If these two lines are in fact the same straight line then we have a line of invariant points. We are given that $(a-1)(d-1) = bc$, so we can use this to eliminate $(a-1)$ from the first equation:

$$\begin{aligned} (a-1)x + by &= 0 \\ \frac{bc}{d-1}x + by &= 0 \quad \text{assuming } d \neq 1 \text{ (we will need to consider this case!)} \\ \frac{c}{d-1}x + y &= 0 \quad \text{we can divide by } b \text{ as we know in this case } b \neq 0 \\ cx + (d-1)y &= 0 \end{aligned}$$

So if we have $(a-1)(d-1) = bc$ and $b \neq 0, d \neq 1$ then the lines $(a-1)x+by = 0$ and $cx+(d-1)y = 0$ are the same line and this is a line of invariant points.

The last case we need to consider is $d = 1$. If $d = 1$ then $(a-1)(d-1) = bc$ implies that either $b = 0$ or $c = 0$. The case where $b = 0$ we have already considered so take $d = 1, c = 0$. This means that we will have $cx + dy = y$ for all x, y so the invariant line here is given by $ax + by = x \implies (a-1)x + by = 0$.

Looking back at this, I could have avoided having to consider two separate cases here by using $(a-1)x + by = 0 \implies (a-1)(d-1)x + b(d-1)y = 0$, i.e. avoiding dividing by $(d-1)$.

Part (iii) - 6 marks

Here L_2 is an invariant line. Using $y = mx + k$ we have:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ mx + k \end{pmatrix} = \begin{pmatrix} ax + b(mx+k) \\ cx + d(mx+k) \end{pmatrix} = \begin{pmatrix} X \\ mX + k \end{pmatrix}$$

Note here the use of x and X — here the point (x, y) is mapped to another point (X, Y) which still lies on L_2, but might be a different point.

Eliminating X gives:

$$\begin{aligned} cx + d(mx+k) &= m\big[ax + b(mx+k)\big] + k \\ (c + dm)x + dk &= (ma + bm^2)x + (mbk + k) \end{aligned}$$

We need this to be true for all values of x (i.e. this is an *identity* rather than an *equation*), so we can equate coefficients of x to get:

$$\begin{aligned} c + dm &= m(a + bm) \\ dk &= k(mb + 1) \end{aligned}$$

We are told that L_2 does not pass through the origin, so $k \neq 0$. This means that we can divide the second equation by k to get $mb + 1 = d \implies bm = (d-1)$. Substituting in for bm in the first equation gives:

$$\begin{aligned} cb + dbm &= bm\big(a + bm\big) \\ cb + d(d-1) &= (d-1)\big(a + (d-1)\big) \\ cb &= (d-1)\big(a + (d-1)\big) - d(d-1) \\ cb &= (d-1)\big(a + (d-1) - d\big) \\ bc &= (a-1)(d-1) \qquad \text{as required.} \end{aligned}$$

Question 4

4 The nth degree polynomial $\mathrm{P}(x)$ is said to be *reflexive* if:

(a) $\mathrm{P}(x)$ is of the form $x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$ where $n \geqslant 1$;

(b) $a_1, a_2, \ldots, a_n$ are real;

(c) the n (not necessarily distinct) roots of the equation $\mathrm{P}(x) = 0$ are $a_1, a_2, \ldots, a_n$.

(i) Find all reflexive polynomials of degree less than or equal to 3.

(ii) For a reflexive polynomial with $n > 3$, show that

$$2a_2 = -a_2^2 - a_3^2 - \cdots - a_n^2.$$

Deduce that, if all the coefficients of a reflexive polynomial of degree n are integers and $a_n \neq 0$, then $n \leqslant 3$.

(iii) Determine all reflexive polynomials with integer coefficients.

Examiner's report

Two thirds of candidates attempted this scoring, on average about half marks. The first part was often well answered by those that used the Vieta equations[2], though a common error was to divide by a potential zero and therefore omit one of the solutions. Those that substituted the three roots into the polynomial equation encountered equations that were more difficult to solve, and the method yielded additional solutions which were often not rejected, so generally those taking this approach did much less well. Part **(ii)** had many good answers from squaring the sum of roots equation, though a common error was made with inaccurate summation notation. Many argued the deduction correctly but likewise many others assumed the a_is were non-zero without a reason. Part **(iii)** was found more difficult. Those candidates that did not realise the significance of the previous deduction were rarely successful and others gave the correct answer, but with no explanation. Those making the inductive argument to factorise powers of x rarely justified that the remaining polynomial was likewise reflexive.

[2] *Vieta's equations* are the formulae that connect the coefficients of a polynomial to sums and products of the roots.

Solution This is a surprisingly top-heavy question with 11 marks available for part **(i)**.

Part (i) - 11 marks

Starting with $n = 1$ (as reflexive polynomials have degree at least 1).

$\mathrm{P}(x)$ has the form $x - a_1$, and this does have root a_1, so anything of the form $x - a_1$ is a reflexive polynomial.

For $n = 2$, $\mathrm{P}(x) = x^2 - a_1x + a_2$ and this has to have roots a_1 and a_2. Substituting in $x = a_1$ gives $a_1^2 - a_1 \times a_1 + a_2 = 0$ which means that $a_2 = 0$. So $\mathrm{P}(x) = x^2 - a_1x$, which has roots a_1 and 0. Anything of the form $\mathrm{P}(x) = x^2 - a_1x$ is a reflexive polynomial.

For $n = 3$, $\mathrm{P}(x) = x^3 - a_1x^2 + a_2x - a_3 = (x - a_1)(x - a_2)(x - a_3)$. By equating coefficients we have:

$$a_1 = a_1 + a_2 + a_3 \tag{1}$$
$$a_2 = a_1a_2 + a_2a_3 + a_3a_1 \tag{2}$$
$$a_3 = a_1a_2a_3 \tag{3}$$

Equation (1) gives $0 = a_2 + a_3$ which if we substitute this into equation (2) gives $a_2 = a_2a_3$ (since $a_1a_2 + a_3a_1 = a_1(a_2 + a_3) = 0$).

$a_2 = a_2a_3 \implies a_2(1 - a_3) = 0$ then either $a_2 = 0$ or a_3=1.

Starting with $a_2 = 0$ this gives $a_3 = -a_2 = 0$ and we get $\mathrm{P}(x) = x^3 - a_1x^2$ (which has roots $a_1, 0, 0$).

If $a_3 = 1$, then $a_2 = -a_3 = -1$ and using equation (3) we have $a_1a_2 = 1 \implies a_1 = -1$. This gives $\mathrm{P}(x) = x^3 + x^2 - x - 1$.

It is really easy to get this last one wrong — remember that $\mathrm{P}(x) = x^3 - a_1x^2 + a_2x - a_3$. It is also useful to check that your answer does actually have roots $1, -1, -1$.

The set of reflexive polynomials of degree less than or equal to 3 is:

$$\begin{aligned}
&\mathrm{P}(x) = x - a_1\\
&\mathrm{P}(x) = x^2 - a_1x\\
&\mathrm{P}(x) = x^3 - a_1x^2\\
&\mathrm{P}(x) = x^3 + x^2 - x - 1 = (x + 1)^2(x - 1)
\end{aligned}$$

Part(ii) - 5 marks

For this part it would be good to be able to find the sum of the squares of the roots. If we have:

$$\mathrm{P}(x) = x^n - a_1x^{n-1} + a_2x^{n-1} + \cdots + a_n = (x - a_1)(x - a_2)\cdots(x - a_n)$$

Then we know that:

$$\begin{aligned}
a_1 &= a_1 + a_2 + a_3 + \cdots + a_n\\
a_1^2 &= (a_1 + a_2 + a_3 + \cdots + a_n)^2
\end{aligned}$$

We also have:

$$\begin{aligned} a_2 &= a_1a_2 + a_1a_3 + \cdots + a_2a_3 + a_2a_4 + \cdots + a_3a_4 + a_3a_5 + \cdots \\ &= \tfrac{1}{2}\sum_{i\neq j} a_ia_j \end{aligned}$$

Note that the sum $\sum_{i\neq j} a_ia_j$ includes each pair twice, i.e. it includes a_1a_2 and a_2a_1 etc.

We now have:

$$\begin{aligned} (a_1 + a_2 + a_3 + \cdots + a_n)^2 &= a_1^2 \\ \sum_{i=1}^{n} a_i^2 + \sum_{i\neq j} a_ia_j &= a_1^2 \\ \cancel{a_1^2} + \sum_{i=2}^{n} a_i^2 + 2a_2 &= \cancel{a_1^2} \\ \implies 2a_2 &= -a_2^2 - a_3^2 - \cdots - a_n^2 \end{aligned}$$

This gives:

$$\begin{aligned} a_2^2 + 2a_2 &= -a_3^2 - \cdots - a_n^2 \\ (a_2+1)^2 - 1 &= -a_3^2 - \cdots - a_n^2 \\ (a_2+1)^2 &= 1 - a_3^2 - \cdots - a_n^2 \end{aligned}$$

Since $(a_2+1)^2 \geqslant 0$ we must have $a_3^2 + \cdots + a_n^2 \leqslant 1$. If the roots are integers then at most one of $a_3, a_4, \cdots a_n$ can be equal to ± 1 and the rest must be 0.

If $a_n = \pm 1$ then $a_3, a_4, \cdots a_{n-1} = 0$. However this gives a contradiction as $a_n = a_1a_2a_3 \cdots a_n$, so if $a_3 = 0$ we must have $a_n = 0$. The only way we can have $a_n \neq 0$ is if $a_n = a_3$ or a_2 or a_1, i.e. we need $n \leqslant 3$.

Part (iii) - 4 marks

In part **(i)** we found all the reflexive polynomials with degree $n \leqslant 3$, and from part **(ii)** we know that any other reflexive polynomial has $a_n = 0$. This means that the polynomial can be written as:

$$\begin{aligned} &x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^{n-1}a_{n-1}x \\ =&x(x^{n-1} - a_1x^{n-2} + a_2x^{n-3} - \cdots + (-1)^{n-1}a_{n-1}) \end{aligned}$$

The roots need to be $x = 0 (= a_n)$ and $x = a_1, a_2, a_3, \cdots a_{n-1}$, which means that the polynomial of degree $n-1$ must also be a reflexive polynomial so either $n - 1 \leqslant 3$ or $a_{n-1} = 0$. This process can be repeated until we reach the polynomial $\mathrm{P}(x) = x^{n-3}(x^3 - a_1x^2 + a_2x - a_3)$.

By looking back at our answers to part **(i)** we can write down the set of reflexive polynomials as:

$$\begin{aligned} \mathrm{P}(x) &= x^r(x - a_1) \\ \mathrm{P}(x) &= x^r(x+1)^2(x-1) \end{aligned}$$

where $r = 0, 1, 2, \cdots$

Question 5

5 **(i)** Let

$$f(x) = \frac{x}{\sqrt{x^2+p}},$$

where p is a non-zero constant. Sketch the curve $y = f(x)$ for $x \geqslant 0$ in the case $p > 0$.

(ii) Let

$$I = \int \frac{1}{(b^2 - y^2)\sqrt{c^2 - y^2}}\, dy,$$

where b and c are positive constants. Use the substitution $y = \dfrac{cx}{\sqrt{x^2+p}}$, where p is a suitably chosen constant, to show that

$$I = \int \frac{1}{b^2 + (b^2 - c^2)x^2}\, dx.$$

Evaluate

$$\int_1^{\sqrt{2}} \frac{1}{(3 - y^2)\sqrt{2 - y^2}}\, dy.$$

[**Note:** $\displaystyle\int \frac{1}{a^2 + x^2}\, dx = \frac{1}{a}\tan^{-1}\frac{x}{a} +$ constant.]

Hence evaluate

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2 - 1)\sqrt{2y^2 - 1}}\, dy.$$

(iii) By means of a suitable substitution, evaluate

$$\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{(3y^2 - 1)\sqrt{2y^2 - 1}}\, dy.$$

Examiner's report

A little more popular than question 1, attempts were only a little less successful than those for question 4. The majority of candidates successfully drew the graph for **(i)**, but common errors were failures to consider asymptotic behaviour and label appropriately.

In the original paper there was a typographical error in the substitution for part **(ii)**. The vast majority used the incorrectly suggested substitution and proceeded as far as possible on the first result of **(ii)** using it; a few realised at this point that there was an error and then obtained the correct result. Candidates who moved on to the evaluation, scored strongly using the quoted result, even if they had not obtained it, and appreciated that the limits were needing to be changed. Most attempting the second evaluation of **(ii)** obtained full marks on that part as they successfully demonstrated that it was the same answer as the previous evaluation. Few attempted part **(iii)**, and most that were successful made substitutions not involving using the previous part of the question.

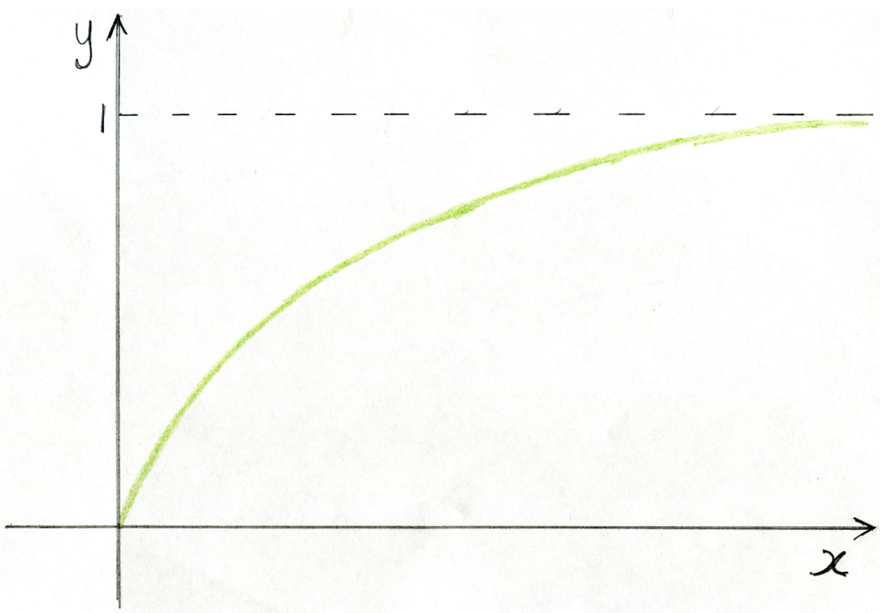
Solution

Part (i) - 3 marks

The curve passes through $(0, 0)$, and as $x \to \infty$, $\mathrm{f}(x) \to 1$. The derivative is given by:

$$\begin{aligned}\mathrm{f}'(x) &= \frac{\sqrt{x^2+p} - x \times \frac{1}{2}\frac{2x}{\sqrt{x^2+p}}}{x^2+p} \\ &= \frac{(x^2+p) - x^2}{(x^2+p)^{3/2}} \\ &= \frac{p}{(x^2+p)^{3/2}}\end{aligned}$$

Hence the gradient is always positive and tends towards 0 as $x \to \infty$.



Part (ii) - 12 marks

Using the substitution we have $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{cp}{(x^2+p)^{3/2}}$ (using the derivative found in part **(i)**). We also have:

$$\begin{aligned} c^2 - y^2 &= c^2 - \frac{c^2x^2}{x^2+p} \\ &= \frac{\cancel{c^2x^2} + c^2p - \cancel{c^2x^2}}{x^2+p} \\ b^2 - y^2 &= \frac{b^2x^2 + b^2p - c^2x^2}{x^2+p} \end{aligned}$$

Using the above have:

$$\begin{aligned} I &= \int \frac{1}{(b^2-y^2)\sqrt{c^2-y^2}}\,\mathrm{d}y \\ &= \int \frac{x^2+p}{b^2x^2+b^2p-c^2x^2} \times \frac{\sqrt{x^2+p}}{c\sqrt{p}} \times \frac{cp}{(x^2+p)^{3/2}}\,\mathrm{d}x \\ &= \int \frac{\sqrt{p}}{b^2p+(b^2-c^2)x^2}\,\mathrm{d}x \end{aligned}$$

And then taking $p = 1$ gives the required result.

Using the given result with $b = \sqrt{3}$ and $c = \sqrt{2}$ we have:

$$\begin{aligned} \int_1^{\sqrt{2}} \frac{1}{(3-y^2)\sqrt{2-y^2}}\,\mathrm{d}y &= \int_1^{\infty} \frac{1}{3+(3-2)x^2}\,\mathrm{d}x \\ &= \int_1^{\infty} \frac{1}{3+x^2}\,\mathrm{d}x \\ &= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_1^{\infty} \\ &= \frac{1}{\sqrt{3}}\left(\frac{\pi}{2} - \frac{\pi}{6}\right) \\ &= \frac{\pi}{3\sqrt{3}} \end{aligned}$$

For the limits we are using $y = \frac{\sqrt{2}x}{\sqrt{x^2+1}}$, which tends to $\sqrt{2}$ as $x \to \infty$.

For the last integral let $y = \dfrac{1}{t}$, so that $\dfrac{\mathrm{d}y}{\mathrm{d}t} = -\dfrac{1}{t^2}$. This gives:

$$\begin{aligned} \int_{\frac{1}{\sqrt{2}}}^{1} \frac{y}{(3y^2-1)\sqrt{2y^2-1}}\,\mathrm{d}y &= \int_{\sqrt{2}}^{1} \frac{\frac{1}{t}}{\left(\frac{3}{t^2}-1\right)\sqrt{\frac{2}{t^2}-1}} \times -\frac{1}{t^2}\,\mathrm{d}t \\ &= -\int_{\sqrt{2}}^{1} \frac{1}{(3-t^2)\sqrt{2-t^2}}\,\mathrm{d}t \\ &= \frac{\pi}{3\sqrt{3}} \end{aligned}$$

Part (iii) - 5 marks

Start by trying to use a substitution of the same form as before, i.e. $y = \dfrac{cx}{\sqrt{x^2+p}}$:

$$\begin{aligned}\int \frac{1}{(3y^2-1)\sqrt{2y^2-1}}\,\mathrm{d}y &= \int \frac{1}{\left(\frac{3c^2x^2}{x^2+p}-1\right)\sqrt{\frac{2c^2x^2}{x^2+p}-1}} \times \frac{cp}{(x^2+p)^{3/2}}\,\mathrm{d}x \\ &= \int \frac{cp}{(3c^2x^2-x^2-p)\sqrt{2c^2x^2-x^2-p}}\,\mathrm{d}x\end{aligned}$$

If we then take $c = \frac{1}{\sqrt{2}}$ and $p = -1$ we get:

$$\int \frac{\frac{-1}{\sqrt{2}}}{\left(\frac{3}{2}x^2 - x^2 + 1\right)\sqrt{x^2-x^2+1}}\,\mathrm{d}x = \frac{-1}{\sqrt{2}}\int \frac{2}{x^2+2}\,\mathrm{d}x$$

Using $y = \frac{x}{\sqrt{2}\sqrt{x^2-1}}$ with the limits $y = \frac{1}{\sqrt{2}}$ and $y = 1$ gives $x = \infty$ and $x = \sqrt{2}$. This means that our integral is now:

$$\begin{aligned}\frac{-1}{\sqrt{2}}\int_{\infty}^{\sqrt{2}} \frac{2}{x^2+2}\,\mathrm{d}x &= \sqrt{2}\int_{\sqrt{2}}^{\infty} \frac{1}{x^2+2}\,\mathrm{d}x \\ &= \sqrt{2}\left[\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_{\sqrt{2}}^{\infty} \\ &= \frac{\pi}{2} - \frac{\pi}{4} \\ &= \frac{\pi}{4}\end{aligned}$$

Question 6

6 The point P in the Argand diagram is represented by the the complex number z, which satisfies

$$zz^* - az^* - a^*z + aa^* - r^2 = 0.$$

Here, r is a positive real number and $r^2 \neq a^*a$. By writing $|z-a|^2$ as $(z-a)(z-a)^*$, show that the locus of P is a circle, C, the radius and the centre of which you should give.

(i) The point Q is represented by ω, and is related to P by $\omega = \dfrac{1}{z}$. Let C' be the locus of Q. Show that C' is also a circle, and give its radius and centre.

If C and C' are the same circle, show that

$$(|a|^2 - r^2)^2 = 1$$

and that either a is real or a is imaginary. Give sketches to indicate the position of C in these two cases.

(ii) Suppose instead that the point Q is represented by ω, where $\omega = \dfrac{1}{z^*}$. If the locus of Q is C, is it the case that either a is real or a is imaginary?

Examiner's report

Half the candidates tried this question, but it was one of the four least successfully attempted. The majority successfully demonstrated that the locus of P was a circle with the correct centre and radius, but few made further significant progress. They usually substituted for z in terms of ω but then failed to rearrange into the necessary form. If they did achieve this, then they were able to score most of the marks up until the very last part which required careful justification to earn full marks.

Solution

Stem - 2 marks

Be careful not to miss the fact that in the stem you are asked to do something!

We have:

$$\begin{aligned}|z-a|^2 &= (z-a)(z-a)^* \\ &= (z-a)(z^*-a^*) \\ &= zz^* - az^* - a^*z + aa^*\end{aligned}$$

and so the given equation becomes $|z-a|^2 - r^2 = 0 \implies |z-a|^2 = r^2$ which is a circle, centre a and radius r.

The given information that $r^2 \neq aa^*$ has not been used here, but it might be worth thinking about it now. The implication of $r^2 \neq aa^*$ is that this circle does not pass through the origin.

Part (i) - 13 marks

$\omega = \frac{1}{z} \implies z = \frac{1}{\omega}$ and so we can substitute this into the equation given in the stem to get:

$$\begin{aligned}\left(\frac{1}{\omega}\right)\left(\frac{1}{\omega^*}\right) - a\left(\frac{1}{\omega^*}\right) - a^*\left(\frac{1}{\omega}\right) + aa^* - r^2 &= 0 \\ 1 - a\omega - a^*\omega^* + (aa^* - r^2)\omega\omega^* &= 0 \\ \frac{1}{aa^*-r^2} - \frac{a}{aa^*-r^2}\omega - \frac{a^*}{aa^*-r^2}\omega^* + \omega\omega^* &= 0\end{aligned}$$

Note that since $aa^* - r^2 \neq 0$ we can divide by this.

Rearranging with the hope of getting a circle equation gives:

$$\begin{aligned}\omega\omega^* - \left(\frac{a^*}{aa^*-r^2}\right)^*\omega - \left(\frac{a^*}{aa^*-r^2}\right)\omega^* + \left(\frac{a^*}{aa^*-r^2}\right)\left(\frac{a^*}{aa^*-r^2}\right)^* \\ = \left(\frac{a^*}{aa^*-r^2}\right)\left(\frac{a^*}{aa^*-r^2}\right)^* - \frac{1}{aa^*-r^2} \\ \left|\omega - \frac{a^*}{aa^*-r^2}\right|^2 &= \frac{a^*a}{(aa^*-r^2)^2} - \frac{1}{aa^*-r^2} \\ \left|\omega - \frac{a^*}{aa^*-r^2}\right|^2 &= \frac{a^*a - (aa^*-r^2)}{(aa^*-r^2)^2} \\ \left|\omega - \frac{a^*}{aa^*-r^2}\right|^2 &= \frac{r^2}{(aa^*-r^2)^2}\end{aligned}$$

Note that $aa^* - r^2$ is real.

Therefore ω is on a circle with centre $\dfrac{a^*}{aa^*-r^2}$ and radius $\left|\dfrac{r}{aa^*-r^2}\right|$.

If the two circles are the same then the centres and radii must be the same, so we have:

$$a = \frac{a^*}{aa^* - r^2}$$
$$r^2 = \frac{r^2}{(aa^* - r^2)^2}$$

Squaring the radii means that you do not have to deal with a modulus sign.

Since we are told that r is positive, $r \neq 0$ and we can divide the second equation above by r^2 to get:

$$(aa^* - r^2)^2 = 1$$

as required.

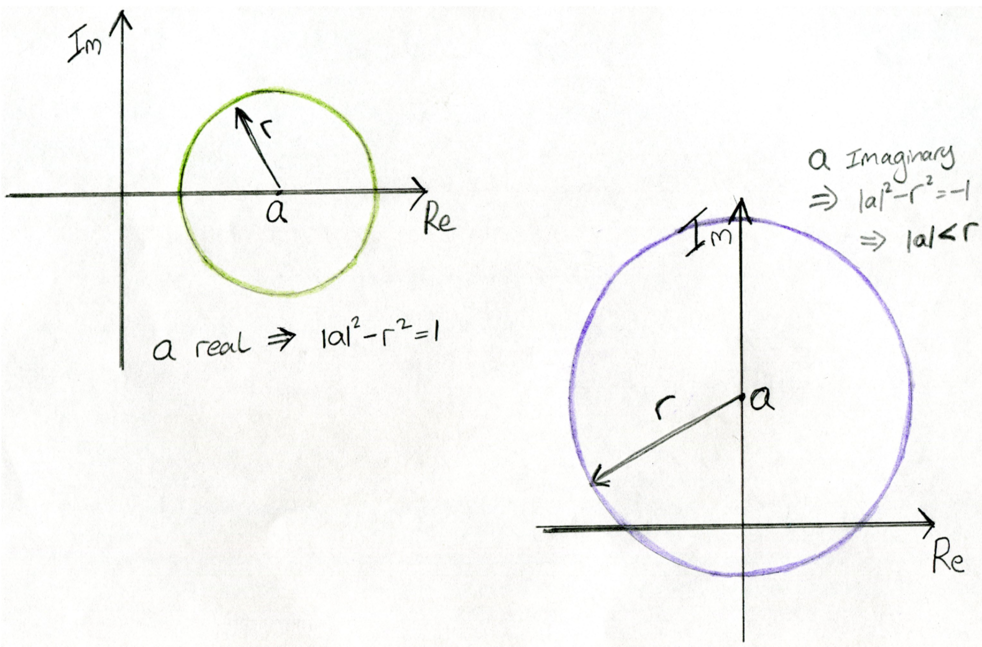
We also need to show that a is real or imaginary (i.e. it does not have both real and imaginary parts). Using the equation for equality of the centres we have:

$$a = \frac{a^*}{aa^* - r^2}$$
$$a(aa^* - r^2) = a^*$$
$$\pm a = a^*$$

Since $(aa^* - r^2)^2 = 1 \implies aa^* - r^2 = \pm 1$.

If $a = \alpha + \mathrm{i}\beta$ then $a = a^* \implies \alpha + \mathrm{i}\beta = \alpha - \mathrm{i}\beta$ and we must have $\beta = 0$ and a is real.
Alternatively $-a = a^* \implies -\alpha - \mathrm{i}\beta = \alpha - \mathrm{i}\beta$ and we must have $\alpha = 0$ and a is imaginary.

For the sketches be a little careful; if a is real then $aa^* - r^2 = 1$, but if a is imaginary then $aa^* - r^2 = -1$.



Part (ii) - 5 marks

If $\omega = \frac{1}{z^*}$ then we have:

$$\left(\frac{1}{\omega^*}\right)\left(\frac{1}{\omega}\right) - a\left(\frac{1}{\omega}\right) - a^*\left(\frac{1}{\omega^*}\right) + aa^* - r^2 = 0$$

$$1 - a\omega^* - a^*\omega + (aa^* - r^2)\omega^*\omega = 0$$

$$\frac{1}{aa^* - r^2} - \frac{a}{aa^* - r^2}\omega^* - \frac{a^*}{aa^* - r^2}\omega + \omega^*\omega = 0$$

$$\left|\omega - \frac{a}{aa^* - r^2}\right|^2 = \frac{aa^*}{(aa^* - r^2)^2} - \frac{1}{aa^* - r^2}$$

$$\left|\omega - \frac{a}{aa^* - r^2}\right|^2 = \frac{r^2}{(aa^* - r^2)^2}$$

In this case if the two circles are the same then $(|a|^2 - r^2)^2 = 1$ like before, but equating the centres gives $a = \dfrac{a}{aa^* - r^2}$. If $|a|^2 - r^2 = 1$ then this becomes $a = a$, so any value of a is possible, so it is not always the case that a is either real or imaginary.

For completeness, if $|a|^2 - r^2 = -1$ then we have $a = -a$ and so $a = 0$.

Question 7

7 The *Devil's Curve* is given by

$$y^2(y^2 - b^2) = x^2(x^2 - a^2)\,,$$

where a and b are positive constants.

(i) In the case $a = b$, sketch the Devil's Curve.

(ii) Now consider the case $a = 2$ and $b = \sqrt{5}$, and $x \geqslant 0$, $y \geqslant 0$.

(a) Show by considering a quadratic equation in x^2 that either $0 \leqslant y \leqslant 1$ or $y \geqslant 2$.

(b) Describe the curve very close to and very far from the origin.

(c) Find the points at which the tangent to the curve is parallel to the x-axis and the point at which the tangent to the curve is parallel to the y-axis.

Sketch the Devil's Curve in this case.

(iii) Sketch the Devil's Curve in the case $a = 2$ and $b = \sqrt{5}$ again, but with $-\infty < x < \infty$ and $-\infty < y < \infty$.

Examiner's report

Attempted by two thirds, the mean score was only about one third marks. Part **(i)** was not very well answered with many appearing to guess one or both of the solutions without managing to factorise, or equivalent. By contrast, part **(ii) (a)** was generally well-answered. Most candidates saw how to do this correctly, though a few tried to treat it as a polynomial in y and take that discriminant which got them nowhere. In part **(b)**, most candidates did not realise what was expected of them so, for example, many wrote $x \to \infty, y \to \infty$ or found the x and y intercepts. Almost all candidates saw what was needed for part **(c)**, however, many made small algebraic errors or didn't check all the cases; the most common error was failure to eliminate the origin. The sketch was generally badly answered, often owing to errors in previous parts or failure to use correctly the information already obtained. Candidates did appreciate what was needed for part **(iii)** and did this well.

Solution

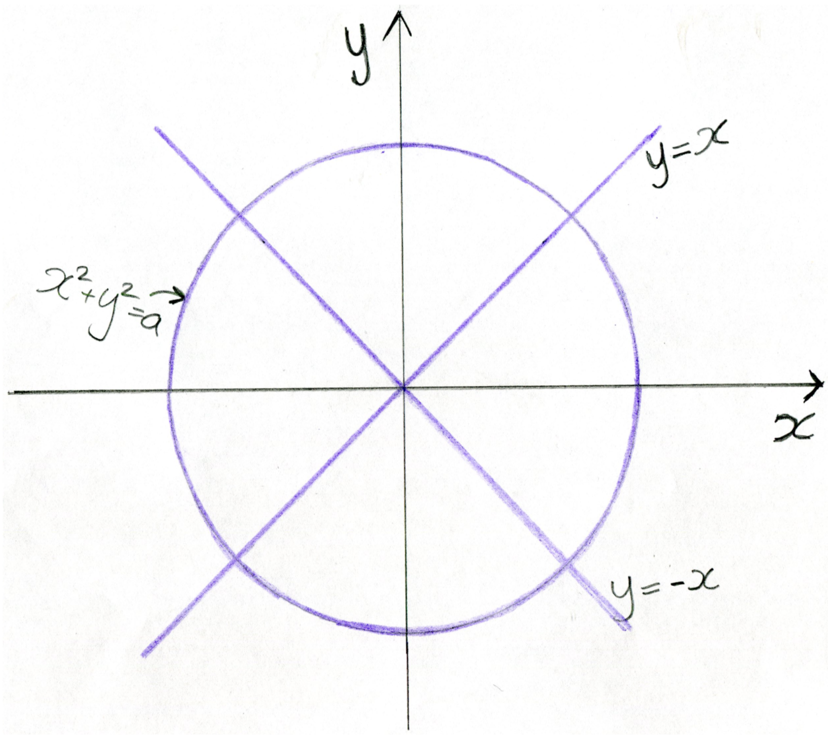
Part (i) - 3 marks

When $a = b$ we have:

$$\begin{aligned}
y^2(y^2 - a^2) &= x^2(x^2 - a^2) \\
y^4 - y^2a^2 &= x^4 - x^2a^2 \\
y^4 - y^2a^2 &= x^4 - x^2a^2 \\
\left(y^2 - \tfrac{1}{2}a^2\right)^2 - \tfrac{1}{4}a^4 &= \left(x^2 - \tfrac{1}{2}a^2\right)^2 - \tfrac{1}{4}a^4 \\
\left(y^2 - \tfrac{1}{2}a^2\right)^2 &= \left(x^2 - \tfrac{1}{2}a^2\right)^2 \\
y^2 - \tfrac{1}{2}a^2 &= \pm\left(x^2 - \tfrac{1}{2}a^2\right)
\end{aligned}$$

Taking the positive sign we have $y^2 = x^2$ which implies that $y = x$ or $y = -x$, so we get a pair of straight lines.

Taking the negative sign we have $y^2 + x^2 = a^2$, which is a circle radius a, centre $(0, 0)$.



Part (ii) (a) - 3 marks

Taking $a = 2$ and $b = \sqrt{5}$ we have:

$$\begin{aligned}
y^2(y^2 - 5) &= x^2(x^2 - 4) \\
x^4 - 4x^2 - y^4 + 5y^2 &= 0 \qquad (*)
\end{aligned}$$

Treating this as a quadratic equation in x^2, we can solve for x^2 as long as the discriminant is greater than or equal to 0, i.e.:

$$\begin{aligned} 16 - 4(5y^2 - y^4) &\geqslant 0 \\ 4y^4 - 20y^2 + 16 &\geqslant 0 \\ y^4 - 5y^2 + 4 &\geqslant 0 \\ (y^2 - 4)(y^2 - 1) &\geqslant 0 \\ (y+2)(y-2)(y+1)(y-1) &\geqslant 0 \end{aligned}$$

A quick sketch of this shows that this is true when $y \leqslant -2$, $-1 \leqslant y \leqslant 1$ and $y \geqslant 2$. However we are told that $y \geqslant 0$, so we need $0 \leqslant y \leqslant 1$ or $y \geqslant 2$.

If you don't get the required answer, it might be a good idea to look back at the information you have been given and see if there is anything you have not used. I had to keep reminding myself to ignore negative solutions in parts **(a), (b)** and **(c)**.

Part (ii) (b) - 2 marks

Very close to the origin we have $x^4 << x^2$ [3] and similar for the y terms, so using $(*)$ we have $-4x^2 + 5y^2 \approx 0$ i.e. $5y^2 \approx 4x^2 \implies \sqrt{5}y \approx \pm 2x$, which is a pair of straight lines passing through the origin.

Far away from the origin $x^4 >> x^2$ and so the equation becomes $x^4 \approx y^4$ i.e. $y \approx \pm x$.

Part (ii) (c) - 7 marks

Differentiating $(*)$ with respect to x gives:

$$\begin{aligned} 4x^3 - 8x - 4y^3\frac{\mathrm{d}y}{\mathrm{d}x} + 10y\frac{\mathrm{d}y}{\mathrm{d}x} &= 0 \\ (4y^3 - 10y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 4x^3 - 8x \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{4x^3 - 8x}{4y^3 - 10y} \\ &= \frac{2x^3 - 4x}{2y^3 - 5y} \end{aligned}$$

When the curve is parallel to the x-axis we have $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ i.e. $2x^3 - 4x = 0 \implies 2x(x^2 - 2)$ and so $x = 0, \pm\sqrt{2}$. However we are taking $x \geqslant 0$, so ignore the negative solution. Using the factorised form of the equation to find y coordinates we have:

- $x = 0$ gives $y^2(y^2 - 5) = 0 \implies y = 0, y = \sqrt{5}$ (remember that $y \geqslant 0$)
- $x = \sqrt{2}$ gives $y^2(y^2 - 5) = -4$ so $y^4 - 5y^2 + 4 = 0 \implies (y^2 - 1)(y^2 - 4) = 0$ and so $y = 1, 2$ (ignoring the negative solutions)

However at the point $(0, 0)$ the gradient is undefined using $\frac{\mathrm{d}y}{\mathrm{d}x}$, instead here use what happens close to the origin using part **(ii)(b)**. The points which are parallel to the x axis are $(0, \sqrt{5})$, $(\sqrt{2}, 1)$ and $(\sqrt{2}, 2)$.

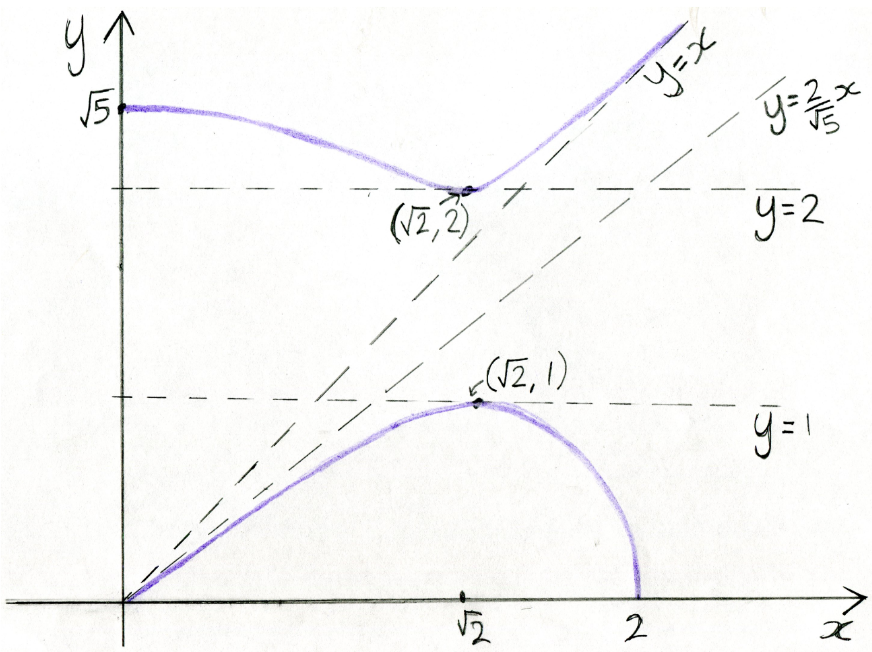
[3] This means that the x^4 term is very much smaller than the x^2 term.

Parallel to the y-axis we have $\dfrac{\mathrm{d}x}{\mathrm{d}y} = 0$ which means $2y^3 - 5y = 0 \implies y = 0, \sqrt{\frac{5}{2}}$.

Substituting into $y^2(y^2-5) = x^2(x^2-4)$ when $y = 0$ we have $x = 0, 2$ and when $y = \sqrt{\frac{5}{2}}$ we have $-\frac{25}{4} = x^4 - 4x^2 \implies 4x^4 - 16x^2 + 25 = 0$ which has a negative discriminant, so there are no real values of x. The only point where the curve is parallel to the y-axis is $(2, 0)$.

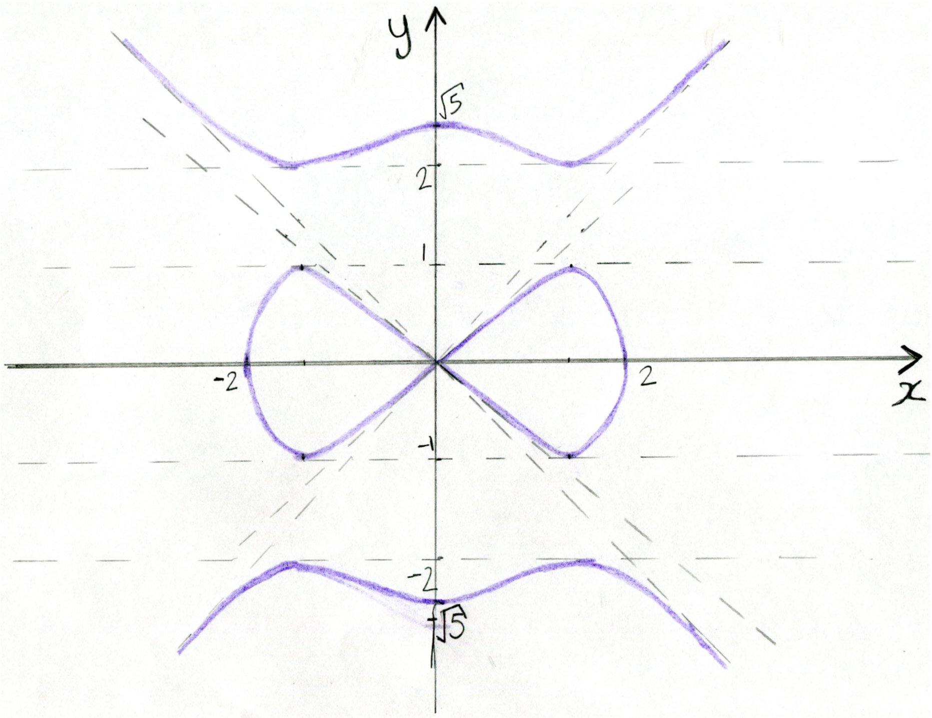
Part (ii) last request - 4 marks

Putting these parts altogether we can sketch the graph in the quadrant $x \geqslant 0, y \geqslant 0$. Draw the asymptotes $y = x$ and $y = \frac{2}{\sqrt{5}}x$ as dotted lines and make sure that the line $y = x$ has a steeper gradient.



Part (iii) - 1 mark

For this last part, the curve is unchanged as $x \to -x$ and $y \to -y$, so the curve has reflectional symmetry across the x-axis and y-axis.



It's quite nice to plot this using Desmos with two sliders a and b. You can then see how the graph looks for different values of a and b.

I spent too long drawing my graph, and you would not be expected to have one that looks this neat. See MEI STEP solutions for a more realistic exam sketch!

Question 8

8 A pyramid has a horizontal rectangular base $ABCD$ and its vertex V is vertically above the centre of the base. The acute angle between the face AVB and the base is α, the acute angle between the face BVC and the base is β and the obtuse angle between the faces AVB and BVC is $\pi - \theta$.

(i) The edges AB and BC are parallel to the unit vectors $\mathbf{i}$ and $\mathbf{j}$, respectively, and the unit vector $\mathbf{k}$ is vertical. Find a unit vector that is perpendicular to the face AVB.

Show that

$$\cos\theta = \cos\alpha\cos\beta\,.$$

(ii) The edge BV makes an angle ϕ with the base. Show that

$$\cot^2\phi = \cot^2\alpha + \cot^2\beta.$$

Show also that

$$\cos^2\phi = \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1-\cos^2\theta} \geqslant \frac{2\cos\theta - 2\cos^2\theta}{1-\cos^2\theta}$$

and deduce that $\phi < \theta$.

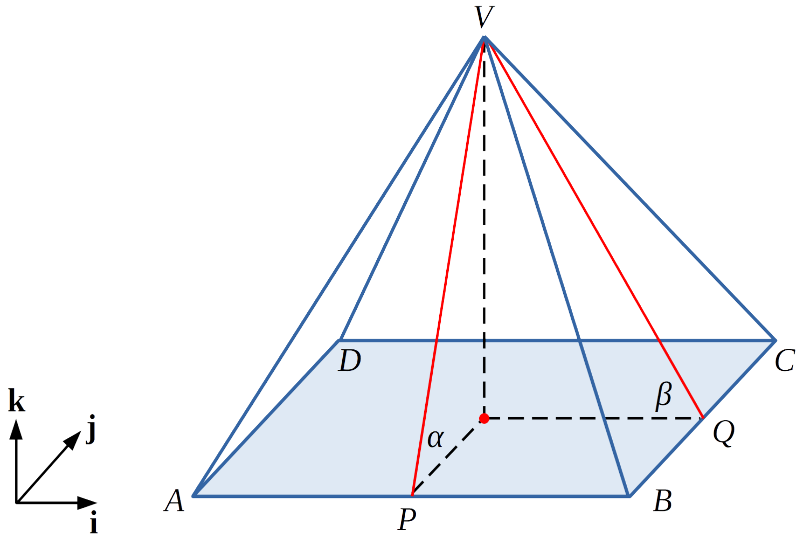
Examiner's report

The least popular of the Pure questions it was attempted by about a quarter of the candidates. It was generally found quite challenging with many attempts receiving little or no credit and so it was one of the four least successfully attempted questions. Some struggled to handle the vectors in part **(i)** with some attempting to divide by vectors or confusing cross and dot products, or normals and tangents, though these sorts of errors were rare. Much more common were errors arising from incorrect signs in the projections onto the base vectors **i**, **j**, and **k** or failure to recognise whether the angle being calculated was θ or $\pi - \theta$. In general, most candidates who took the time to establish a clear vector space set-up did rather well, not just in **(i)** but in **(ii)** as well. Most who attempted the first result of **(ii)** did so successfully. The most challenging part of the question was found to be obtaining the expression for $\cos^2\phi$ which required several small insights relating to trigonometric identities and a fair amount of calculation. A pleasing proportion attempting the calculation did so successfully but again a sound coordinate based set-up rendered it manageable. A number of candidates failed to justify properly the, often elementary, steps for the final part losing credit by so failing to do. Overall the question was largely algebraic rather than geometric, but the best solutions used the interplay between these two aspects to great effect.

Solution

Part (i) - 5 marks

It is probably a good idea to draw a diagram. It is really hard to show the angle between AVB and BVC (so I haven't!).



Let P be the midpoint of AB and let Q be the midpoint of BC.

We have:

$$\overrightarrow{PV} = \begin{pmatrix} 0 \\ \lambda\cos\alpha \\ \lambda\sin\alpha \end{pmatrix}$$

where λ is the length of PV. A unit vector perpendicular to AVB is $\mathbf{p} = \begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix}$.

It is reasonably obvious that $\overrightarrow{PV} \cdot \mathbf{p} = 0$. The reason that the question says "a unit vector" is that you could have picked $-\mathbf{p}$, or even have picked a vector with an $\mathbf{i}$ component (though this would have been a brave choice).

We also have:

$$\overrightarrow{QV} = \begin{pmatrix} -\mu\cos\beta \\ 0 \\ \mu\sin\beta \end{pmatrix}$$

and so a unit vector perpendicular to BVC is $\mathbf{q} = \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix}$.

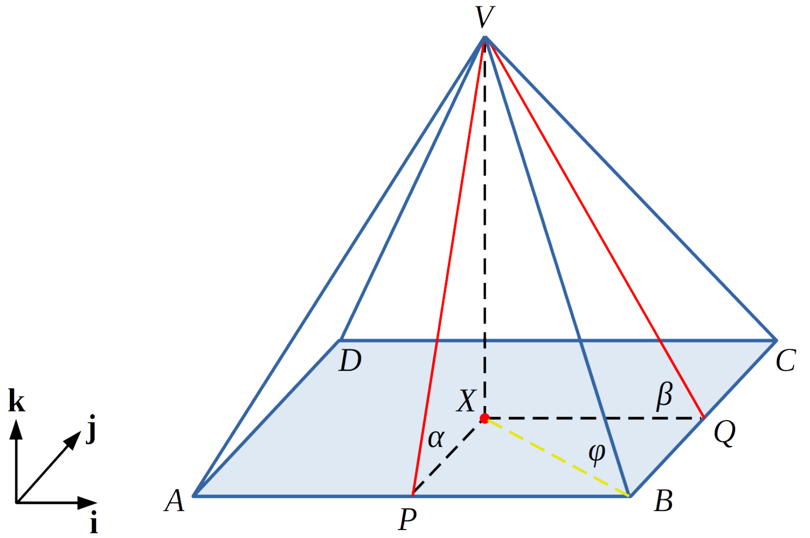
We are told that $\pi - \theta$ is the obtuse angle between the two faces. This means that θ is an acute angle, and this will be equal to the acute angle between $\mathbf{p}$ and $\mathbf{q}$. Since $\mathbf{p}$ and $\mathbf{q}$ are unit vectors this means we have:

$$\begin{pmatrix} 0 \\ -\sin\alpha \\ \cos\alpha \end{pmatrix} \cdot \begin{pmatrix} \sin\beta \\ 0 \\ \cos\beta \end{pmatrix} = 1 \times 1 \times \cos\theta$$

Hence we have $\cos\alpha\cos\beta = \cos\theta$ as required.

Part (ii) - 15 marks

Let the centre of the base of the pyramid be X, and add in angle ϕ to your diagram.



Using rectangle $PBQX$ we have $PB^2 + PX^2 = BX^2$. If $VX = h$ then we also have:

$$PX = h\cot\alpha$$
$$PB = QX = h\cot\beta$$
$$BX = h\cot\phi$$

Combining these gives $\cot^2\alpha + \cot^2\beta = \cot^2\phi$.

We are next asked to find a relationship for $\cos^2\phi$, so start by manipulating our expression for $\cot^2\phi$:

$$\begin{aligned}
\cot^2\phi &= \cot^2\alpha + \cot^2\beta \\
\tan^2\phi &= \frac{1}{\cot^2\alpha + \cot^2\beta} \\
\sec^2\phi - 1 &= \frac{1}{\cot^2\alpha + \cot^2\beta} \qquad \text{using } \tan^2\phi = \sec^2\phi - 1 \\
\sec^2\phi &= \frac{1 + \cot^2\alpha + \cot^2\beta}{\cot^2\alpha + \cot^2\beta} \\
\cos^2\phi &= \frac{\cot^2\alpha + \cot^2\beta}{\cot^2\alpha + \cot^2\beta + 1} \\
\cos^2\phi &= \frac{\tan^2\beta + \tan^2\alpha}{\tan^2\beta + \tan^2\beta + \tan^2\alpha\tan^2\beta} \qquad \text{multiplying by } \tan^2\alpha\tan^2\beta \\
\cos^2\phi &= \frac{\sec^2\beta + \sec^2\alpha - 2}{\sec^2\beta + \sec^2\alpha - 2 + (\sec^2\alpha - 1)(\sec^2\beta - 1)} \\
\cos^2\phi &= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\alpha\cos^2\beta}{\cos^2\alpha + \cos^2\beta - 2\cos^2\alpha\cos^2\beta + (1 - \cos^2\alpha)(1 - \cos^2\beta)} \qquad \text{multiplying by } \cos^2\alpha\cos^2\beta \\
\cos^2\phi &= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{\cancel{\cos^2\alpha} + \cancel{\cos^2\beta} - 2\cos^2\theta + 1 - \cancel{\cos^2\alpha} - \cancel{\cos^2\beta} + \cos^2\alpha\sin^2\beta} \qquad \text{using } \cos\theta = \cos\alpha\cos\beta \\
\cos^2\phi &= \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\theta}
\end{aligned}$$

We now need to show the inequality. Note that:

$$\begin{aligned}
(\cos\alpha - \cos\beta)^2 &\geqslant 0 \\
\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta &\geqslant 0 \\
\cos^2\alpha + \cos^2\beta &\geqslant 2\cos\alpha\cos\beta \\
\cos^2\alpha + \cos^2\beta &\geqslant 2\cos\theta
\end{aligned}$$

and so we have:

$$\cos^2\phi = \frac{\cos^2\alpha + \cos^2\beta - 2\cos^2\theta}{1 - \cos^2\theta} \geqslant \frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta}$$

as required.

Looking at the RHS of the inequality we have:

$$\begin{aligned}
\frac{2\cos\theta - 2\cos^2\theta}{1 - \cos^2\theta} &= \frac{2\cos\theta(1 - \cos\theta)}{(1 - \cos\theta)(1 + \cos\theta)} \\
&= \frac{2\cos\theta}{1 + \cos\theta} \\
&= \frac{2}{1 + \cos\theta} \times \cos\theta
\end{aligned}$$

Since θ is acute we know that $1 - \cos\theta \neq 0$. We also know that $1 + \cos\theta \leqslant 2$ so $\dfrac{2}{1 + \cos\theta} \geqslant 1$.

We now have:

$$\begin{aligned}\cos^2\phi &\geqslant \frac{2}{1+\cos\theta}\times\cos\theta\\ &\geqslant \cos\theta\\ &\geqslant \cos^2\theta \qquad \text{as } 0\leqslant\cos\theta\leqslant 1\end{aligned}$$

Since both $\cos\phi$ and $\cos\theta$ are positive this means we have $\cos\phi > \cos\theta$ and hence $\phi < \theta$ (as $\cos\theta$ is a decreasing function for $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

Question 9

9 In this question, $\mathbf{i}$ and $\mathbf{j}$ are perpendicular unit vectors and $\mathbf{j}$ is vertically upwards.

A smooth hemisphere of mass M and radius a rests on a smooth horizontal table with its plane face in contact with the table. The point A is at the top of the hemisphere and the point O is at the centre of its plane face.

Initially, a particle P of mass m rests at A. It is then given a small displacement in the positive $\mathbf{i}$ direction. At a later time t, when the particle is still in contact with the hemisphere, the hemisphere has been displaced by $-s\,\mathbf{i}$ and $\angle AOP = \theta$.

(i) Let $\mathbf{r}$ be the position vector of the particle at time t with respect to the initial position of O. Write down an expression for $\mathbf{r}$ in terms of a, θ and s and show that

$$\dot{\mathbf{r}} = (a\dot{\theta}\cos\theta - \dot{s})\,\mathbf{i} - a\dot{\theta}\sin\theta\,\mathbf{j}.$$

Show also that

$$\dot{s} = (1-k)\,a\,\dot{\theta}\cos\theta,$$

where $k = \dfrac{M}{m+M}$, and deduce that

$$\dot{\mathbf{r}} = a\dot{\theta}(k\cos\theta\,\mathbf{i} - \sin\theta\,\mathbf{j}).$$

(ii) Show that

$$a\dot{\theta}^2\left(k\cos^2\theta + \sin^2\theta\right) = 2g(1-\cos\theta).$$

(iii) At time T, when $\theta = \alpha$, the particle leaves the hemisphere. By considering the component of $\ddot{\mathbf{r}}$ parallel to the vector $\sin\theta\,\mathbf{i} + k\cos\theta\,\mathbf{j}$, or otherwise, show that at time T

$$a\dot{\theta}^2 = g\cos\alpha.$$

Find a cubic equation for $\cos\alpha$ and deduce that $\cos\alpha > \frac{2}{3}$.

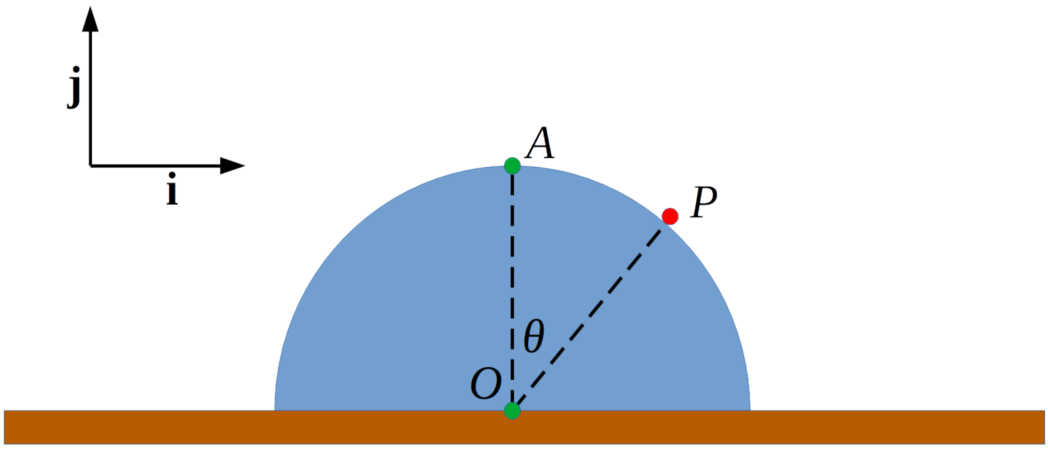
Examiner's report

Although this was the least frequently attempted question, it was tried by just over 10% scoring on average marginally under one third marks. The position and velocity in **(i)** were successful for most, and those considering momentum (or centre of mass) found the next result easy, although those considering forces struggled. Most struggled with **(ii)**, for whilst considering energy, typically they forgot the kinetic energy of the hemisphere. Part **(iii)** evoked a number of approaches, which were usually unsuccessful, and most could not make use of the suggested method. Even candidates who were unsuccessful with the rest of the question were able to obtain the cubic equation, though hardly any could justify the final inequality.

Solution

The hemisphere is free to slide on the table (both are smooth), so as the particle moves on the hemisphere it exerts a force on the hemisphere which causes it to slide.

Your diagram might look like this:



Part (i) - 6 marks

At time t point O has moved to $\begin{pmatrix} -s \\ 0 \end{pmatrix}$. In relation to the position of O at time t, P will be at $\begin{pmatrix} a\sin\theta \\ a\cos\theta \end{pmatrix}$. Combining these gives:

$$\mathbf{r} = \begin{pmatrix} a\sin\theta - s \\ a\cos\theta \end{pmatrix} = (a\sin\theta - s)\mathbf{i} + a\cos\theta\mathbf{j}$$

This was a "write down" so you did not need to show any justification.

Differentiation with respect to time gives:

$$\dot{\mathbf{r}} = (a\dot{\theta}\cos\theta - \dot{s})\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}$$

For the next request note that k is in terms of m and M, so it might be an idea to look at momentum. Conservation of horizontal momentum gives:

$$\begin{aligned} m(a\dot{\theta}\cos\theta - \dot{s}) - M\dot{s} &= 0 \\ ma\dot{\theta}\cos\theta &= (m+M)\dot{s} \\ \implies \dot{s} &= \frac{m}{m+M}a\dot{\theta}\cos\theta \\ \dot{s} &= \left(1 - \frac{M}{m+M}\right)a\dot{\theta}\cos\theta \\ \dot{s} &= (1-k)a\dot{\theta}\cos\theta \end{aligned}$$

Substituting this expression for $\dot{s}$ into the expression for $\dot{\mathbf{r}}$ gives:

$$\begin{aligned}\dot{\mathbf{r}} &= \left(a\dot{\theta}\cos\theta - (1-k)a\dot{\theta}\cos\theta\right)\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}\\ &= ak\dot{\theta}\cos\theta\mathbf{i} - a\dot{\theta}\sin\theta\mathbf{j}\\ &= a\dot{\theta}(k\cos\theta\mathbf{i} - \sin\theta\mathbf{j})\end{aligned}$$

Note that $\dot{\mathbf{r}}$ is the velocity of the particle — we will use this in the conservation of energy equation below.

Part (ii) - 5 marks

The energy of the system at the start is mga (as neither particle nor hemisphere are moving, the only energy is the potential energy of the particle). At time t there will the kinetic energy of the hemisphere and particle to include. This gives:

$$\begin{aligned}mga &= mga\cos\theta + \tfrac{1}{2}M\dot{s}^2 + \tfrac{1}{2}m\left[(a\dot{\theta}k\cos\theta)^2 + (a\dot{\theta}\sin\theta)^2\right]\\ &= mga\cos\theta + \tfrac{1}{2}M(1-k)^2a^2\dot{\theta}^2\cos^2\theta + \tfrac{1}{2}ma^2\dot{\theta}^2\left[k^2\cos^2\theta + \sin^2\theta\right]\\ 2mga(1-\cos\theta) &= a^2\dot{\theta}^2\left[M(1-k)^2\cos^2\theta + m(k^2\cos^2\theta + \sin^2\theta)\right]\\ 2g(1-\cos\theta) &= a\dot{\theta}^2\left[\frac{M}{m}(1-k)^2\cos^2\theta + (k^2\cos^2\theta + \sin^2\theta)\right]\\ 2g(1-\cos\theta) &= a\dot{\theta}^2\left[\left(\frac{M}{m}\times\left(\frac{m}{m+M}\right)^2 + k^2\right)\cos^2\theta + \sin^2\theta\right]\\ 2g(1-\cos\theta) &= a\dot{\theta}^2\left[\left(\frac{Mm}{(m+M)^2} + \frac{M^2}{(m+M)^2}\right)\cos^2\theta + \sin^2\theta\right]\\ 2g(1-\cos\theta) &= a\dot{\theta}^2\left[\left(\frac{M(m+M)}{(m+M)^2}\right)\cos^2\theta + \sin^2\theta\right]\\ 2g(1-\cos\theta) &= a\dot{\theta}^2\left[k\cos^2\theta + \sin^2\theta\right]\end{aligned}$$

Part (iii) - 9 marks

This part suggests considering a component of $\ddot{\mathbf{r}}$, so start by finding $\ddot{\mathbf{r}}$:

$$\begin{aligned}\dot{\mathbf{r}} &= \left[a\dot{\theta}k\cos\theta\right]\mathbf{i} - \left[a\dot{\theta}\sin\theta\right]\mathbf{j}\\ \ddot{\mathbf{r}} &= \left[a\ddot{\theta}k\cos\theta - a(\dot{\theta})^2k\sin\theta\right]\mathbf{i} - \left[a\ddot{\theta}\sin\theta + a(\dot{\theta})^2\cos\theta\right]\mathbf{j}\\ &= a\ddot{\theta}\begin{pmatrix}k\cos\theta\\ -\sin\theta\end{pmatrix} - a(\dot{\theta})^2\begin{pmatrix}k\sin\theta\\ \cos\theta\end{pmatrix}\end{aligned}$$

The particle loses contact when $\ddot{\mathbf{r}} = -g\mathbf{j}$ which happens when $\theta = \alpha$. This means we have:

$$a\ddot{\theta}\begin{pmatrix}k\cos\alpha\\ -\sin\alpha\end{pmatrix} - a(\dot{\theta})^2\begin{pmatrix}k\sin\alpha\\ \cos\alpha\end{pmatrix} = \begin{pmatrix}0\\ -g\end{pmatrix} \qquad (*)$$

(where $\ddot{\theta}$ and $\dot{\theta}$ are the values that they take when $\theta = \alpha$).

The question says "considering the component of $\ddot{\mathbf{r}}$ parallel to the vector $\sin\theta\mathbf{i} + k\cos\theta\mathbf{j}$", so it would be nice to be able to ignore the first vector of $(*)$. Taking the dot product with $\begin{pmatrix}\sin\alpha\\ k\cos\alpha\end{pmatrix}$

will eliminate this first vector and we get:

$$\begin{aligned}-a(\dot{\theta})^2\begin{pmatrix}k\sin\alpha\\ \cos\alpha\end{pmatrix}\cdot\begin{pmatrix}\sin\alpha\\ k\cos\alpha\end{pmatrix} &= \begin{pmatrix}0\\ -g\end{pmatrix}\cdot\begin{pmatrix}\sin\alpha\\ k\cos\alpha\end{pmatrix}\\ -a(\dot{\theta})^2\left[k\sin^2\alpha+k\cos^2\alpha\right] &= -gk\cos\alpha\\ a(\dot{\theta})^2 &= g\cos\alpha\end{aligned}$$

Using the result $2g(1-\cos\theta) = a\dot{\theta}^2\left[k\cos^2\theta+\sin^2\theta\right]$ and substituting in $a(\dot{\theta})^2 = g\cos\alpha$ when $\theta=\alpha$ gives:

$$\begin{aligned}2\not{g}(1-\cos\alpha) &= \not{g}\cos\alpha\left[k\cos^2\alpha+\sin^2\alpha\right]\\ 2-2\cos\alpha &= k\cos^3\alpha+\cos\alpha-\cos^3\alpha\\ (1-k)\cos^3\alpha &= 3\cos\alpha-2\end{aligned}$$

Since $k=\frac{M}{m+M}$ we have $k<1$ and as $0<\alpha<\frac{\pi}{2}$ we have $(1-k)\cos^3\alpha>0$ and so $3\cos\alpha-2>0 \implies \cos\alpha>\frac{2}{3}$.

Question 10

10 Two identical smooth spheres P and Q can move on a smooth horizontal table. Initially, P moves with speed u and Q is at rest. Then P collides with Q. The direction of travel of P before the collision makes an acute angle α with the line joining the centres of P and Q at the moment of the collision. The coefficient of restitution between P and Q is e where $e < 1$.

As a result of the collision, P has speed v and Q has speed w, and P is deflected through an angle θ.

(i) Show that

$$u \sin\alpha = v \sin(\alpha + \theta)$$

and find an expression for w in terms of v, θ and α.

(ii) Show further that

$$\sin\theta = \cos(\theta + \alpha)\sin\alpha + e\sin(\theta + \alpha)\cos\alpha$$

and find an expression for $\tan\theta$ in terms of $\tan\alpha$ and e.

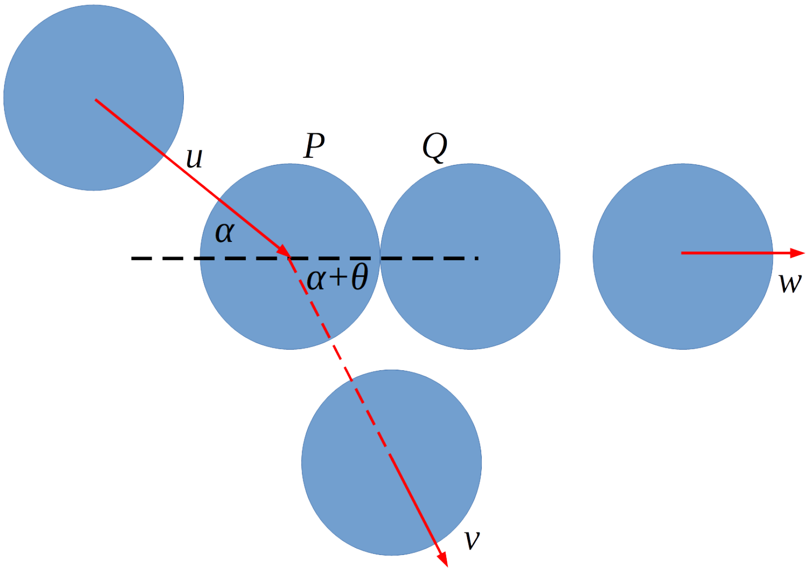
Find, in terms of e, the maximum value of $\tan\theta$ as α varies.

Examiner's report

Comfortably the most popular applied question on the paper with two fifths of candidates trying it, it was also one of the most successfully attempted on the whole paper with an average score just shy of half marks. A significant number of candidates struggled to set up the problem correctly, but those that did generally obtained the first result of **(i)**. Then common mistakes were using Newton's Law of Impact in this part and failure to express $\dot{\theta}$ in terms of the specified variables. In part **(ii)**, most candidates correctly applied Newton's Law of Impact, but depending on their expression for $\dot{\theta}$ in **(i)** had varying levels of success obtaining the required expression. A lot of candidates did manage to gain most marks in the final part even if they had struggled earlier. Alongside trigonometric and algebraic mistakes, a common mistake was failing to express $\tan\theta$ in terms of $\tan\alpha$, as against other trigonometric ratios, and e. The commonest approach for the final result was to use differentiation, although a few candidates successfully used the AM-GM inequality. However, fewer than a handful of candidates justified the value being a global as against local maximum.

Solution

As usual with a mechanics question, it is a god idea to start with a diagram! The one below shows the information given in the stem.



Part (i) - 7 marks

Let the masses of P and Q be m.[4]

The spheres are smooth, and there is no friction with the table so the momentum of P perpendicular to the line joining the centres (sometimes called the "line of centres" or "LOC") is unchanged. This gives:

$$mu\sin\alpha = mv\sin(\alpha+\theta) \implies u\sin\alpha = v\sin(\alpha+\theta)$$

Using momentum along the LOC we have:

$$mu\cos\alpha = mv\cos(\alpha+\theta) + mw \implies u\cos\alpha = v\cos(\alpha+\theta) + w$$

The question asks for w in terms of v, α and θ, so use the first result to eliminate u to get:

$$\begin{aligned}
\frac{v\sin(\alpha+\theta)}{\sin\alpha}\cos\alpha &= v\cos(\alpha+\theta) + w \\
w\sin\alpha &= v\big[\sin(\alpha+\theta)\cos\alpha - \cos(\alpha+\theta)\sin\alpha\big] \\
w\sin\alpha &= v\big[\sin\big((\alpha+\theta)-\alpha\big)\big] \\
w &= v\frac{\sin\theta}{\sin\alpha}
\end{aligned}$$

When I first did this question I expanded $\sin(\alpha+\theta)$ and $\cos(\alpha+\theta)$, which works but is perhaps a little less elegant. You don't get extra marks for elegance though, so if you had done it the expansion way and it worked you don't need to re-do your working!

[4] Actually we could WLOG make the masses both equal to 1.

Part (ii) - 13 marks

Using Newton's Experimental Law in the direction of the line connecting the centres we have:

$$w - v\cos(\alpha+\theta) = eu\cos\alpha$$

I always think of NEL as "speed of separation $= e\times$ speed of approach". This way I (usually) don't make a mistake with the negative signs!

The result we are trying to show has none of u, v and w present. In part **(i)** we had found u and w in terms of v, so substitute these in:

$$\begin{aligned} w - v\cos(\alpha+\theta) &= eu\cos\alpha \\ v\frac{\sin\theta}{\sin\alpha} - v\cos(\alpha+\theta) &= ev\frac{\sin(\alpha+\theta)}{\sin\alpha}\times\cos\alpha \\ \sin\theta - \cos(\alpha+\theta)\sin\alpha &= e\sin(\alpha+\theta)\cos\alpha \\ \sin\theta &= \cos(\alpha+\theta)\sin\alpha + e\sin(\alpha+\theta)\cos\alpha \end{aligned}$$

as required.

Expanding the compound angle expressions and then dividing by $\cos\theta$ gives:

$$\begin{aligned} \sin\theta &= \left[\cos\alpha\cos\theta - \sin\alpha\sin\theta\right]\sin\alpha + e\left[\sin\alpha\cos\theta + \cos\alpha\sin\theta\right]\cos\alpha \\ \tan\theta &= \left[\cos\alpha - \sin\alpha\tan\theta\right]\sin\alpha + e\left[\sin\alpha + \cos\alpha\tan\theta\right]\cos\alpha \\ &= \cos\alpha\sin\alpha - \sin^2\alpha\tan\theta + e\sin\alpha\cos\alpha + e\cos^2\alpha\tan\theta \end{aligned}$$

Gathering terms in $\tan\theta$ gives:

$$\tan\theta\left(1+\sin^2\alpha - e\cos^2\alpha\right) = (1+e)\sin\alpha\cos\alpha$$

Dividing by $\cos^2\alpha$ gives:

$$\begin{aligned} \tan\theta\left(\sec^2\alpha + \tan^2\alpha - e\right) &= (1+e)\tan\alpha \\ \tan\theta\left(1+\tan^2\alpha+\tan^2\alpha - e\right) &= (1+e)\tan\alpha \\ \tan\theta &= \frac{(1+e)\tan\alpha}{1-e+2\tan^2\alpha} \end{aligned}$$

To make life a little easier, use $\tan\alpha = t$:

$$\tan\theta = \frac{(1+e)t}{1-e+2t^2}$$

To find the maximum differentiate $\tan\theta$ with respect to $t(=\tan\alpha)$.

$$\frac{\mathrm{d}}{\mathrm{d}t}(\tan\theta) = \frac{(1-e+2t^2)(1+e) - 4t(1+e)t}{(1-e+2t^2)^2}$$

At the maximum we have $\dfrac{\mathrm{d}}{\mathrm{d}t}\tan\theta = 0$ and so:

$$\begin{aligned}(1-e+2t^2)(1+e) - 4t(1+e)t &= 0\\ 1-e+2t^2-4t^2 &= 0\\ 2t^2 &= 1-e\\ t &= \sqrt{\frac{1-e}{2}}\end{aligned}$$

Since $0 \leqslant \alpha < \frac{1}{2}\pi$ we have $\tan\alpha = t \geqslant 0$ and so we only take the positive root.

Substituting this into the expression for $\tan\theta$ gives:

$$\begin{aligned}\tan\theta &= \frac{(1+e)t}{1-e+2t^2}\\ &= \frac{(1+e)\times\sqrt{\frac{1-e}{2}}}{1-e+2\times\frac{1-e}{2}}\\ &= \frac{(1+e)\times\sqrt{\frac{1-e}{2}}}{2-2e}\\ &= \frac{(1+e)}{2-2e}\times\sqrt{\frac{1-e}{2}}\\ &= \frac{(1+e)}{2\sqrt{2}\sqrt{1-e}}\end{aligned}$$

There is only one stationary point, and from the expression for $\tan\theta$ in terms of t we can see that when $t = 0$, $\tan\theta = 0$ and as $t \to \infty$, $\tan\theta \to 0$, and the curve is continuous so this stationary point must be a maximum point.

Question 11

11 The number of customers arriving at a builders' merchants each day follows a Poisson distribution with mean λ. Each customer is offered some free sand. The probability of any given customer taking the free sand is p.

(i) Show that the number of customers each day who take sand follows a Poisson distribution with mean $p\lambda$.

(ii) The merchant has a mass S of sand at the beginning of the day. Each customer who takes the free sand gets a proportion k of the remaining sand, where $0 \leqslant k < 1$. Show that by the end of the day the expected mass of sand taken is

$$\big(1 - \mathrm{e}^{-kp\lambda}\big)S.$$

(iii) At the beginning of the day, the merchant's bag of sand contains a large number of grains, exactly one of which is made from solid gold. At the end of the day, the merchant's assistant takes a proportion k of the remaining sand. Find the probability that the assistant takes the golden grain. Comment on the case $k = 0$ and on the limit $k \to 1$.

In the case $p\lambda > 1$ find the value of k which maximises the probability that the assistant takes the golden grain.

Examiner's report

The least successful question with a mean score of just under one quarter marks, it was attempted by a fifth of the candidates. Many candidates assumed for **(i)** that the number of customers taking sand would follow a Poisson distribution without giving a proof and some thought that checking the mean equalled the variance was sufficient. For **(ii)**, some assumed $\mathrm{E}\big[\mathrm{f}(x)\big] = \mathrm{f}\big(\mathrm{E}[X]\big)$ which may have been with an eye to the 'show that' so e.g. '$Y \sim \mathrm{Po}(\lambda p)$, $\mathrm{e}^{-k\lambda p} = \mathrm{E}[\mathrm{e}^{-kY}]$'. For **(iii)**, a minority used the law of total probability and were generally successful. A popular approach was to spot that P(assistant gets sand) = E[proportion of sand they take], however few were able to express this as a correct probabilistic statement. In particular, some treated the amount of sand taken/remaining as a deterministic constant equal to its mean. Many candidates struggled to differentiate $k\mathrm{e}^{-k\lambda p}$ with respect to k, and few gave a valid justification why the stationary point was a maximum.

Solution

Part (i) - 5 marks

The probability that i customers arrive on a day is $\mathrm{P}(N=i)=\frac{\mathrm{e}^{-\lambda}\lambda^n}{n!}$. If X is the number of customers who take sand then we have:

$$\begin{aligned}\mathrm{P}(X=r) &= \mathrm{P}(N=r)\times\mathrm{P}(\text{all } r \text{ take the sand})+\mathrm{P}(N=r+1)\times\mathrm{P}(r \text{ of the } r+1 \text{ take sand})\\ &\quad +\mathrm{P}(N=r+2)\times\mathrm{P}(r \text{ of the } r+2 \text{ take the sand})\cdots\\ &= \frac{\mathrm{e}^{-\lambda}\lambda^r}{r!}p^r+\frac{\mathrm{e}^{-\lambda}\lambda^{r+1}}{(r+1)!}(r+1)p^r(1-p)+\frac{\mathrm{e}^{-\lambda}\lambda^{r+2}}{(r+1)!}\binom{r+2}{r}p^r(1-p)^2+\cdots\\ &= \sum_{i=r}^{\infty}\frac{\lambda^i\mathrm{e}^{-\lambda}}{i!}\times\binom{i}{r}p^r(1-p)^{i-r}\end{aligned}$$

Extracting terms which are independent of i from the sum gives:

$$\begin{aligned}&\mathrm{e}^{-\lambda}p^r\sum_{i=r}^{\infty}\frac{\lambda^i}{i!}\times\binom{i}{r}(1-p)^{i-r}\\ =&\mathrm{e}^{-\lambda}p^r\sum_{i=r}^{\infty}\frac{\lambda^i}{\cancel{i!}}\times\left(\frac{\cancel{i!}}{r!(i-r)!}\right)(1-p)^{i-r}\\ =&\frac{\mathrm{e}^{-\lambda}p^r\lambda^r}{r!}\sum_{i=r}^{\infty}\frac{\lambda^{i-r}}{(i-r)!}(1-p)^{i-r}\\ =&\frac{\mathrm{e}^{-\lambda}p^r\lambda^r}{r!}\sum_{k=0}^{\infty}\frac{\lambda^k}{(k)!}(1-p)^k \qquad \text{using } k=i-r\\ =&\frac{\mathrm{e}^{-\lambda}(\lambda p)^r}{r!}\sum_{k=0}^{\infty}\frac{[\lambda(1-p)]^k}{(k)!}\\ =&\frac{\mathrm{e}^{-\lambda}(\lambda p)^r}{r!}\times\mathrm{e}^{\lambda(1-p)}\\ =&\frac{\mathrm{e}^{-\lambda p}(\lambda p)^r}{r!}\end{aligned}$$

which means that X is a Poisson distribution with mean λp.

Part (ii) - 6 marks

The set up here is to ensure that there is always some sand available for those who want it. For example, if $k=0.5$ each customer would get 50% of the sand remaining, and no-one ever takes the last bit. (We are assuming that this won't get down to individual grains of sand.)

If 1 customer takes sand then kS will have been taken, leaving $(1-k)S$ sand left. If 2 customers take sand then $(k+k(1-k))S$ will have been taken and there will be $(1-k)^2S$ left. If r customers take sand then $k(1+(1-k)+(1-k)^2+\cdots+(1-k)^{r-1})S$ will have been taken and there will be $(1-k)^rS$ left.

We can sum the geometric series for the r^{th} case to get $\dfrac{k(1-(1-k)^r)}{1-(1-k)}S = [1-(1-k)^r]S$ sand taken — or more simply use the fact that if r people have taken sand then there will be $(1-k)^r S$ sand left and so the amount taken must be $[1-(1-k)^r]S$.

Using $\mathrm{E}(X) = \sum x \times \mathrm{P}(X = x)$ we have:

$$\begin{aligned}
\text{Expected mass of sand taken} &= \sum_{r=0}^{\infty}[1-(1-k)^r]S \times \frac{\mathrm{e}^{-\lambda p}(\lambda p)^r}{r!} \\
&= S\sum_{r=0}^{\infty}\frac{\mathrm{e}^{-\lambda p}(\lambda p)^r}{r!} - S\sum_{r=0}^{\infty}(1-k)^r\frac{\mathrm{e}^{-\lambda p}(\lambda p)^r}{r!} \\
&= S - S\sum_{r=0}^{\infty}\frac{\mathrm{e}^{-\lambda p}(\lambda p(1-k))^r}{r!} \\
&= S - S\mathrm{e}^{-\lambda pk}\sum_{r=0}^{\infty}\frac{\mathrm{e}^{-\lambda p(1-k)}(\lambda p(1-k))^r}{r!} \\
&= S - S\mathrm{e}^{-\lambda pk} \\
&= S(1-\mathrm{e}^{-k\lambda p})
\end{aligned}$$

We know that the sum of the probabilities of a Poisson distribution is equal to 1, so we have $\sum \dfrac{\mathrm{e}^{-\mu}\mu^r}{r!} = 1$. This has been used twice above, once with $\mu = \lambda p$ and once with $\mu = \lambda p(1-k)$.

To double check this is reasonable, when $k = 0$ we would expect no sand to be taken, and $S(1-\mathrm{e}^0) = 0$ as expected.

Part (iii) - 9 marks

If r customers have taken sand, then there will be $(1-k)^r S$ left. The assistant will take $k(1-k)^r S$ sand. The probability that this will contain the golden grain in this case $\dfrac{k(1-k)^r S}{S} = k(1-k)^r$.

To find the overall probability we need to sum over all the possibilities:

$$\begin{aligned}
\mathrm{P}(\text{Assistant takes the golden grain}) &= \sum_{r=0}^{\infty}\mathrm{P}(r \text{ customers take sand}) \times k(1-k)^r \\
&= \sum_{r=0}^{\infty}\frac{\mathrm{e}^{-\lambda p}(\lambda p)^r}{r!} \times k(1-k)^r \\
&= k\mathrm{e}^{-\lambda p}\sum_{r=0}^{\infty}\frac{[(1-k)\lambda p]^r}{r!} \\
&= k\mathrm{e}^{-\lambda p} \times \mathrm{e}^{(1-k)\lambda p} \\
&= k\mathrm{e}^{-k\lambda p}
\end{aligned}$$

If $k = 0$ then no-one (including the assistant) takes any sand, so the probability should be 0. We have $0 \times \mathrm{e}^0 = 0$ as expected.

As $k \to 1$ the probability that the assistant takes the sand tends to $e^{-\lambda p}$. From part **(i)** this is the same as the probability that no customers take sand. This makes sense as when $k \to 1$, the only way that there will be any sand left for the assistant is if no customers take sand. Hence the probability that the assistant takes the golden grain is the same as the probability that no customers took any sand.

Question 12

12 The set S is the set of all integers from 1 to n. The set T is the set of all distinct subsets of S, including the empty set $\varnothing$ and S itself. Show that T contains exactly 2^n sets.

The sets $A_1,\ A_2,\ \ldots,\ A_m$, which are not necessarily distinct, are chosen randomly and independently from T, and for each k $(1 \leqslant k \leqslant m)$, the set A_k is equally likely to be any of the sets in T.

(i) Write down the value of $\mathrm{P}(1 \in A_1)$.

(ii) By considering each integer separately, show that $\mathrm{P}(A_1 \cap A_2 = \varnothing) = (\frac{3}{4})^n$.

Find $\mathrm{P}(A_1 \cap A_2 \cap A_3 = \varnothing)$ and $\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_m = \varnothing)$.

(iii) Find $\mathrm{P}(A_1 \subseteq A_2)$, $\mathrm{P}(A_1 \subseteq A_2 \subseteq A_3)$ and $\mathrm{P}(A_1 \subseteq A_2 \subseteq \cdots \subseteq A_m)$.

Examiner's report

A quarter of the candidates attempted this scoring on average about half marks, making it the second most successfully attempted question. Candidates using the approaches suggested by the question tended to make good progress. Many produced correct solutions using various combinatorial arguments, some of which were easier to generalise than others and not all were well-explained. In part **(ii)**, a few candidates used arguments not permitted by the wording of the question. In part **(iii)**, some obtained the first result by the nice method $\mathrm{P}(A_1 \subseteq A_2) = \mathrm{P}(A_1 \cap A_2' = \varnothing)$ which is of course the first result of **(ii)** but this was not easy to generalise. Several incorrectly assumed $\mathrm{P}(A_1 \subseteq A_2 \subseteq A_3) = \mathrm{P}(A_1 \subseteq A_2) \times \mathrm{P}(A_2 \subseteq A_3)$ "by independence".

Solution

In the examiner's report it says that this was the second most successfully attempted question — do have a look at the Probability and Mechanics questions! There is actually a request in the stem of the question, be careful not to miss these out.

Stem - 1 mark

Each integer can either be in or out of a particular subset. There are n integers altogether so the total number of distinct subsets is $2 \times 2 \times \cdots \times 2 = 2^n$.

Part (i) - 1 mark

This is a "write down", so no working required. $\mathrm{P}(1 \in A_1) = \frac{1}{2}$.

Part (ii) - 7 marks

Here we are looking for the probability that there are no common integers in both A_1 and A_2. The probability that a particular integer is is both A_1 and A_2 is $\mathrm{P}(k \in A_1 \cap m \in A_2) = \frac{1}{4}$. This means that the probability that k is **not** in both the sets is $\left(\frac{3}{4}\right)$.

If $A_1 \cap A_2 = \varnothing$ then 1 cannot be in both the sets, and 2 cannot be in both the sets, etc. We have $\mathrm{P}(A_1 \cap A_2 = \varnothing) = \frac{3}{4} \times \frac{3}{4} \times \frac{3}{4} \times \cdots \times \frac{3}{4} = \left(\frac{3}{4}\right)^n$.

The probability that k is in all three subsets A_1, A_2 and A_3 is $\frac{1}{8}$, so that probability that it is **not** in all three is $\frac{7}{8}$.

We have:

$$\mathrm{P}(A_1 \cap A_2 \cap A_3 = \varnothing) = \left(\frac{7}{8}\right)^n$$

and similarly:

$$\mathrm{P}(A_1 \cap A_2 \cap \cdots \cap A_m = \varnothing) = \left(1 - \frac{1}{2^m}\right)^n$$

Part (iii) - 11 marks

For $A_1 \subseteq A_2$ then each integer k is either in both the subsets, is in neither of the subsets, or is in A_2 but not A_1. The only possibility that we cannot have is that k is in A_1 but not in A_2. The probability that k is in A_1 but not in A_2 is $\frac{1}{4}$, so the probability that k does what we want is $\frac{3}{4}$. This is the same for all the integers so we have $\mathrm{P}(A_1 \subseteq A_2) = \left(\frac{3}{4}\right)^n$.

For $A_1 \subseteq A_2 \subseteq A_3$ each integer k has to obey one of the following:

A_1	A_2	A_3
✓	✓	✓
✗	✓	✓
✗	✗	✓
✗	✗	✗

This means that 4 out of the 8 options for k placement do what we want. Hence

$$\mathrm{P}(A_1 \subseteq A_2 \subseteq A_3) = \left(\frac{4}{8}\right)^n$$

For $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_m$ each integer k has to obey one of the following:

A_1	A_2	A_3	$\cdots$	A_m
✓	✓	✓	$\cdots$	✓
✗	✓	✓	$\cdots$	✓
✗	✗	✓	$\cdots$	✓
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
✗	✗	✗	$\cdots$	✗

There are $m+1$ rows here, and there are 2^m options for k to be in or out of the m subsets. Hence the probability that k is in an "allowed" state is $\dfrac{m+1}{2^m}$ and:

$$\mathrm{P}(A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots \subseteq A_m) = \left(\frac{m+1}{2^m}\right)^n$$