STEP Support Programme

2021 STEP 3 Worked Paper

General comments

These solutions have a lot more words in them than you would expect to see in an exam script and in places I have tried to explain some of my thought processes as I was attempting the questions. What you will not find in these solutions is my crossed out mistakes and wrong turns, but please be assured that they did happen!

You can find the examiners report and mark schemes for this paper from the Cambridge Assessment Admissions Testing website. These are the general comments for the STEP 2021 exam from the Examiner's report:

"*The total entry was a marginal increase from that of 2019, that of 2020 having been artificially reduced. Comfortably more than 90% attempted one of the questions*[1]*, four others were very popular, and a sixth was attempted by 70%. Every question was attempted by at least 10% of the candidature. 85% of candidates attempted no more than 7 questions, though very nearly all the candidates made genuine attempts on at most six questions (the extra attempts being at times no more than labelling a page or writing only the first line or two). Generally, candidates should be aware that when asked to "Show that" they must provide enough working to fully substantiate their working, and that they should follow the instructions in a question, so if it says "Hence", they should be using the previous work in the question in order to complete the next part. Likewise, candidates should be careful when dividing or multiplying, that things are positive, or at other times non-zero.*"

Please send any corrections, comments or suggestions to step@maths.org.

[1] This, as is usually the case, was Question 1

Question 1

1 **(i)** A curve has parametric equations

$$x = -4\cos^3 t\,, \quad y = 12\sin t - 4\sin^3 t\,.$$

Find the equation of the normal to this curve at the point

$$(-4\cos^3\phi\,,\ 12\sin\phi - 4\sin^3\phi),$$

where $0 < \phi < \frac{1}{2}\pi$.

Verify that this normal is a tangent to the curve

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$$

at the point $(8\cos^3\phi\,,\ 8\sin^3\phi)$.

(ii) A curve has parametric equations

$$x = \cos t + t\sin t\,, \quad y = \sin t - t\cos t.$$

Find the equation of the normal to this curve at the point

$$(\cos\phi + \phi\sin\phi\,,\ \sin\phi - \phi\cos\phi),$$

where $0 < \phi < \frac{1}{2}\pi$.

Determine the perpendicular distance from the origin to this normal, and hence find the equation of a curve, independent of ϕ, to which this normal is a tangent.

Examiner's report

This was the most popular question by a fair margin, being attempted by 93%, and equally was comfortably the most successful with a mean mark of slightly over 15/20. Generally, most found the equation of the normal in part **(i)** correctly, though the more successful candidates simplified their answer sensibly at this point and similarly with other results in the question. A number of candidates forgot the negative sign when obtaining a perpendicular gradient and merely attempted to use the reciprocal. Most used implicit differentiation in order to arrive at an expression for the gradient of the tangent to the second curve in part **(i)**, though parametric differentiation was probably simpler. There was an equal split between those that obtained the equation of the tangent to the second curve and demonstrated that it was the same as that for the normal to the first curve, and those that demonstrated that the point given parametrically was on the normal and that the gradient of the normal and the tangent were the same.

In part **(ii)**, surprisingly, some candidates made errors with the initial differentiation. Those that simplified their equation of the normal profited from the easier working, whichever way they then tried to obtain the perpendicular distance. About three quarters of the candidates found this distance by first finding the intersection of the normal with a perpendicular line through the origin. However, using the formula for the perpendicular distance of a point from a line was simpler. A range of other methods for this distance were seen; briefly, these were **(a)** simple trigonometry having sketched the normal, the axes and line's intercepts, **(b)** expressing the normal equation as the scalar product of vectors, **(c)** minimising by differentiation, or completing the square, of the distance of a general point on the normal from the origin or **(d)** by equating two expressions for the area of the triangle formed by the normal and the two axes. Errors in this part arose from unsimplified working complicating the issue (as already mentioned), overlooking the modulus sign in the distance formula, or calculating the distance from the origin to a point on the curve. The final requirement for the equation of a curve to which the normal found is a tangent was either not spotted by some candidates who had otherwise answered the question perfectly, or the requirement was overlooked.

Solution

(i) Differentiating x and y with respect to t gives:

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}y} &= 12\cos^2 t \sin t \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= 12\cos t - 12\sin^2 t\cos t \\
&= 12\cos t\big(1-\sin^2 t\big) \\
&= 12\cos^3 t \\
\implies \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{12\cos^3 t}{12\cos^2 t\sin t} \ \text{(for } \cos t \neq 0, \sin t \neq 0) \\
&= \cot t
\end{aligned}$$

Since $0 < \phi < \frac{\pi}{2}$ we have $\cos\phi \neq 0$ and $\sin\phi \neq 0$, so the gradient of the curve is well defined here. The gradient of the curve is equal to $\cos\phi$ at this point, so the gradient of the normal is equal to $-\tan\phi$. Therefore the equation of the normal to the curve at the point where $t=\phi$ is:

$$\begin{aligned}
y - \big(12\sin\phi - 4\sin^3\phi\big) &= -\tan\phi\big(x + 4\cos^3\phi\big) \\
y + (\tan\phi)x &= -4\tan\phi\cos^3\phi + 12\sin\phi - 4\sin^3\phi \\
y + (\tan\phi)x &= \sin\phi\big(12 - 4\sin^2\phi - 4\cos^2\phi\big) \\
y + (\tan\phi)x &= 8\sin\phi
\end{aligned}$$

To verify that this normal is a tangent to the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at the given point we can show that the point lies on the normal and that the gradient of the curve at that point is equal to the gradient of the normal.

Substituting $x = 8\cos^3\phi$ and $y = 8\sin^3\phi$ into the left hand side of $y + (\tan\phi)x = 8\sin\phi$ gives:

$$\begin{aligned}&y + (\tan\phi)x\\ =&8\sin^3\phi + 8\tan\phi\cos^3\phi\\ =&8\sin\phi(\sin^2\phi+\cos^2\phi)\\ =&8\sin\phi\end{aligned}$$

Hence the point $\left(8\cos^3\phi,\, 8\sin^3\phi\right)$ lies on the normal $y + (\tan\phi)x = 8\sin\phi$.

Differentiating $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ gives:

$$\begin{aligned}\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{\mathrm{d}y}{\mathrm{d}x} &= 0\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= -\left(\frac{y}{x}\right)^{\frac{1}{3}} \text{ (for } x \neq 0)\end{aligned}$$

Substituting $x = 8\cos^3\phi$ and $y = 8\sin^3\phi$ gives:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\left(\frac{8\sin^3\phi}{8\cos^3\phi}\right)^{\frac{1}{3}} = \frac{\sin\phi}{\cos\phi} = \tan\phi$$

Hence the gradient of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$ at the point $\left(8\cos^3\phi,\, 8\sin^3\phi\right)$ is the same as the gradient of the normal $y + (\tan\phi)x = 8\sin\phi$. Hence this line is the tangent to $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$. Alternatively you could find the equation of the tangent to the curve by differentiation and show that this is the same line as the normal found in the first part of **(i)**.

(ii) Differentiating with respect to t gives:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= -\sin t + \sin t + t\cos t = t\cos t\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= \cos t - \cos t + t\sin t = t\sin t\\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \tan t \text{ (for } t, \cos t \neq 0)\end{aligned}$$

The equation of the normal is:

$$\begin{aligned}y - \left(\sin\phi - \phi\cos\phi\right) &= -\cot\phi\left(x - (\cos\phi + \phi\sin\phi)\right)\\ y - \sin\phi + \cancel{\phi\cos\phi} &= -(\cot\phi)x + \cot\phi\cos\phi + \cancel{\phi\cos\phi}\\ y + (\cot\phi)x &= \sin\phi + \frac{\cos^2\phi}{\sin\phi}\\ y + (\cot\phi)x &= \frac{\sin^2\phi + \cos^2\phi}{\sin\phi}\\ y + (\cot\phi)x &= \operatorname{cosec}\phi\end{aligned}$$

The equation of the perpendicular line which passes through the origin is $y = (\tan\phi)x$.

Solving the equations simultaneously gives:

$$
\begin{aligned}
(\tan\phi)x + (\cot\phi)x &= \operatorname{cosec}\phi \\
\left(\frac{\sin\phi}{\cos\phi} + \frac{\cos\phi}{\sin\phi}\right)x &= \frac{1}{\sin\phi} \\
\frac{1}{\cos\phi\sin\phi}x &= \frac{1}{\sin\phi} \\
x &= \cos\phi
\end{aligned}
$$

This then gives $y = (\tan\phi)x = \sin\phi$.

The distance between the origin and the point $(\cos\phi, \sin\phi)$ is $\sqrt{\cos^2\phi + \sin^2\phi} = 1$. Therefore the equation of a curve to which the normal $y + (\cot\phi)x = \operatorname{cosec}\phi$ is a tangent is $x^2 + y^2 = 1$.

Question 2

2 **(i)** Let

$$x = \frac{a}{b-c}\,, \quad y = \frac{b}{c-a} \text{ and } z = \frac{c}{a-b}\,,$$

where a, b and c are distinct real numbers.

Show that

$$\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and use this result to deduce that $yz + zx + xy = -1$.

Hence show that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} \geqslant 2\,.$$

(ii) Let

$$x = \frac{2a}{b+c}\,, \quad y = \frac{2b}{c+a} \text{ and } z = \frac{2c}{a+b}\,,$$

where a, b and c are positive real numbers.

Using a suitable matrix, show that $xyz + yz + zx + xy = 4$.

Hence show that

$$(2a+b+c)(a+2b+c)(a+b+2c) > 5(b+c)(c+a)(a+b)\,.$$

Show further that

$$(2a+b+c)(a+2b+c)(a+b+2c) > 7(b+c)(c+a)(a+b)\,.$$

Examiner's report

This was the fourth most popular question being attempted by very nearly four fifths of the candidates. It was the third most successful with a mean mark of just over 9/20, though very few achieved full marks. With four "Show that"s, marks were frequently lost for lack of proper justification, and with inequalities to demonstrate involving fractional quantities, positivity was often not considered, let alone proved, or stated as relevant.

Even if candidates stated $\det(\mathbf{M}) = 0$, which they sometimes didn't, only a minority of candidates realised that they had to justify using $\det(\mathbf{M}) = 0$, and of these only some could do so convincingly; there were a number of incorrect arguments used.

Some candidates sacrificed marks by, for example, attempting to show that $\dfrac{a^2}{(b-c)^2} + \dfrac{b^2}{(c-a)^2} + \dfrac{c^2}{(a-b)^2} \geqslant 2$ via a purely algebraic approach rather than using the result just found (i.e. ignoring the "hence").

For the very last part of the question, candidates used a variety of methods in order to explain why $x + y + z > 2$. The most common method was to express the sum in terms of a, b and c and then show that this was greater than 2, but approaches using the AM-GM inequality or by splitting into different cases were sometimes used successfully.

Solution

As mentioned in the examiners report, there are lots of places in the question where you are asked to "Show that", or "Hence show that" or "deduce". If asked to show a given result you must show sufficient detail to justify your derivation of the given result. A "Hence" or "Deduce" means that you must use the previous result or results to do the next part of the question.

(i) We have:

$$\begin{aligned}
\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} &= \begin{pmatrix} a - bx + cx \\ ay + b - cy \\ -az + bz + c \end{pmatrix} \\
&= \begin{pmatrix} a + (c-b)x \\ b + (a-c)y \\ c + (b-a)z \end{pmatrix} \\
&= \begin{pmatrix} a + (c-b) \times \frac{a}{b-c} \\ b + (a-c) \times \frac{b}{c-a} \\ c + (b-a) \times \frac{c}{a-b} \end{pmatrix} \\
&= \begin{pmatrix} a - a \\ b - b \\ c - c \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

There are probably a few more lines of working here than you need, but this first part of the question is a "show that" so you do need to show some working to get the marks! An answer of the form

$$\begin{pmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a - bx + cx \\ ay + b - cy \\ -az + bz + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

would not gain all of the marks as there is no evidence of the given expressions for x, y, z being used, and it has not been convincingly shown that the result is the zero vector.

We have an equation of the form $\mathbf{Mx} = \mathbf{0}$. If $\mathbf{M}$ is invertible (i.e. $\mathbf{M}$ has an inverse) then we can write this as $\mathbf{x} = \mathbf{M}^{-1}\mathbf{0} = \mathbf{0}$. This implies that if the inverse of $\mathbf{M}$ exists then we have $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, however we are told at the beginning of the question that a, b and c are *distinct*, so they cannot all be equal to 0. Hence the inverse of $\mathbf{M}$ does not exist and so we have $\det \mathbf{M} = 0$.[2]

Using $\det \mathbf{M} = 0$ gives us:

$$\begin{aligned} \begin{vmatrix} 1 & -x & x \\ y & 1 & -y \\ -z & z & 1 \end{vmatrix} &= 0 \\ 1\begin{vmatrix} 1 & -y \\ z & 1 \end{vmatrix} - (-x)\begin{vmatrix} y & -y \\ -z & 1 \end{vmatrix} + x\begin{vmatrix} y & 1 \\ -z & z \end{vmatrix} &= 0 \\ 1 + yz + xy - \cancel{xyz} + \cancel{xyz} + xz &= 0 \\ \implies yz + zx + xy &= -1 \end{aligned}$$

The last request made in this part is equivalent to showing that $x^2 + y^2 + z^2 \geqslant 2$. We know that $(x + y + z)^2 \geqslant 0$ (as x, y, z are real - but you did not need to state this!). This gives:

$$\begin{aligned} (x+y+z)^2 &\geqslant 0 \\ x^2 + y^2 + z^2 + 2xy + 2yz + 2zx &\geqslant 0 \\ x^2 + y^2 + z^2 &\geqslant -2(xy + yz + zx) \\ \therefore \quad x^2 + y^2 + z^2 &\geqslant 2 \quad \text{using } yz + zx + xy = -1 \end{aligned}$$

(ii) Rearranging the given relationships for x, y, z gives:

$$\begin{aligned} x(b+c) = 2a &\implies \quad 2a - xb - xc = 0 \\ y(c+a) = 2b &\implies -ya + 2b - yc = 0 \\ z(a+b) = 2c &\implies -za - zb + 2c = 0 \end{aligned}$$

The three linear equations here can be written in matrix form as:

$$\begin{pmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

This time we are told that a, b and c are positive, which means that they cannot be 0. Therefore we have $\det \mathbf{M} = 0$ as before.

[2] Alternatively you can argue that the volume scale factor of the transformation represented by $\mathbf{M}$ must be 0, as a, b, c cannot all be 0, and so we must have $\det \mathbf{M} = 0$. This was sometimes argued as "transformation reduces space to a point" which is equivalent to saying the volume scale factor is 0 (but still needs justification that a, b, c are not all 0).

This gives:

$$\begin{vmatrix} 2 & -x & -x \\ -y & 2 & -y \\ -z & -z & 2 \end{vmatrix} = 0$$

$$2\begin{vmatrix} 2 & -y \\ -z & 2 \end{vmatrix} - (-x)\begin{vmatrix} -y & -y \\ -z & 2 \end{vmatrix} - x\begin{vmatrix} -y & 2 \\ -z & -z \end{vmatrix} = 0$$

$$8 - 2yz - 2xy - xyz - xyz - 2xz = 0$$

$$\implies xyz + xy + yz + zx = 4$$

For the next part, we can divide the LHS throughout by $(a+b)(b+c)(c+a)$ to get:

$$\begin{aligned}\frac{(2a+b+c)(a+2b+c)(a+b+2c)}{(a+b)(b+c)(c+a)} &= \frac{(2a+b+c)}{(b+c)} \times \frac{(a+2b+c)}{(c+a)} \times \frac{(a+b+2c)}{(a+b)} \\ &= \left(\frac{2a}{b+c}+1\right)\left(\frac{2b}{c+a}+1\right)\left(\frac{2c}{a+b}+1\right) \\ &= (x+1)(y+1)(z+1)\end{aligned}$$

Expanding $(x+1)(y+1)(z+1)$ gives:

$$\begin{aligned}(x+1)(y+1)(z+1) &= (xy+x+y+1)(z+1) \\ &= xyz + xz + yz + z + xy + x + y + 1 \\ &= (xyz + xy + yz + zx) + (x+y+z) + 1 \\ &= 4 + 1 + (x+y+z) \\ &> 5 \quad \text{as } x, y, z \text{ are all positive because } a, b, c \text{ are positive}\end{aligned}$$

So we now have:

$$\begin{aligned}&(x+1)(y+1)(z+1) > 5 \\ \implies &(2a+b+c)(a+2b+c)(a+b+2c) > 5(b+c)(c+a)(a+b) \\ &\qquad \text{as } (b+c), (c+a), (a+b) \text{ are all positive}\end{aligned}$$

To show the last part it is sufficient to show that $x+y+z > 2$ (as then we will have shown that $(x+1)(y+1)(z+1) > 7$).

Substituting the given expressions for x, y and z gives us:

$$\begin{aligned}x+y+z &= \frac{2a}{b+c} + \frac{2b}{c+a} + \frac{2c}{a+b} \\ &= \frac{2a(c+a)(a+b) + 2b(b+c)(a+b) + 2c(b+c)(c+a)}{(b+c)(c+a)(a+b)} \\ &= \frac{(2a^3+2a^2b+2a^2c+2abc) + (2b^3+2b^2a+2b^2c+2abc) + (2c^3+2c^2a+2c^2b+2abc)}{a^2b+a^2c+b^2a+b^2c+c^2b+c^2a+2abc} \\ &= \frac{2(a^2b+a^2c+b^2a+b^2c+c^2b+c^2a+2abc)}{a^2b+a^2c+b^2a+b^2c+c^2b+c^2a+2abc} + \frac{2a^3+2b^3+2c^3+2abc}{a^2b+a^2c+b^2a+b^2c+c^2b+c^2a+2abc} \\ &= 2 + \frac{2a^3+2b^3+2c^3+2abc}{a^2b+a^2c+b^2a+b^2c+c^2b+c^2a+2abc} \\ &> 2 \quad \text{as } a, b, c \text{ are all positive}\end{aligned}$$

Hence we have $(x+1)(y+1)(z+1) > 7$ and multiplying throughout by $(a+b)(b+c)(c+a)$ (which is positive) gives the required result.

Alternatively, we have $x = \dfrac{2a}{b+c} > \dfrac{2a}{a+b+c}$, as $a > 0$. Similarly we have $y > \dfrac{2b}{a+b+c}$ and $z > \dfrac{2c}{a+b+c}$. Hence we have:

$$x + y + z > \frac{2a}{a+b+c} + \frac{2b}{a+b+c} + \frac{2c}{a+b+c} = 2$$

I tend to think of the first way as the "brute force and ignorance way", and the second way as slightly neater, but needs you to be able to spot "a trick" (it's not really a trick as such, but does need a little bit of inspiration). Both ways (and there are other methods as well!) are perfectly acceptable and you don't get any bonus points for "style".

Question 3

3 **(i)** Let $I_n = \displaystyle\int_0^{\beta} (\sec x + \tan x)^n \, dx$, where n is a non-negative integer and $0 < \beta < \frac{\pi}{2}$.

For $n \geqslant 1$, show that

$$\frac{1}{2}(I_{n+1} + I_{n-1}) = \frac{1}{n}\big((\sec\beta + \tan\beta)^n - 1\big).$$

Show also that

$$I_n < \frac{1}{n}\big((\sec\beta + \tan\beta)^n - 1\big).$$

(ii) Let $J_n = \displaystyle\int_0^{\beta} (\sec x \cos\beta + \tan x)^n \, dx$, where n is a non-negative integer and $0 < \beta < \frac{\pi}{2}$.

For $n \geqslant 1$, show that

$$J_n < \frac{1}{n}\big((1 + \tan\beta)^n - \cos^n\beta\big).$$

Examiner's report

Whilst this was the second most popular question, being attempted by 84%, it was the fifth most successful with a mean mark a little below 8/20. Most candidates scored full marks for successfully obtaining the first result of part **(i)**, and many gained nearly full credit for obtaining the second result of that part. As is nearly always true, the rule of thumb that it is usually easier to prove that something is greater than (or less than) zero applied here, and so those that considered $\frac{1}{2}(I_{n+1} + I_{n-1}) - I_n$ (and a similar expression for part **(ii)**) generally fared better. A small, but not insignificant number of candidates solved part **(ii)** by a direct method and were generally successful if they did so. Common errors when considering inequalities were failure to fully justify positivity of integrals in both parts, incorrect flows of logic, obtaining weak rather than strict inequalities, and stating inequalities that were inconsistent with the claimed ranges of validity. Otherwise, use of induction or integration by parts caused difficulties, and a number expected, when replicating the first part of working in **(ii)** from part **(i)**, that there would again be an equation, and overlooked the extra term that arose in **(ii)**. Some did not understand that $\sec x \cos\beta \leqslant 1$ in part **(ii)**.

Solution

(i) We have:

$$\begin{aligned}\frac{1}{2}(I_{n+1}-I_{n-1}) &= \frac{1}{2}\int_0^\beta \left[(\sec x+\tan x)^{n+1}+(\sec x+\tan x)^{n-1}\right]\mathrm{d}x\\ &= \frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left[(\sec x+\tan x)^2+1\right]\mathrm{d}x\\ &= \frac{1}{2}\int_0^\beta (\sec x+\tan x)^{n-1}\left[\sec^2 x+2\sec x\tan x+\tan x^2+1\right]\mathrm{d}x\\ &= \int_0^\beta (\sec x+\tan x)^{n-1}[\sec^2 x+\sec x\tan x]\,\mathrm{d}x\end{aligned}$$

Having a quick look at the solution suggests that considering $(\sec x+\tan x)^n$ might be useful. Differentiating $\sec x+\tan x$ gives $\sec x\tan x+\sec^2 x$, and so we have:

$$\begin{aligned}\frac{1}{2}(I_{n+1}-I_{n-1}) &= \left[\frac{1}{n}(\sec x+\tan x)^n\right]_0^\beta\\ &= \frac{1}{n}(\sec\beta+\tan\beta)^n-\frac{1}{n}(\sec 0+\tan 0)^n\\ &= \frac{1}{n}\big((\sec\beta+\tan\beta)^n-1\big)\end{aligned}$$

The second request in this part asks us to show that $I_n<\frac{1}{2}(I_{n+1}+I_{n-1})$. We could rearrange this to the equivalent $I_{n+1}-2I_n+I_{n-1}>0$ (or $I_{n+1}-I_n>I_n-I_{n-1}$).

$$\begin{aligned}I_{n+1}-2I_n+I_{n-1} &= \int_0^\beta (\sec x+\tan x)^{n-1}\left[(\sec x+\tan x)^2-2(\sec x+\tan x)+1\right]\mathrm{d}x\\ &= \int_0^\beta (\sec x+\tan x)^{n-1}[(\sec x+\tan x)-1]^2\,\mathrm{d}x\end{aligned}$$

We know that $[(\sec x+\tan x)-1]^2\geqslant 0$, and in fact when $0<x<\beta$ we have $[(\sec x+\tan x)-1]^2>0$ (remember that $0<\beta<\frac{\pi}{2}$). We also have $\sec x,\tan x>0$ for $0<x<\beta$, and so $(\sec x+\tan x)^{n-1}>0$.

Hence we are integrating something positive, and so the result is positive. Hence we have $I_{n+1}-2I_n+I_{n-1}>0$ and so $I_n<\frac{1}{2}(I_{n+1}+I_{n-1})$.

When trying to show than an inequality is true it is often a good idea to try to rearrange it so that you are trying to show that something is positive (or negative).

(ii) This looks similar to the previous part, so start by trying the same method as suggested in part **(i)**.

$$\begin{aligned}\frac{1}{2}(J_{n+1}+J_{n-1}) &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[(\sec x\cos\beta+\tan x)^2+1\right]\mathrm{d}x\\ &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta+\tan x)^{n-1}\left[\sec^2 x\cos^2\beta+2\sec x\cos\beta\tan x+\tan^2 x+1\right]\mathrm{d}x\end{aligned}$$

Hopefully we can integrate in a similar way to before, so consider what happens when you differentiate $\sec x \cos\beta + \tan x$:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sec x \cos\beta + \tan x) = \sec x \tan x \cos\beta + \sec^2 x$$

So it would be good if we could simplify the square brackets to get something of the form $\sec x \tan x \cos\beta + \sec^2 x$.

$$\begin{aligned}
\frac{1}{2}(J_{n+1} + J_{n-1}) &= \frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\big[\sec^2 x\cos^2\beta + 2\sec x\cos\beta\tan x + \tan^2 x + 1\big]\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\big[\sec^2 x(1-\sin^2\beta) + 2\sec x\cos\beta\tan x + \sec^2 x\big]\,\mathrm{d}x \\
&= \int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\big[\sec x\tan x\cos\beta + \sec^2 x\big]\,\mathrm{d}x \\
&\qquad\qquad - \frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\sec^2 x\sin^2\beta\,\mathrm{d}x \\
&= \frac{1}{n}\big[(\sec x\cos\beta + \tan x)^n\big]_0^\beta - \frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\sec^2 x\sin^2\beta\,\mathrm{d}x \\
&= \frac{1}{n}\big[(\sec\beta\cos\beta + \tan\beta)^n - \cos^n\beta\big] - \frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\sec^2 x\sin^2\beta\,\mathrm{d}x \\
&= \frac{1}{n}\big[(1 + \tan\beta)^n - \cos^n\beta\big] - \frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\sec^2 x\sin^2\beta\,\mathrm{d}x
\end{aligned}$$

In the range $0 < x < \beta$, we know that $\sec x, \tan x > 0$ and we also know that $\sin\beta, \cos\beta > 0$. Hence $\frac{1}{2}\int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\sec^2 x\sin^2\beta\,\mathrm{d}x > 0$ and so

$$\frac{1}{2}(J_{n+1} + J_{n-1}) < \frac{1}{n}\big[(1+\tan\beta)^n - \cos^n\beta\big]$$

Now consider $J_{n+1} - 2J_n + J_{n-1}$:

$$\begin{aligned}
J_{n+1} - 2J_n + J_{n-1} &= \int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\big[(\sec x\cos\beta + \tan x)^2 - 2(\sec x\cos\beta + \tan x) + 1\big]\,\mathrm{d}x \\
&= \int_0^\beta (\sec x\cos\beta + \tan x)^{n-1}\big[(\sec x\cos\beta + \tan x) - 1\big]^2\,\mathrm{d}x \\
&> 0 \text{ as } \sec x, \tan x, \cos\beta > 0
\end{aligned}$$

Therefore we have $J_{n+1} - 2J_n + J_{n-1} > 0$, and so $J_n < \frac{1}{2}(J_{n+1} + J_{n-1})$. Hence we have $J_n < \frac{1}{n}\big[(1+\tan\beta)^n - \cos^n\beta\big]$ as required.

Question 4

4 Let $\mathbf{n}$ be a vector of unit length and Π be the plane through the origin perpendicular to $\mathbf{n}$. For any vector $\mathbf{x}$, the *projection* of $\mathbf{x}$ onto the plane Π is defined to be the vector $\mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\,\mathbf{n}$.

The vectors $\mathbf{a}$ and $\mathbf{b}$ each have unit length and the angle between them is θ, which satisfies $0 < \theta < \pi$. The vector $\mathbf{m}$ is given by $\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

(i) Show that $\mathbf{m}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

(ii) The vector $\mathbf{c}$ also has unit length. The angle between $\mathbf{a}$ and $\mathbf{c}$ is α, and the angle between $\mathbf{b}$ and $\mathbf{c}$ is β. Both angles are acute and non-zero.

Let $\mathbf{a}_1$ and $\mathbf{b}_1$ be the projections of $\mathbf{a}$ and $\mathbf{b}$, respectively, onto the plane through the origin perpendicular to $\mathbf{c}$. Show that $\mathbf{a}_1 \cdot \mathbf{c} = 0$ and, by considering $|\mathbf{a}_1|^2 = \mathbf{a}_1 \cdot \mathbf{a}_1$, show that $|\mathbf{a}_1| = \sin\alpha$.

Show also that the angle ϕ between $\mathbf{a}_1$ and $\mathbf{b}_1$ satisfies

$$\cos\phi = \frac{\cos\theta - \cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

(iii) Let $\mathbf{m}_1$ be the projection of $\mathbf{m}$ onto the plane through the origin perpendicular to $\mathbf{c}$. Show that $\mathbf{m}_1$ bisects the angle between $\mathbf{a}_1$ and $\mathbf{b}_1$ if and only if

$$\alpha = \beta \quad \text{or} \quad \cos\theta = \cos(\alpha - \beta).$$

Examiner's report

Comfortably the least popular Pure question on the paper, it was attempted by just very slightly more than a third of the candidates, which made it almost exactly the same popularity as the most popular Probability and Statistics question. With a mean score of less than 7/20, it was seventh most successful. Those candidates who engaged with the given definition of projection and followed the structure of the question generally did correct calculations of dot products and recognised the relevance of their calculations. Several candidates assumed properties of a projection, not realising that the purpose of this question was to prove properties of a projection given only a single definition. Many of these implicitly made the assumptions when drawing geometric diagrams and arguing geometrically.

Solution

(i) We are told in the stem (the bit of the question before the question-parts start) that $\mathbf{a}$ and $\mathbf{b}$ both have unit length (i.e. $|\mathbf{a}| = |\mathbf{b}| = 1$). Let α be the angle between $\mathbf{m}$ and $\mathbf{a}$ and let β be the angle between $\mathbf{m}$ and $\mathbf{b}$, so we are being asked to show that $\alpha = \beta$.

We have:

$$\mathbf{m} \cdot \mathbf{a} = |\mathbf{m}||\mathbf{a}| \cos\alpha = |\mathbf{m}| \cos\alpha$$
$$\mathbf{m} \cdot \mathbf{b} = |\mathbf{m}||\mathbf{b}| \cos\beta = |\mathbf{m}| \cos\beta$$

Using the given definition for $\mathbf{m}$ we have:

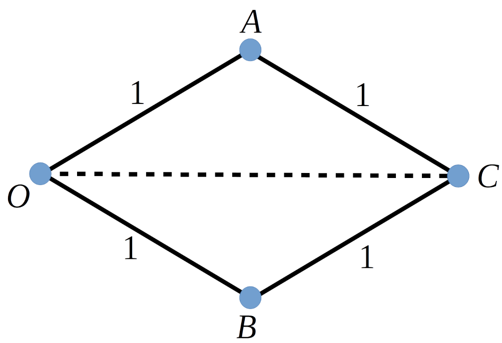
$$\mathbf{m} \cdot \mathbf{a} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}) \cdot \mathbf{a} = \tfrac{1}{2}(\mathbf{a}\cdot\mathbf{a} + \mathbf{b}\cdot\mathbf{a}) = \tfrac{1}{2}(1 + \mathbf{b}\cdot\mathbf{a})$$
$$\mathbf{m} \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{a}+\mathbf{b}) \cdot \mathbf{b} = \tfrac{1}{2}(\mathbf{a}\cdot\mathbf{b} + \mathbf{b}\cdot\mathbf{b}) = \tfrac{1}{2}(\mathbf{a}\cdot\mathbf{b} + 1)$$

Note that $\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = 1$, as we are told that $\mathbf{a}$ has unit length. Similarly $\mathbf{b}\cdot\mathbf{b} = 1$.

Since the dot product is commutative (i.e. $\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$) then these are the same, and so we have $|\mathbf{m}|\cos\alpha = |\mathbf{m}|\cos\beta \implies \cos\alpha = \cos\beta$. Since we know that $0 < \alpha, \beta < \pi$ (as we are given that $0 < \theta < \pi$), then $\cos\alpha = \cos\beta \implies \alpha = \beta$ and so $\mathbf{m}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

There are (as usual!) other ways to complete this question. One possible geometrical argument could be:

Let $OACB$ be a quadrilateral where $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c} = \mathbf{a}+\mathbf{b}$. Therefore we have $\overrightarrow{AC} = \mathbf{b}$ and $\overrightarrow{BC} = \mathbf{a}$. Since $\mathbf{a}$ and $\mathbf{b}$ have unit length then all of the sides of the quadrilateral have length 1, and so $OACB$ is a rhombus.



Considering triangles OAC and OBC, we have $OA = OB = 1$, $CA = BC = 1$ and OC is a common shared side and so by the SSS condition the triangles are congruent. Hence the angles $\angle AOC$ and $\angle BOC$ are the same and the line OC bisects $\angle AOB$. $\overrightarrow{OC} = \mathbf{a}+\mathbf{b}$ is parallel to $\mathbf{m} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$ therefore $\mathbf{m}$ bisects $\mathbf{a}$ and $\mathbf{b}$.

Note that you cannot just state that the diagonal of a rhombus bisects the angle, because essentially this is what you are being asked to prove in the question!

(ii) Note that α and β are now defined. If this happened before part **(i)** then we would have had to use different labels for those angles.

In this part we need to use the definition of projection given in the stem of the question. Since the plane we are projecting onto is perpendicular to $\mathbf{c}$, and $\mathbf{c}$ is a unit vector, we have $\mathbf{a_1} = \mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{c}$.

$$\begin{aligned}\mathbf{a_1} \cdot \mathbf{c} &= \big[\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{c}\big] \cdot \mathbf{c} \\ &= \mathbf{a} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{c})|\mathbf{c}|^2 \\ &= \mathbf{a} \cdot \mathbf{c} - (\mathbf{a} \cdot \mathbf{c}) = 0\end{aligned}$$

Considering $\mathbf{a_1} \cdot \mathbf{a_1}$ we have:

$$\begin{aligned}\mathbf{a_1} \cdot \mathbf{a_1} &= \big[\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{c}\big] \cdot \big[\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{c}\big] \\ &= \mathbf{a} \cdot \mathbf{a} - 2(\mathbf{a} \cdot \mathbf{c})\mathbf{c} \cdot \mathbf{a} + (\mathbf{a} \cdot \mathbf{c})^2 \\ &= \mathbf{a} \cdot \mathbf{a} - (\mathbf{a} \cdot \mathbf{c})^2 \\ &= 1 - (|\mathbf{a}||\mathbf{c}| \cos \alpha)^2 \\ &= 1 - \cos^2 \alpha \\ &= \sin^2 \alpha\end{aligned}$$

Hence we have $|\mathbf{a_1}|^2 = \sin^2 \alpha$, and since α is acute, we know that $\sin \alpha > 0$ therefore $|\mathbf{a_1}| = \sin \alpha$.

Similarly, we have $\mathbf{b_1} \cdot \mathbf{c} = 0$ and $|\mathbf{b_1}| = \sin \beta$ (you don't need to repeat all of the working out again!).

We have:

$$\begin{aligned}\mathbf{a_1} \cdot \mathbf{b_1} &= \big[\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{c}\big] \cdot \big[\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{c}\big] \\ &= \mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{c})\mathbf{c} \cdot \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{c} \cdot \mathbf{a} + (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{c})|\mathbf{c}|^2 \\ &= \mathbf{a} \cdot \mathbf{b} - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{c}) \\ &= \cos \theta - \cos \alpha \cos \beta\end{aligned}$$

Remember that $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$

Using $\mathbf{a_1} \cdot \mathbf{b_1} = |\mathbf{a_1}||\mathbf{b_1}| \cos \phi$ gives $\mathbf{a_1} \cdot \mathbf{b_1} = \sin \alpha \sin \beta \cos \phi$ and so we have:

$$\begin{aligned}\sin \alpha \sin \beta \cos \phi &= \cos \theta - \cos \alpha \cos \beta \\ \implies \cos \phi &= \frac{\cos \theta - \cos \alpha \cos \beta}{\sin \alpha \sin \beta}\end{aligned}$$

as required.

Since α and β are acute and non-zero, i.e. $0 < \alpha, \beta < \frac{\pi}{2}$ we have $\sin \alpha \sin \beta \neq 0$, and so we can divide both sides by this. Always be careful to check that the things you want to divide by are non zero (or if it is an inequality, that they are positive).

(iii) The projection of $\mathbf{m}$ is given by:

$$\begin{aligned}\mathbf{m_1} &= \mathbf{m} - (\mathbf{m}\cdot\mathbf{c})\mathbf{c}\\ &= \tfrac{1}{2}(\mathbf{a}+\mathbf{b}) - \left[\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}\right]\mathbf{c}\\ &= \tfrac{1}{2}\Big[\big(\mathbf{a}-(\mathbf{a}\cdot\mathbf{c})\mathbf{c}\big) + \big(\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{c}\big)\Big]\\ &= \tfrac{1}{2}(\mathbf{a_1}+\mathbf{b_1})\end{aligned}$$

The angle between $\mathbf{m_1}$ and $\mathbf{a_1}$ is given by $\cos\alpha_1 = \dfrac{\mathbf{m_1}\cdot\mathbf{a_1}}{|\mathbf{m_1}||\mathbf{a_1}|}$, and the angle between $\mathbf{m_1}$ and $\mathbf{b_1}$ is given by $\cos\beta_1 = \dfrac{\mathbf{m_1}\cdot\mathbf{a_1}}{|\mathbf{m_1}||\mathbf{a_1}|}$.
Hence $\mathbf{m_1}$ bisects the angle between $\mathbf{a_1}$ and $\mathbf{b_1}$ if, and only if:

$$\begin{aligned}\frac{\mathbf{m_1}\cdot\mathbf{a_1}}{\cancel{|\mathbf{m_1}|}|\mathbf{a_1}|} &= \frac{\mathbf{m_1}\cdot\mathbf{b_1}}{\cancel{|\mathbf{m_1}|}|\mathbf{b_1}|}\\ (\mathbf{m_1}\cdot\mathbf{a_1})|\mathbf{b_1}| &= (\mathbf{m_1}\cdot\mathbf{b_1})|\mathbf{a_1}|\\ \cancel{\tfrac{1}{2}}\big[(\mathbf{a_1}+\mathbf{b_1})\cdot\mathbf{a_1}\big]|\mathbf{b_1}| &= \cancel{\tfrac{1}{2}}\big[(\mathbf{a_1}+\mathbf{b_1})\cdot\mathbf{b_1}\big]|\mathbf{a_1}|\\ \big[|\mathbf{a_1}|^2 + (\mathbf{a_1}\cdot\mathbf{b_1})\big]|\mathbf{b_1}| &= \big[(\mathbf{a_1}\cdot\mathbf{b_1}) + |\mathbf{b_1}|^2\big]|\mathbf{a_1}|\\ \big[\sin^2\alpha + \sin\alpha\sin\beta\cos\phi\big]\sin\beta &= \big[\sin\alpha\sin\beta\cos\phi + \sin^2\beta\big]\sin\alpha\\ \big[\sin\alpha + \sin\beta\cos\phi\big]\sin\alpha\sin\beta &= \big[\sin\alpha\cos\phi + \sin\beta\big]\sin\alpha\sin\beta\\ \big[\sin\alpha + \sin\beta\cos\phi\big] &= \big[\sin\alpha\cos\phi + \sin\beta\big]\end{aligned}$$

Note that we can divide throughout by $\sin\alpha\sin\beta$ as before.

Taking all the terms to one side gives:

$$\begin{aligned}\sin\alpha - \sin\beta + \sin\beta\cos\phi - \sin\alpha\cos\phi &= 0\\ (\sin\alpha - \sin\beta)(1-\cos\phi) &= 0\end{aligned}$$

Therefore $\mathbf{m_1}$ bisects the angle between $\mathbf{a_1}$ and $\mathbf{b_1}$ if, and only if $\sin\alpha = \sin\beta$ or $\cos\phi = 1$.

The condition $\sin\alpha = \sin\beta$ implies that $\alpha = \beta$ (as both angles are acute). For the second condition we have:

$$\begin{aligned}\cos\phi &= 1\\ \frac{\cos\theta - \cos\alpha\cos\beta}{\sin\alpha\sin\beta} &= 1\\ \cos\theta - \cos\alpha\cos\beta &= \sin\alpha\sin\beta\\ \cos\theta &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ \cos\theta &= \cos(\alpha-\beta)\end{aligned}$$

as required.

Question 5

5 Two curves have polar equations $r = a + 2\cos\theta$ and $r = 2 + \cos 2\theta$, where $r \geqslant 0$ and a is a constant.

(i) Show that these curves meet when

$$2\cos^2\theta - 2\cos\theta + 1 - a = 0\,.$$

Hence show that these curves touch if $a = \frac{1}{2}$ and find the other two values of a for which the curves touch.

(ii) Sketch the curves $r = a + 2\cos\theta$ and $r = 2 + \cos 2\theta$ on the same diagram in the case $a = \frac{1}{2}$. Give the values of r and θ at the points at which the curves touch and justify the other features you show on your sketch.

(iii) On two further diagrams, one for each of the other two values of a, sketch both the curves $r = a + 2\cos\theta$ and $r = 2 + \cos 2\theta$. Give the values of r and θ at the points at which the curves touch and justify the other features you show on your sketch.

Examiner's report

A handful of candidates more attempted this question than question **2**, but with marginally less success than question **4**. Nearly every candidate obtained the very first result and many then obtained $a = \frac{1}{2}$ from considering the discriminant. Finding the other values of a (1 and 5) caused many candidates difficulty which could have been overcome had they considered equating expressions for $\dfrac{\mathrm{d}r}{\mathrm{d}\theta}$. In the diagrams, the curve representing the second equation was often drawn as an ellipse, or with cusps rather than smooth indentations. On the other hand, touching points were usually well drawn. It seemed that many appreciated that the curves had symmetry but seldom referred to this in their justification. Similarly, many might have earned credit, but didn't, for indicating values of r for important points such as where the curves met the initial line or the line perpendicular to it. Few candidates found the angles of the cusp in the first two cases (especially with struggling to deal with $\cos^{-1}\left(-\frac{1}{4}\right)$, as opposed to $\cos^{-1}\left(-\frac{1}{2}\right)$).

Solution

(i) Where the curves meet we can equate the expressions for r to get:

$$\begin{aligned} a + 2\cos\theta &= 2 + \cos 2\theta \\ a + 2\cos\theta &= 2 + 2\cos^2\theta - 1 \\ \implies 2\cos^2\theta - 2\cos\theta + 1 - a &= 0 \end{aligned} \qquad (*)$$

If the curves are to touch then they must meet, and the gradients of the two curves must be the same. Equating expressions for the gradients of the curves gives:

$$-2\sin\theta = -2\sin 2\theta = -4\sin\theta\cos\theta$$

This means that the values of θ for which the gradients are the same are when either $\sin\theta = 0$, or when $-2 = -4\cos\theta \implies \cos\theta = \frac{1}{2}$.

When $\sin\theta = 0$ then $\cos\theta = \pm 1$. Substituting $\cos\theta = 1$ into $(*)$ gives $2 - 2 + 1 - 1 = 0 \implies a = 1$, substituting $\cos\theta = -1$ into $(*)$ gives $2 + 2 + 1 - a = 0 \implies a = 5$ and substituting $\cos\theta = \frac{1}{2}$ into $(*)$ gives $\frac{1}{2} - 1 + 1 - a = 0 \implies a = \frac{1}{2}$.

(ii) When $a = \frac{1}{2}$ the points where they touch are where we have $\cos\theta = \frac{1}{2}$, which are at $\theta = \pm\frac{1}{3}\pi$. At these points we have $r = \frac{3}{2}$.

Functions of $\cos\theta$ are symmetrical about the x-axis, which applies to both of the curves. The second curve is a function of $\cos 2\theta = 1 - 2\sin^2\theta$, so this is also symmetrical about the y-axis.

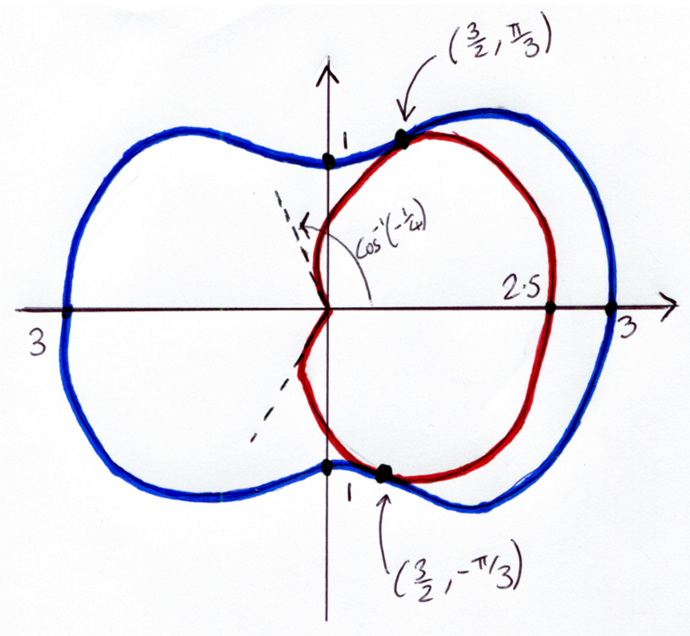
Considering $r = a + 2\cos\theta = \frac{1}{2} + 2\cos\theta$, the fact that we are told that $r \geqslant 0$ means that we must have $\cos\theta \geqslant -\frac{1}{4}$, and so we have $-\cos^{-1}\left(-\frac{1}{4}\right) < \theta < \cos^{-1}\left(-\frac{1}{4}\right)$. For $r = 2 + \cos 2\theta$ there are no restrictions on θ as r is always positive.

For $r = a + 2\cos\theta$ the maximum value of r is $2\frac{1}{2}$, which occurs when $\theta = 0$. The minimum allowed value is 0, which we have discussed above. The values of θ for which $r = 0$ form tangents to the curve at the origin, i.e. at $\theta = \pm\cos^{-1}\left(-\frac{1}{4}\right)$. This forms a "cusp".

For $r = 2 + \cos 2\theta$ the maximum value if 3, which occurs when $\theta = 0, \pi$ and the minimum is 1 which occurs when $\theta = \pm\frac{\pi}{2}$.

The intercepts of $r = 2 + \cos 2\theta$ with the y-axis have already been discussed (they are where the minimum value of r is). For $r = a + 2\cos\theta$, when $\theta = \pm\frac{\pi}{2}$ we have $r = \frac{1}{2}$.

This gives us enough information to draw the graph!



You would not necessarily have to show all the working above to justify the graph, but make sure you indicate the key features (where it crosses the axes, max/min etc.) in some way.

If you find graph sketching a little tricky then putting some notes on the side (such as "this graph is supposed to be symmetrical in the x-axis") will help make it clear to your examiner what your intentions are.

To help make these graphs clear when scanning, I went over the lines using Sharpies (other brands of pen are available). It is usually best to sketch graphs in pencil, but make sure that your pencil is not too fine and faint in case the scripts are going to be scanned (as was the case in 2021).

You can use a graphic calculator, or website such as Desmos to check your graph - but be careful as these often allow negative radii!

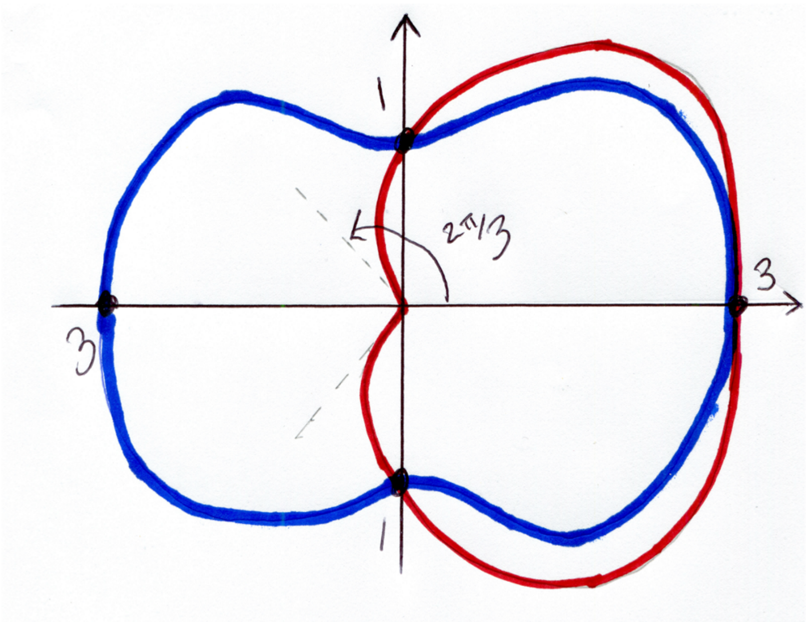
(iii) In this part, all of the previous work done for $r = 2 + \cos 2\theta$ still holds — you don't need to repeat any justification of points in this part as well!

When $a = 1$ we have $\cos\theta = 1$ (from the work done in part **i**) and so $\theta = 0$. Therefore they touch at the point $(3, 0)$.

The curve $r = 1 + 2\cos\theta$ passes through the points $(1, \pm\frac{\pi}{2})$, as does the other curve, so they meet at these points. They cannot meet anywhere other than these three points, and at the points $(1, \pm\frac{\pi}{2})$ the graphs cross (they don't touch!).

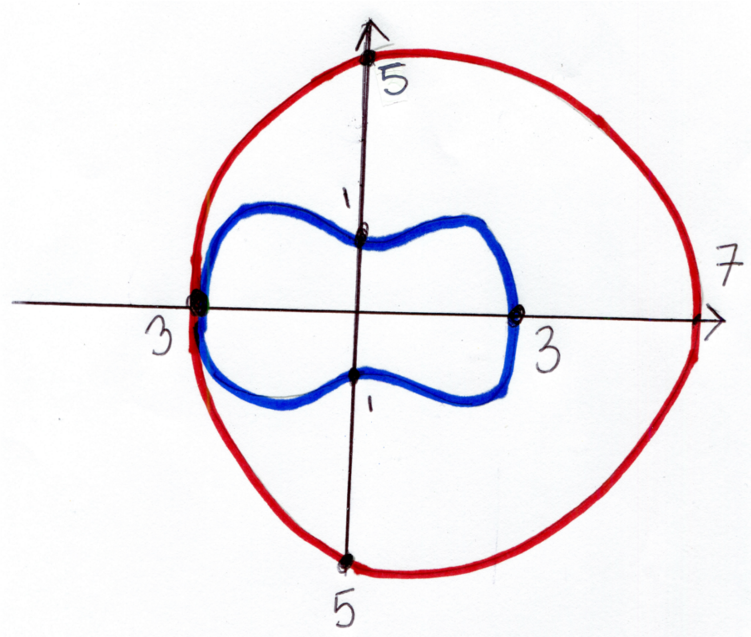
Solving for $r = 0$ gives $1+2\cos\theta = 0$, so there are tangents to the origin at $\theta = \pm\cos^{1}\left(-\frac{1}{2}\right) = \pm\frac{2\pi}{3}$ (note that these are less steep than the tangents for part **(iii)**). The graph only exists for $-\frac{2\pi}{3} \leqslant \theta \leqslant \frac{2\pi}{3}$.

We now have enough info to draw this graph:



When $a = 5$ the curves touch when $\cos\theta = -1$, so when $\theta = \pi$. Substituting $a = 5$ into $(*)$ gives $2\cos^2\theta - 2\cos\theta - 4 = 0$ which factorises to give $2(\cos\theta - 2)(\cos\theta + 1) = 0$. The point where the curves touch is where $\cos\theta = -1$, and $\cos\theta = 2$ has no real solutions, so the two curves do not meet anywhere else.

The curve $r = 5+2\cos\theta$ satisfies $r > 0$ for all values of θ. The maximum value of r is 7 which occurs when $\theta = 0$ and the minimum value is 3 which occurs when $\theta = \pi$. When $\theta = \pm\frac{\pi}{2}$ we have $r = 5$.



You can investigate the graphs using this Desmos page. Note that these graphs do not restrict r to being non-negative! If you change the ϕ slider, a point on the curve will show you when the radius is negative or not. In the STEP question the graphs were restricted to regions where $r \geqslant 0$.

In the marksheme supplied by Cambridge Assessment, the solution to part **(i)** has the statement that the curves touch if $\cos\theta = \pm 1$, which might be a little mystifying on first reading!

The equation $2\cos^2\theta - 2\cos\theta + 1 - a = 0$, for $-\pi < \theta \leqslant \pi$, has up to 4 distinct roots (for example if $\cos\theta = \frac{1}{2}$ then we have 2 solutions for θ, i.e. $\theta = \pm\frac{\pi}{3}$). For most values of $\cos\theta$ there will be 2 possible values of θ, the exceptions being $\cos\theta = 1$ (where $\theta = 0$) and $\cos\theta = -1$ (where $\theta = \pi$). These situations count as "repeated roots", and if the equation formed when curves meet has a repeated root, then the curves touch here. Substituting $\cos\theta = \pm 1$ into the equation gives $a = 1, 5$. To find the last value of a, the equation will also have repeated roots when the discriminant of the quadratic is equal to 0 — this gives the final value of a.

Question 6

6 **(i)** For $x \neq \tan\alpha$, the function f_α is defined by

$$\mathrm{f}_\alpha(x) = \tan^{-1}\left(\frac{x\tan\alpha + 1}{\tan\alpha - x}\right)$$

where $0 < \alpha < \frac{1}{2}\pi$.

Show that $\mathrm{f}_\alpha'(x) = \dfrac{1}{1+x^2}$.

Hence sketch $y = \mathrm{f}_\alpha(x)$.

On a separate diagram, sketch $y = \mathrm{f}_\alpha(x) - \mathrm{f}_\beta(x)$ where $0 < \alpha < \beta < \frac{1}{2}\pi$.

(ii) For $0 \leqslant x \leqslant 2\pi$ and $x \neq \frac{1}{2}\pi, \frac{3}{2}\pi$, the function $\mathrm{g}(x)$ is defined by

$$\mathrm{g}(x) = \tanh^{-1}(\sin x) - \sinh^{-1}(\tan x).$$

For $\frac{1}{2}\pi < x < \frac{3}{2}\pi$, show that $\mathrm{g}'(x) = 2\sec x$.

Use this result to sketch $y = \mathrm{g}(x)$ for $0 \leqslant x \leqslant 2\pi$.

Examiner's report

The seventh most popular question, it was attempted by almost 70% of candidates. However, it was fourth most successful with a mean just short of 8/20. Most candidates successfully differentiated f_α correctly to obtain the required result. Many then sketched a shifted arctan graph but frequently failed to appreciate that there were two branches to the curve with a discontinuity at $x = \tan\alpha$, and also often forgot that the range of the function is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. In addition, few candidates labelled all the requisite values of intercepts, the discontinuity, the asymptote, and the range on the axes. Few consequently sketched $\mathrm{f}_\alpha(x) - \mathrm{f}_\beta(x)$ correctly.

In part **(ii)**, many candidates incorrectly manipulated the negative sign when differentiating g, which then meant that although they sketched the section of the graph for $\frac{\pi}{2} < x < \frac{3\pi}{2}$, they did not wonder why the negative sign arose and hence failed to sketch the two constant segments of the function.

Solution

(i) The result $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan^{-1}x = \dfrac{1}{1+x^2}$ is included in the list of required formulae for STEP (on page 44 of the STEP specifications), so you can just quote it. It can also be derived reasonably quickly:

$$\begin{aligned} y &= \tan^{-1}x \\ \tan y &= x \\ \frac{\mathrm{d}}{\mathrm{d}x}(\tan y) &= 1 \\ \sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ (1+\tan^2 y)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ (1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} &= 1 \\ \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{1}{1+x^2} \end{aligned}$$

Using this, remembering that $\tan\alpha$ is a constant, gives:

$$\begin{aligned} \mathrm{f}'_\alpha(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(\frac{x\tan\alpha+1}{\tan\alpha-x}\right)\right] \\ &= \frac{1}{1+\left(\frac{x\tan\alpha+1}{\tan\alpha-x}\right)^2}\times\frac{\tan\alpha(\tan\alpha-x)+(x\tan\alpha+1)}{(\tan\alpha-x)^2} \\ &= \frac{\tan^2\alpha+1}{(\tan\alpha-x)^2+(x\tan\alpha+1)^2} \\ &= \frac{\tan^2\alpha+1}{\tan^2\alpha-\cancel{2x\tan\alpha}+x^2+x^2\tan^2\alpha+\cancel{2x\tan\alpha}+1} \\ &= \frac{\tan^2\alpha+1}{(\tan^2\alpha+1)+x^2(\tan^2\alpha+1)} \\ &= \frac{1}{1+x^2} \end{aligned}$$

Since we have $\mathrm{f}'_\alpha(x) = \dfrac{1}{1+x^2}$, we can integrate to get $\mathrm{f}_\alpha(x) = \tan^{-1}(x)+c$. When $x=0$, we can substitute this into the given definition of $\mathrm{f}_\alpha(x)$ to get:

$$\begin{aligned} \mathrm{f}_\alpha(0) &= \tan^{-1}\left(\frac{0-1}{\tan\alpha-0}\right) \\ &= \tan^{-1}\left(\frac{1}{\tan\alpha}\right) \\ &= \tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right) \\ &= \tan^{-1}\left(\frac{\sin(\frac{\pi}{2}-\alpha)}{\cos(\frac{\pi}{2}-\alpha)}\right) \\ &= \frac{\pi}{2}-\alpha \end{aligned}$$

This seems to imply that the equation of the graph can be written in the form $f_\alpha(x) = \tan^{-1} x + \frac{\pi}{2} - \alpha$, however this is a problem when $x = \tan\alpha$ because the original $f_\alpha(x)$ is undefined here (the denominator is equal to 0).

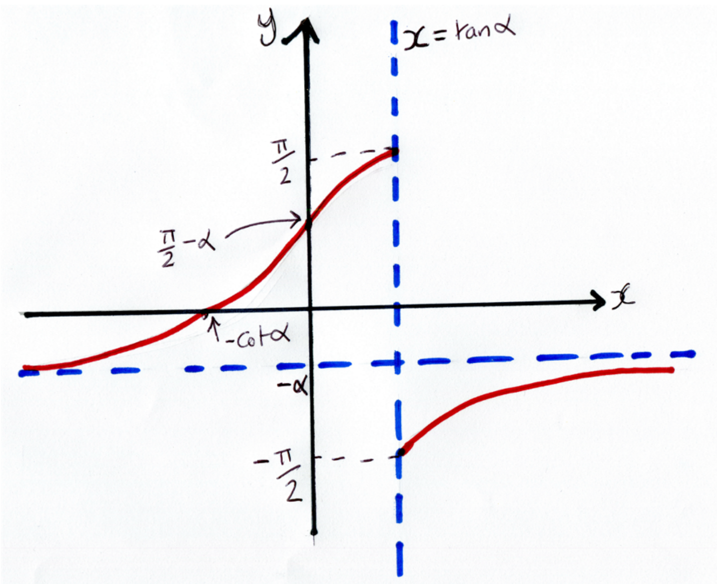
Using the original definition of $f_\alpha(x)$, as $x \to \tan\alpha$ from below we have $\tan\alpha - x \to 0^{+}$[3]. This means that we have $\dfrac{x\tan\alpha + 1}{\tan\alpha - x} \to \infty$ as $x \to \tan\alpha$ from below, and so we have $\tan^{-1}\left(\dfrac{x\tan\alpha+1}{\tan\alpha - x}\right) \to \dfrac{\pi}{2}$.

In a similar way, as $x \to \tan\alpha$ from above we have $\tan\alpha - x \to 0^-$. Therefore as $x \to \tan\alpha$ from above we have $\tan^{-1}\left(\dfrac{x\tan\alpha+1}{\tan\alpha - x}\right) \to -\dfrac{\pi}{2}$.

Note that this is not a vertical asymptote as the gradient of the graph is "well behaved" everywhere, $\dfrac{1}{1+x^2}$ does not tend to infinity. Instead it is a discontinuity of the graph, so it looks as if a graph of $y = \tan^{-1} x$ has been cut and part of it moved downwards. We have $f_\alpha(x) = \tan^{-1} x + \frac{\pi}{2} - \alpha$ for $x < \tan\alpha$ and $f_\alpha(x) = \tan^{-1} x - \frac{\pi}{2} - \alpha$ for $x > \tan\alpha$.

We almost have enough detail to sketch the graph. At the point where the graph crosses the x axis we have $x\tan\alpha + 1 = 0$ (from the first definition of $f_\alpha(x)$; setting the numerator equal to 0), i.e. $x = -\dfrac{1}{\tan\alpha} = -\cos\alpha$.

There is a horizontal asymptote as $x \to \pm\infty$, when we have $f_\alpha(x) \to \tan^{-1}(-\tan\alpha) = -\alpha$. We can now draw the graph:



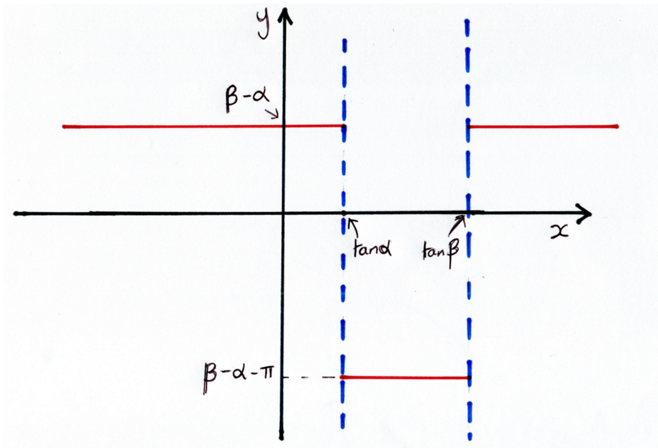
This Desmos page shows an interactive version of the graph.

For the next part, if $x < \tan\alpha$ then neither graph has met the discontinuity yet, so $f_\alpha(x) - f_\beta(x) = \left(\tan^{-1} x + \frac{\pi}{2} - \alpha\right) - \left(\tan^{-1} x + \frac{\pi}{2} - \beta\right) = \beta - \alpha$.

[3] This means that x is tending to 0 through positive values of x (sometimes we say "x tends to 0 from above").

In a similar way, when $x > \tan\beta$, both graphs have passed through the discontinuity and do we have $\mathrm{f}_\alpha(x) - \mathrm{f}_\beta(x) = \left(\tan^{-1}x - \frac{\pi}{2} - \alpha\right) - \left(\tan^{-1}x - \frac{\pi}{2} - \beta\right) = \beta - \alpha$.

For $\tan\alpha < x < \tan\beta$ we have $\mathrm{f}_\alpha(x) - \mathrm{f}_\beta(x) = \left(\tan^{-1}x - \frac{\pi}{2} - \alpha\right) - \left(\tan^{-1}x + \frac{\pi}{2} - \beta\right) = \beta - \alpha - \pi$.



This Desmos page shows an interactive version of the difference between the two curves. When exploring remember that you need $\alpha < \beta$.

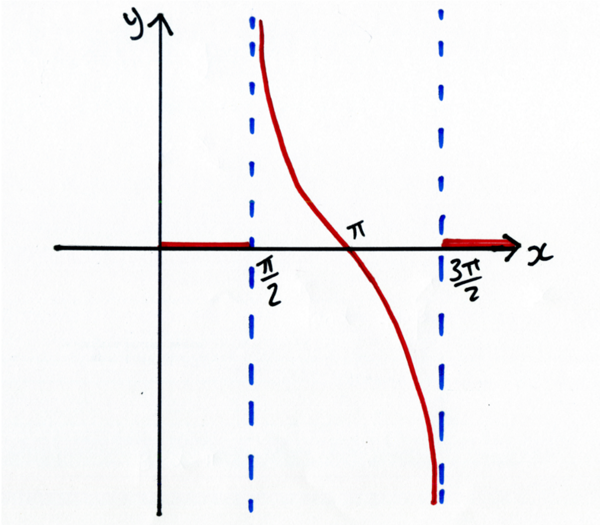
(ii) There are standard differentiation results for $\dfrac{\mathrm{d}}{\mathrm{d}x}\tanh^{-1}(x) = \dfrac{1}{1-x^2}$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\sinh^{-1}(x) = \dfrac{1}{\sqrt{1+x^2}}$. Using these gives:

$$\begin{aligned}
\mathrm{g}(x) &= \tanh^{-1}(\sin x) - \sinh^{-1}(\tan x)\\
\mathrm{g}'(x) &= \frac{1}{1-\sin^2 x} \times \cos x - \frac{1}{\sqrt{1+\tan^2 x}} \times \sec^2 x\\
&= \frac{\cos x}{\cos^2 x} - \frac{\sec^2 x}{|\sec x|}\\
&= \sec x - |\sec x|
\end{aligned}$$

When $\frac{1}{2}\pi < x < \frac{3}{2}x$, $\sec x < 0$ and so $|\sec x| = -\sec x$. This means that in this range we have $\mathrm{g}'(x) = 2\sec x$ (and this is negative!). For $0 \leqslant x < \frac{1}{2}\pi$ and $\frac{3}{2}\pi < x \leqslant 2\pi$ we have $\mathrm{g}'(x) = \sec x - \sec x = 0$.

Using $\mathrm{g}(x) = \tanh^{-1}(\sin x) - \sinh^{-1}(\tan x)$ we have $\mathrm{g}(0) = \mathrm{g}(\pi) = \mathrm{g}(2\pi) = 0$, and so we have $\mathrm{g}(x) = 0$ throughout the ranges $0 \leqslant x < \frac{1}{2}\pi$ and $\frac{3}{2}\pi < x \leqslant 2\pi$.

For $\frac{1}{2}\pi < x < \frac{3}{2}x$ the gradient is negative throughout and as $x \to \frac{1}{2}\pi^+$ and $x \to \frac{3}{2}\pi^-$ we have $\mathrm{g}'(x) \to -\infty$. The graph is least steep when $x = \pi$, when we have $\mathrm{g}'(\pi) = -2$.


y
x
π/2
π
3π/2

Question 7

7 **(i)** Let

$$z = \frac{e^{i\theta} + e^{i\phi}}{e^{i\theta} - e^{i\phi}},$$

where θ and ϕ are real, and $\theta - \phi \neq 2n\pi$ for any integer n. Show that

$$z = i\cot\left(\tfrac{1}{2}(\phi - \theta)\right)$$

and give expressions for the modulus and argument of z.

(ii) The distinct points A and B lie on a circle with radius 1 and centre O. In the complex plane, A and B are represented by the complex numbers a and b, and O is at the origin. The point X is represented by the complex number x, where $x = a + b$ and $a + b \neq 0$. Show that OX is perpendicular to AB.

If the distinct points A, B and C in the complex plane, which are represented by the complex numbers a, b and c, lie on a circle with radius 1 and centre O, and $h = a+b+c$ represents the point H, then H is said to be the *orthocentre* of the triangle ABC.

(iii) The distinct points A, B and C lie on a circle with radius 1 and centre O. In the complex plane, A, B and C are represented by the complex numbers a, b and c, and O is at the origin.

Show that, if the point H, represented by the complex number h, is the orthocentre of the triangle ABC, then either $h = a$ or AH is perpendicular to BC.

(iv) The distinct points A, B, C and D (in that order, anticlockwise) all lie on a circle with radius 1 and centre O. The points P, Q, R and S are the orthocentres of the triangles ABC, BCD, CDA and DAB, respectively. By considering the midpoint of AQ, show that there is a single transformation which maps the quadrilateral $ABCD$ on to the quadrilateral $QRSP$ and describe this transformation fully.

Examiner's report

This was the least successfully attempted Pure question with a mean score under 6/20. It was less than 4% more popular than question **4**. The first part of this question was generally well attempted, with a significant number of candidates being able to correctly verify the algebraic identity utilising a number of different approaches. There were some very neat solutions, but candidates who multiplied throughout by the complex conjugate and managed to keep track of the ensuing algebra were also often successful. Candidates must make sure that when they are trying to show a given result that they fully justify their solution – in this case some candidates missed out several steps of working and so did not gain full credit. Many candidates recognised that the form of z meant that the number was purely imaginary, but only a few candidates gained full credit for this part of the question with many omitting the modulus signs on the cot term for the modulus, or omitting the second possible angle. Some candidates were confused by the angles present in the given form of z and gave the argument as $\frac{1}{2}(\phi - \theta)$.

In part **(ii)**, the approach using the result from part **(i)** often did not score full marks due to the fact candidates would divide by quantities without explaining why they were non-zero. Some attempted this question with vector methods without clearly setting up that they were treating a, b as vectors rather than complex numbers. They were often unclear as to whether they were actually considering vectors, or considering complex numbers, which was particularly apparent in attempts to take the dot product of vectors without including the "dot" symbol. A number of candidates attempted to work out the gradients of the two line segments and show they multiplied to give -1: unfortunately, none recognised that a number of special cases were not taken care of with this method (cases where the lines were horizontal and vertical) and so did not score highly. Some candidates took a geometrical approach which needed to be fully explained to be convincing. For part **(iii)**, many were more successful than for **(ii)**: they recognised that part **(ii)** could be applied to give the result, and those who did generally gained full, or nearly full, credit. Vector approaches and considering the gradients of the line segments were used again in this part, with some candidates repeating the work they had done in the previous part, with the same pitfalls. Many omitted the case "if $b + c = 0$ then $h = a$". Part **(iv)** was not attempted by a significant proportion. Of those who did attempt it, a significant number gained full credit. The most common mistake for this part of the question was candidates giving the transformation as "reflection through a point", which did not gain them credit as this is not considered to be a "Single transformation" as requested (each point is reflected through a different line). Another common mistake was the miscalculation of the midpoint of AQ as $\frac{1}{2}(b + c + d - a)$ or as $\frac{1}{4}(a + b + c + d)$.

Solution

(i) There are lots of ways in which you can approach this first part! One possible way is to start by "realising" the denominator:

$$\begin{aligned} z &= \frac{e^{i\theta} + e^{i\phi}}{e^{i\theta} - e^{i\phi}} \\ &= \frac{(e^{i\theta} + e^{i\phi})(e^{-i\theta} - e^{-i\phi})}{(e^{i\theta} - e^{i\phi})(e^{-i\theta} - e^{-i\phi})} \\ &= \frac{\cancel{1} + e^{i(\phi-\theta)} - e^{i(\theta-\phi)} - \cancel{1}}{1 - e^{i(\phi-\theta)} - e^{i(\theta-\phi)} + 1} \\ &= \frac{\left[\cos(\phi-\theta) + i\sin(\phi-\theta)\right] - \left[\cos(\theta-\phi) + i\sin(\theta-\phi)\right]}{2 - \left[\cos(\phi-\theta) + i\sin(\phi-\theta)\right] - \left[\cos(\theta-\phi) + i\sin(\theta-\phi)\right]} \\ &= \frac{2i\sin(\phi-\theta)}{2 - 2\cos(\phi-\theta)} \\ &= i \times \frac{\sin(\phi-\theta)}{1 - \cos(\phi-\theta)} \\ &= i \times \frac{2\sin\frac{1}{2}(\phi-\theta)\cos\frac{1}{2}(\phi-\theta)}{2\sin^2\frac{1}{2}(\phi-\theta)} \\ &= i \times \frac{\cos\frac{1}{2}(\phi-\theta)}{\sin\frac{1}{2}(\phi-\theta)} \\ &= i\cot\tfrac{1}{2}(\phi-\theta) \end{aligned}$$

When "realising" the denominator you need to be careful - a common error here would be to multiply throughout by $(e^{i\theta} + e^{i\phi})$.

Another approach is:

$$\begin{aligned} z &= \frac{e^{i\theta} + e^{i\phi}}{e^{i\theta} - e^{i\phi}} \\ &= \frac{e^{i\theta} + e^{i\phi}}{e^{i\theta} - e^{i\phi}} \times \frac{e^{-\frac{i}{2}(\theta+\phi)}}{e^{-\frac{i}{2}(\theta+\phi)}} \\ &= \frac{e^{i\left(\frac{\theta-\phi}{2}\right)} + e^{-i\left(\frac{\theta-\phi}{2}\right)}}{e^{i\left(\frac{\theta-\phi}{2}\right)} - e^{-i\left(\frac{\theta-\phi}{2}\right)}} \\ &= \frac{2\cos\left(\frac{\theta-\phi}{2}\right)}{2i\sin\left(\frac{\theta-\phi}{2}\right)} \\ &= -i\cot\left(\frac{\theta-\phi}{2}\right) \\ &= i\cot\left(\frac{\phi-\theta}{2}\right) \end{aligned}$$

The question states that $\theta - \phi \neq 2n\pi$ - this is to ensure that the denominator is non-zero!

The very last part of the question asks you to find expressions for the modulus and the argument of z. The first thing to note is that $\cot\left(\frac{\phi-\theta}{2}\right)$ is a real number, and so z has the form $\mathrm{i}b$, i.e. it is purely imaginary.

This means that the argument is either $\frac{\pi}{2}$ or $-\frac{\pi}{2}$ (or $\frac{3\pi}{2}$ if you prefer), and the modulus is $\left|\cot\left(\frac{\phi-\theta}{2}\right)\right|$ (Note that the modulus must be positive).

Do make sure that you have checked that you have actually answered all of the requests in each part of the question - it is very easy to miss out something!

(ii) The parts of a STEP question are often linked - so it is always a good idea to check if you can use anything done in a previous part to help you do the next part. Here we are told that A and B are points on the unit circle, which means that A could be represented by the complex number $a = \mathrm{e}^{\mathrm{i}\alpha}$ and B by $b = \mathrm{e}^{\mathrm{i}\beta}$. We then have OX represented by $x = a + b = \mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{\mathrm{i}\beta}$ and AB represented by $b - a = \mathrm{e}^{\mathrm{i}\beta} - \mathrm{e}^{\mathrm{i}\alpha}$.

We then have:

$$\begin{aligned}\arg x - \arg AB &= \arg\left(\frac{x}{AB}\right)\\ &= \arg\left(\frac{b+a}{b-a}\right)\\ &= \arg\left(\frac{\mathrm{e}^{\mathrm{i}\beta}+\mathrm{e}^{\mathrm{i}\alpha}}{\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}}\right)\end{aligned}$$

The expression inside the argument on the last line has the same form as z in the previous part. We are told that A and B are distinct, so we know that $\mathrm{e}^{\mathrm{i}\alpha} \neq \mathrm{e}^{\mathrm{i}\beta}$. We are also told that $a + b \neq 0$, and so $\mathrm{e}^{\mathrm{i}\alpha} + \mathrm{e}^{\mathrm{i}\beta} \neq 0$. This means that our "$z$" is not equal to zero, and since the denominator is not zero we can use the result from the first part to say that:

$$\arg x - \arg AB = \arg\left(\frac{\mathrm{e}^{\mathrm{i}\beta}+\mathrm{e}^{\mathrm{i}\alpha}}{\mathrm{e}^{\mathrm{i}\beta}-\mathrm{e}^{\mathrm{i}\alpha}}\right) = \pm\frac{\pi}{2}$$

and hence the two lines are perpendicular.

There are lots of different ways of approaching this part, but in all methods you need to be careful of "special cases". For example, you might like to use vectors and show that $\mathbf{r}\cdot\mathbf{x} = 0$, but you must rule out the cases $\mathbf{r} = \mathbf{0}$ and $\mathbf{x} = \mathbf{0}$ before you can conclude that $\mathbf{r}$ and $\mathbf{x}$ are perpendicular.

Between parts **(ii)** and **(iii)** there is a "substem". Anything written here will hold in the later parts of the question (in this case, parts **(iii)** and **(iv)**).

(iii) Now we are given three (distinct!) points on the unit circle, and we also have $h = a + b + c$.

We have AH represented by $(a + b + c) - a = b + c$ and BC represented by $b - c$.

If $b + c = 0$ then we have $h = a + b + c = a$.

If $b + c \neq 0$, and since we know that $b \neq c$, then we have:

$$\frac{AH}{BC} \to \frac{b+c}{c-b}$$

This has the same form as the expression in part **(ii)** and so if $b + c \neq 0$ then BC is perpendicular to AH.

(iv) We have:

$$p = a + b + c$$
$$q = b + c + d$$
$$r = c + d + a$$
$$s = d + a + b$$

The midpoint of AQ is located at $\dfrac{a + (b + c + d)}{2}$ and similarly this is also the midpoint of BR, CS and DP. This means that the lines segments AQ, BR, CS and DP all intersect at the same place and that point is in the centre of each line segment.

This means that the transformation is an enlargement, scale factor -1, with centre of enlargement at the midpoint of AQ. Alternately we could describe this as a rotation of angle π about the midpoint of AQ. The quadrilaterals are modelled on this Geogebra page.

"Reflection in a point" was not allowed as a description of the transformation.

Question 8

8 A sequence $x_1,\ x_2, \ldots$ of real numbers is defined by $x_{n+1} = x_n^2 - 2$ for $n \geqslant 1$ and $x_1 = a$.

(i) Show that if $a > 2$ then $x_n \geqslant 2 + 4^{n-1}(a-2)$.

(ii) Show also that $x_n \to \infty$ as $n \to \infty$ if and only if $|a| > 2$.

(iii) When $a > 2$, a second sequence $y_1,\ y_2, \ldots$ is defined by

$$y_n = \frac{Ax_1x_2\cdots x_n}{x_{n+1}},$$

where A is a positive constant and $n \geqslant 1$.

Prove that, for a certain value of a, with $a > 2$, which you should find in terms of A,

$$y_n = \frac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$$

for all $n \geqslant 1$.

Determine whether, for this value of a, the second sequence converges.

Examiner's report

Fifth most popular (77%), this was fourth least successful with a mean mark of six and a half. There were very few perfect attempts and a sizeable number of attempts failed to get any marks. Induction in both parts **(i)** and **(iii)** was generally executed very well, however marks were frequently lost for logical imprecision. A very common cause of lost marks was a lack of care with inequalities involving potentially negative numbers. In part **(i)**, almost no candidates noticed that squaring the inequality required noting the non-negativity of the lower bound. Many candidates also had trouble with the base case, some because they were mistakenly thinking $4^0 = 0$. In part **(ii)**, many candidates lost marks when attempting to show that the sequence $|x_n|$ remains bounded in the case $|a| < 2$, by not excluding the possibility that x_2 goes below -2 and hence diverges to positive infinity. Another common error in part **(ii)** was failing to make the link to the inequality in part **(i)**. Many candidates tried to show divergence to infinity by showing that the sequence was increasing. In part **(iii)** most candidates worked back from the required result to find a suitable value for a. The inductive calculation was generally performed well, however plenty of candidates failed to show that their value of a worked and was greater than 2. When solving equations, it should either be checked that all the steps are reversible (in this case they were not because of a possible division by zero) or that the claimed solution does in fact work. Most attempts at the final section on convergence were informal but successful.

Solution

(i) Assume the result is true when $n = k$, so we have:

$$x_k \geqslant 2 + 4^{k-1}(a-2) \qquad (*)$$

Since we are told that $a > 2$ we have $2 + 4^{k-1}(a-2) > 0$, and so both sides of $(*)$ are positive. Hence we can legitimately square both sides of $(*)$ to get:

$$x_k^2 \geqslant \left[2 + 4^{k-1}(a-2)\right]^2$$

Using $x_{n+1} = x_n^2 - 2$ we have:

$$\begin{aligned} x_{k+1} &= x_k^2 - 2 \\ &\geqslant \left[2 + 4^{k-1}(a-2)\right]^2 - 2 \\ &= 4 + 2 \times 2 \times 4^{k-1}(a-2) + \left[4^{k-1}(a-2)\right]^2 - 2 \\ &= 2 + 4^k(a-2) + 4^{2(k-1)}(a-2)^2 \\ &> 2 + 4^k(a-2) \end{aligned}$$

and so we have $x_{k+1} \geqslant 2 + 4^{(k+1)-1}(a-2)$.

When $n = 1$ we have $x_1 = a$. Considering $2 + 4^{n-1}(a-2)$ when $n = 1$ gives $2 + (a-2) = a$, hence when $n = 1$ we also have $x_n \geqslant 2 + 4^{n-1}(a-2)$. Therefore since the result is true when $n = 1$, and if the result is true when $n = k$ the result is also true for $n = k+1$, we can conclude that the result is true for all integers $n \geqslant 1$.

(ii) If $a > 2$, then we know that $x_n \geqslant 2 + 4^{n-1}(a-2)$ which tends to infinity as n tends to infinity, i.e. if $a > 2$ then $x_n \to \infty$ as $n \to \infty$.

If $a < -2$ then all the terms after x_1 will be the same as in the sequence with $x_1 = |a|$, as $x_2 = x_1^2 - 2 = |a|^2 - 2$. If $a < -2$, then $|a| > 2$ therefore as $n \to \infty$, $x_n \to \infty$.

We have now shown that **if** $|a| > 2$ then $x_n \to \infty$ as $n \to \infty$. However this was an "if and only if", so we need to show that if $|a| \leqslant 2$ then the sequence does not tend to infinity.

If $|a| \leqslant 2$ then $|x_1| \leqslant 2$. If $|x_k| \leqslant 2$ then $x_{k+1} = x_k^2 - 2$ lies between -2 and 2 (as we have $0 \leqslant x_k^2 \leqslant 4$), so we also have $|x_{k+1}| \leqslant 2$. Hence by induction we know that if $|a| \leqslant 2$ then $-2 \leqslant x_n \leqslant 2$ for all $n \geqslant 1$, and so x_n is bounded and cannot tend to infinity.

(iii) Assume that $y_k = \dfrac{\sqrt{x_{k+1}^2 - 4}}{x_{k+1}}$, and let's try to show that y_{k+1} has a similar form. We have:

$$\begin{aligned} y_{k+1} &= \frac{Ax_1x_2\cdots x_kx_{k+1}}{x_{k+2}} \\ &= y_k \times \frac{x_{k+1}^2}{x_{k+2}} \\ &= \frac{\sqrt{x_{k+1}^2 - 4}}{x_{k+1}} \times \frac{x_{k+1}^2}{x_{k+2}} \\ &= \frac{x_{k+1}}{x_{k+2}} \times \sqrt{x_{k+1}^2 - 4} \end{aligned}$$

From the definition of sequence x_n given in the stem we have $x_{k+2} = x_{k+1}^2 - 2$. We can use this to write the x_{k+1} terms as functions of x_{k+2}, i.e. we have $x_{k+1}^2 - 4 = x_{k+2} - 2$ and $x_{k+1} = \sqrt{x_{k+2} + 2}$ (since $a > 2$ we know that all of the sequence is positive, so take the positive root!).

This means we can write:

$$\begin{aligned} y_{k+1} &= \frac{\sqrt{x_{k+2} + 2}}{x_{k+2}} \times \sqrt{x_{k+2} - 2} \\ &= \frac{\sqrt{{x_{k+2}}^2 - 4}}{x_{k+2}} \end{aligned}$$

Which has the same form as the expression for y_k.

If y_1 is going to have the same form then we need:

$$\begin{aligned} \frac{Ax_1}{x_2} &= \frac{\sqrt{x_2^2 - 4}}{x_2} \\ \implies A^2x_1^2 &= x_2^2 - 4 \\ A^2a^2 &= (a^2 - 2)^2 - 4 \\ A^2a^2 &= a^4 - 4a^2 + 4 - 4 \\ \implies A^2 &= a^2 - 4 \\ a &= \pm\sqrt{A^2 + 4} \end{aligned}$$

Since we want a positive, let $a = \sqrt{A^2 + 4}$.

Using $x_1 = a$ and $x_2 = a_1^2 - 2 = a^2 - 2$. Since $a = \sqrt{A^2 + 4}$, and $A > 0$ (the important bit is that $A \neq 0$) then we have $a > 2$.

Therefore with this value of a we have $y_n = \dfrac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$ for all $n \geqslant 1$.

Since $a > 2$, then as $n \to \infty$, $x_n \to \infty$ (from part **(ii)**). We can write $y_n = \sqrt{1 - \frac{4}{x_{k+1}^2}}$ and as as $n \to \infty$, $y_n \to 1$ and so the second sequence converges.

An alternative approach to this part is shown below. It's important to note that the work in part **(i)** implies that $x_n > 0$ (in fact we have $x_n > 2$) for all $n \geqslant 1$). This is used in various places!

If we can write $y_n = \dfrac{\sqrt{x_{n+1}^2 - 4}}{x_{n+1}}$, then equating the two expressions for y_n gives:

$$\begin{aligned} \frac{Ax_1x_2\cdots x_n}{x_{n+1}} &= \frac{\sqrt{x_{n+1}^2-4}}{x_{n+1}} \\ x_{n+1} \neq 0 \implies Ax_1x_2\cdots x_n &= \sqrt{x_{n+1}^2 - 4} \\ &= \sqrt{(x_n^2-2)^2 - 4} \\ &= \sqrt{x_n^4 - 4x_n^2} \\ Ax_1x_2\cdots x_n &= x_n\sqrt{x_n^2-4} \qquad \big[\text{since } x_n > 0\big] \\ x_n \neq 0 \implies Ax_1x_2\cdots x_{n-1} &= \sqrt{x_n^2-4} \end{aligned}$$

Repeating this process gives:

$$\begin{aligned} Ax_1 &= \sqrt{x_2^2-4} \\ Ax_1 &= x_1\sqrt{x_1^2-4} \\ x_1 \neq 0 \implies A &= \sqrt{x_1^2-4} \\ x_1 = a \implies A &= \sqrt{a^2-4} \end{aligned}$$

Rearranging, and using the fact that $a > 2$, gives $a = \sqrt{A^2+4}$.

This means that we have shown that **if** $y_n = \dfrac{\sqrt{x_{n+1}^2-4}}{x_{n+1}}$ **then** we must have $a = \sqrt{A^2+4}$ (or in other words it is necessary to have $a = \sqrt{A^2+4}$). However this is not the statement we were asked to prove — we actually need to show that **if** $a = \sqrt{A^2+4}$ **then** $y_n = \dfrac{\sqrt{x_{n+1}^2-4}}{x_{n+1}}$.

You could argue this by carefully explaining why each of the steps is reversible, or instead substitute $a = \sqrt{A^2+4} \implies A = \sqrt{a^2-4}$ into $y_n = \dfrac{Ax_1x_2\cdots x_n}{x_{n+1}}$ and show that this value of a **does** give the required form of y_n (or in other words it is *necessary* and *sufficient* to have $a = \sqrt{A^2+4}$).

Question 9

9 An equilateral triangle ABC has sides of length a. The points P, Q and R lie on the sides BC, CA and AB, respectively, such that the length BP is x and QR is parallel to CB. Show that

$$(\sqrt{3}\cot\phi + 1)(\sqrt{3}\cot\theta + 1)x = 4(a - x)\,,$$

where $\theta = \angle CPQ$ and $\phi = \angle BRP$.

A horizontal triangular frame with sides of length a and vertices A, B and C is fixed on a smooth horizontal table. A small ball is placed at a point P inside the frame, in contact with side BC at a distance x from B. It is struck so that it moves round the triangle PQR described above, bouncing off the frame at Q and then R before returning to point P. The frame is smooth and the coefficient of restitution between the ball and the frame is e.

Show that

$$x = \frac{ae}{1+e}\,.$$

Show further that if the ball continues to move round PQR after returning to P, then $e = 1$.

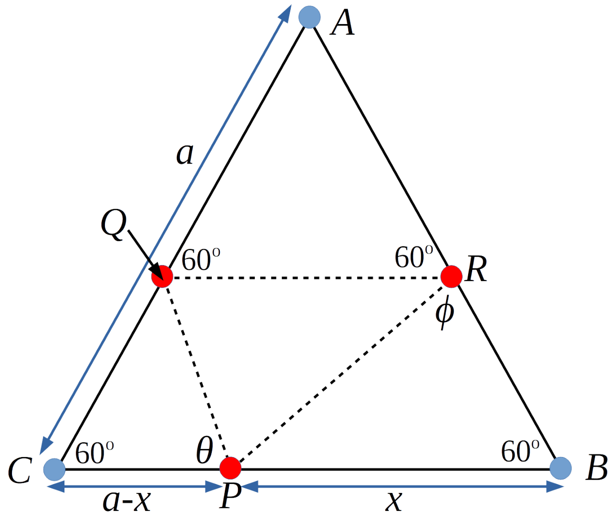
Examiner's report

Just over a fifth attempted this but it had the dubious distinction of being the least successful question with a mean score a little over 4/20. There were a number of alternative methods used for the first result, and those that were successful usually applied the sine rule or dropped perpendiculars. However, some candidates drew a triangle with angles found and wrote down sine or cosine rules with no indication of how they were to be combined thus earning very little credit. Candidates who understood the concept of restitution were usually able to complete the second part of the question without any problems. Many candidates failed on the last part of the question by trying to give verbose intuition-based arguments instead of finding a third restitution equation

Solution

When I first drew a diagram, I had the line AC horizontal, but when I went to place the points P, Q and R I realised that I really wanted the line CB to be horizontal, so I redrew it! You may find that you can cope with the triangle in a different orientation, but I found it much easier to see what was happening with CB at "the bottom" of the triangle.

First thing to draw is a diagram. If drawing by hand I would probably use $\frac{\pi}{3}$ rather than 60°, but my drawing package does not display fractions very nicely.



Looking at the required result, there is an "x" and an "$a - x$" present, so we probably want to consider the triangles PQC and PRB. Using the sine rule in these two triangles we have:

$$\frac{PQ}{\sin 60^\circ} = \frac{a-x}{\sin(120^\circ - \theta)}$$

$$\text{and} \quad \frac{PR}{\sin 60^\circ} = \frac{x}{\sin\phi}$$

It would be nice not to have PQ and PR present. Using the sine rule in triangle RPQ gives:

$$\frac{PR}{\sin\theta} = \frac{PQ}{\sin(120^\circ - \phi)}$$

$$PR\sin(120^\circ - \phi) = PQ\sin\theta$$

Replacing PR and PQ with the expressions from the first two triangles gives:

$$\frac{x\,\cancel{\sin 60^\circ}}{\sin\phi} \times \sin(120^\circ - \phi) = \frac{(a-x)\cancel{\sin 60^\circ}}{\sin(120^\circ - \theta)} \times \sin\theta$$

$$x\sin(120^\circ - \phi)\sin(120^\circ - \theta) = (a-x)\sin\theta\sin\phi$$

$$x\left[\sin(120^\circ)\cos\phi - \sin\phi\cos(120^\circ)\right]\left[\sin(120^\circ)\cos\theta - \sin\theta\cos(120^\circ)\right] = (a-x)\sin\theta\sin\phi$$

$$x\left[\frac{\sqrt{3}}{2}\cos\phi + \frac{1}{2}\sin\phi\right]\left[\frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta\right] = (a-x)\sin\theta\sin\phi$$

$$x\left[\sqrt{3}\frac{\cos\phi}{\sin\phi} + 1\right]\left[\sqrt{3}\frac{\cos\theta}{\sin\theta} + 1\right] = 4(a-x)$$

$$x(\sqrt{3}\cot\phi + 1)(\sqrt{3}\cot\theta + 1) = 4(a-x)$$

Let the initial speed of the ball be v_1, the speed between points Q and R be v_2 and the speed between points R and P be v_3.

It is a good idea to explain your notation! Using subscripts has the advantage that you don't run out of letters — the speeds could be $u, v, w, \cdots$, but at some point you might want a fourth speed, but x is usually used elsewhere. Also do bear in mind your handwriting style — if you are someone who finds that their "u"'s and "v"'s look similar (especially under stress) then you would probably be better going for v_1 and v_2.

The same advice goes for m and M (often a favourite pairing in STEP exams) — in this case you may want to redefine the variables at the start of the question, e.g. start by writing "Let $m = m_1$ and $M = m_2$".

There is no friction, so the component of speed parallel to the sides of the frame does not change (no change in momentum). The change of speed perpendicular to the side of the frame obeys Newton's Law of Restitution.

$$\begin{aligned} \text{Parallel to } CA: \quad & v_1 \cos(120^\circ - \theta) = v_2 \cos 60^\circ \\ \text{Perpendicular to } CA: \quad & ev_1 \sin(120^\circ - \theta) = v_2 \sin 60^\circ \end{aligned}$$

Dividing the second by the first gives:

$$\begin{aligned} e\tan(120^\circ - \theta) &= \tan 60^\circ \\ e\frac{\tan(120^\circ) - \tan\theta}{1 + \tan(120^\circ)\tan\theta} &= \tan 60^\circ \\ e\frac{-\sqrt{3} - \tan\theta}{1 - \sqrt{3}\tan\theta} &= \sqrt{3} \\ e(-\sqrt{3} - \tan\theta) &= \sqrt{3}(1 - \sqrt{3}\tan\theta) \\ e(\sqrt{3} + \tan\theta) &= 3\tan\theta - \sqrt{3} \\ \tan\theta &= \frac{\sqrt{3}(e+1)}{3-e} \end{aligned}$$

For the collision with side AB we have:

$$\begin{aligned} \text{Parallel to } AB: \quad & v_2 \cos 60^\circ = v_3 \cos\phi \\ \text{Perpendicular to } AB: \quad & ev_2 \sin 60^\circ = v_3 \sin\phi \\ \implies & e\tan 60^\circ = \tan\phi \\ & \tan\phi = \sqrt{3}e \end{aligned}$$

Substituting for $\cot\theta$ and $\cos\phi$ into the first result required in this question gives:

$$\begin{aligned} x(\sqrt{3}\cot\phi + 1)(\sqrt{3}\cot\theta + 1) &= 4(a-x) \\ x\left(\sqrt{3}\frac{1}{\sqrt{3}e} + 1\right)\left(\sqrt{3} \times \frac{(3-e)}{\sqrt{3}(e+1)} + 1\right) &= 4(a-x) \\ x\left(\frac{1}{e} + 1\right)\left(\frac{(3-e)}{(e+1)} + 1\right) &= 4(a-x) \\ x\left(\frac{\cancel{1+e}}{e}\right)\left(\frac{(3-e)+(e+1)}{\cancel{(e+1)}}\right) &= 4(a-x) \\ x\left(\frac{4}{e}\right) &= 4(a-x) \\ x &= e(a-x) \\ x(1+e) &= ae \\ x &= \frac{ae}{1+e} \end{aligned}$$

Let the speed after returning to P be v_4. Similarly to before we have:

$$\begin{aligned}\text{Parallel to } BC: &\quad v_3\cos(120^\circ-\phi)=v_4\cos\theta\\ \text{Perpendicular to } BC: &\quad ev_3\sin(120^\circ-\phi)=v_4\sin\theta\\ &\implies e\tan(120^\circ-\phi)=\tan\theta\end{aligned}$$

Simplifying this gives:

$$\begin{aligned}\tan\theta&=e\left(\frac{\tan 120^\circ-\tan\phi}{1+\tan 120^\circ\tan\phi}\right)\\ \tan\theta&=e\left(\frac{-\sqrt{3}-\tan\phi}{1-\sqrt{3}\tan\phi}\right)\\ \tan\theta&=e\left(\frac{-\sqrt{3}-\sqrt{3}e}{1-\sqrt{3}\times\sqrt{3}e}\right)\\ \tan\theta&=e\left(\frac{\sqrt{3}(1+e)}{3e-1}\right)\end{aligned}$$

Equating this with the previous expression for $\tan\theta$ gives:

$$\begin{aligned}e\left(\frac{\sqrt{3}(1+e)}{3e-1}\right)&=\frac{\sqrt{3}(e+1)}{3-e}\\ \frac{e}{3e-1}&=\frac{1}{3-e}\\ e(3-e)&=3e-1\\ 3e-e^2&=3e-1\\ e^2&=1\\ \implies\quad e&=1 \text{ because } e\geqslant 0\end{aligned}$$

Question 10

10 The origin O of coordinates lies on a smooth horizontal table and the x- and y-axes lie in the plane of the table. A cylinder of radius a is fixed to the table with its axis perpendicular to the x-y plane and passing through O, and with its lower circular end lying on the table. One end, P, of a light inextensible string PQ of length b is attached to the bottom edge of the cylinder at $(a, 0)$. The other end, Q, is attached to a particle of mass m, which rests on the table.

Initially PQ is straight and perpendicular to the radius of the cylinder at P, so that Q is at (a, b). The particle is then given a horizontal impulse parallel to the x-axis so that the string immediately begins to wrap around the cylinder. At time t, the part of the string that is still straight has rotated through an angle θ, where $a\theta < b$.

(i) Obtain the Cartesian coordinates of the particle at this time.

Find also an expression for the speed of the particle in terms of θ, $\dot{\theta}$, a and b.

(ii) Show that

$$\dot{\theta}(b - a\theta) = u\,,$$

where u is the initial speed of the particle.

(iii) Show further that the tension in the string at time t is

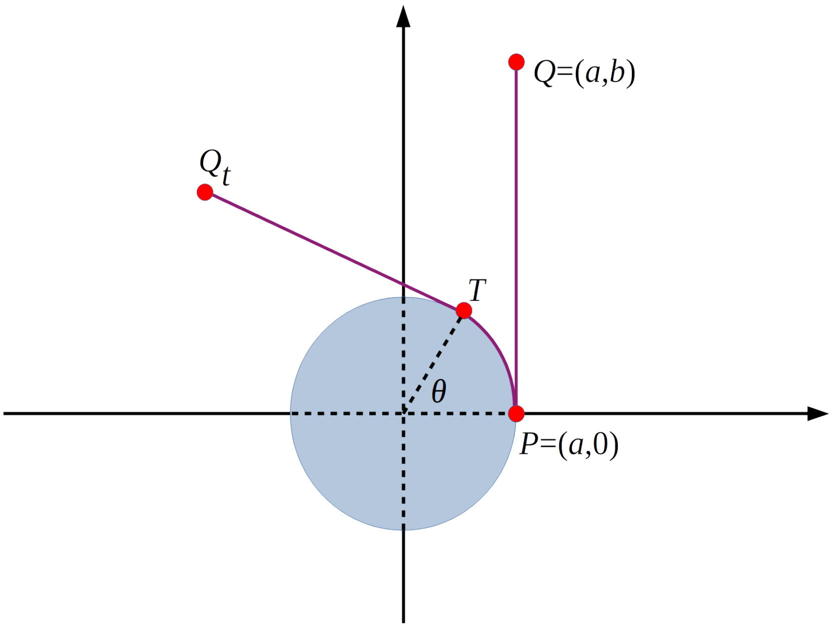
$$\frac{mu^2}{\sqrt{b^2 - 2aut}}\,.$$

Examiner's report

Whilst this was the least popular question, being attempted by a tenth of the candidature, it was the sixth most successful with a mean over 7/20. Part **(i)** was successfully attempted by many candidates, by correctly finding the coordinates of the particle and then using differentiation and Pythagoras to find the speed as required. In part **(ii)**, many understood that they could use conservation of energy even though they failed to justify it. Many used the appropriate circular motion formula in part **(iii)**, but then stumbled as they lacked justification of the evaluation of their constant of integration, or the choice of sign when taking the square root. Quite a few struggled to find the link between $b - a\theta$ and the given answer, and some attempted to jump to the given answer!

Solution

As in a lot of mechanics questions, it is a good idea to read the set up a couple of times! Here we have a cylinder standing up on one of its flat ends on a smooth table. Probably the best diagram to draw is one looking down on the table. Like a lot of circular motion questions, the letter "a" is used for the radius. Don't confuse this with acceleration!



(i) Let Q_t be the position of the particle at time t, and let T be the point where the string leaves contact with the cylinder.

Point T is at $(a\cos\theta, a\sin\theta)$. The length of the arc PT is $a\theta$, and so the the length of the straight segment Q_tT is $b - a\theta$. The angle that Q_tT makes with the horizontal is $\frac{\pi}{2} - \theta$. This means that we have:

$$\overrightarrow{TQ_t} = \begin{pmatrix} -(b-a\theta)\cos\left(\frac{\pi}{2}-\theta\right) \\ (b-a\theta)\sin\left(\frac{\pi}{2}-\theta\right) \end{pmatrix} = \begin{pmatrix} -(b-a\theta)\sin(\theta) \\ (b-a\theta)\cos(\theta) \end{pmatrix}$$

The coordinates of Q_t are given by $\big(a\cos\theta - (b-a\theta)\sin\theta,\ a\sin\theta + (b-a\theta)\cos\theta\big)$.

The horizontal and vertical components of velocity can be found by differentiating the x and y coordinates of the position of Q_t with respect to time. This gives:

$$\begin{aligned} \dot{x} &= -a\dot{\theta}\sin\theta - (b-a\theta)\dot{\theta}\cos\theta + a\dot{\theta}\sin\theta \\ \dot{y} &= a\dot{\theta}\cos\theta - (b-a\theta)\dot{\theta}\sin\theta - a\dot{\theta}\cos\theta \\ \implies \dot{x} &= -(b-a\theta)\dot{\theta}\cos\theta \\ \dot{y} &= -(b-a\theta)\dot{\theta}\sin\theta \end{aligned}$$

The speed of the particle is given by $\sqrt{\dot{x}^2+\dot{y}^2}$, which is:

$$\begin{aligned} \sqrt{\dot{x}^2+\dot{y}^2} &= \sqrt{(b-a\theta)^2\dot{\theta}^2\cos^2\theta + (b-a\theta)^2\dot{\theta}^2\cos^2\theta} \\ &= (b-a\theta)\dot{\theta}\sqrt{\cos^2\theta+\sin^2\theta} \qquad \big[\text{since } (b-a\theta)\dot{\theta} > 0\big] \\ &= (b-a\theta)\dot{\theta} \end{aligned}$$

(ii) There are no external forces acting on this system, so energy is conserved. Only kinetic energy needs to be considered as the particle does not change height - it is being held up by the frictionless table - and the string is inextensible so there is no elastic potential energy. Conservation of kinetic energy gives:

$$\tfrac{1}{2}mu^2 = \tfrac{1}{2}m\left[(b-a\theta)\dot{\theta}\right]^2$$

and since both u and $(b-a\theta)\dot{\theta}$ are positive this means we have $u = (b-a\theta)\dot{\theta}$.

(iii) The acceleration of the particle along the line of the string is given by $\dfrac{v^2}{r}$, and at time t the speed is u (as it is constant — as shown in part **(ii)**), and the radius of the circle the particle is moving at at the instant when time is t is $b - a\theta$.

Hence the force along the string, or tension, is given by $\dfrac{mu^2}{b-a\theta}$.

The question wants us to find this in terms of t rather than θ. From part **(ii)** we know that $(b-a\theta)\dot{\theta} = u$. Integrating this with respect to t gives:

$$\begin{aligned}\int \left[b\dot{\theta} - a\theta\dot{\theta}\right] \mathrm{d}t &= ut + c \\ b\theta - \tfrac{1}{2}a\theta^2 &= ut + c\end{aligned}$$

When $t = 0$ we have $\theta = 0$, and so $c = 0$. this gives:

$$\begin{aligned}b\theta - \tfrac{1}{2}a\theta^2 &= ut \\ a\theta^2 - 2b\theta &= -2ut \\ (a\theta)^2 - 2ab\theta &= -2aut \\ (b-a\theta)^2 - b^2 &= -2aut\end{aligned}$$

The reason for rearranging in this way is that we want to change the denominator of $b - a\theta$ into one of the form $\sqrt{b^2 - 2aut}$ — the fact that we are aiming for a square root suggests that trying to complete the square might be a good idea!

We now have $(b-a\theta)^2 = b^2 - 2aut$, and taking positive square roots gives $b - a\theta = \sqrt{b^2 - 2aut}$.

Hence the tension in the string is equal to:

$$\frac{mu^2}{b-a\theta} = \frac{mu^2}{\sqrt{b^2-2aut}}$$

Question 11

11 The continuous random variable X has probability density function

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geqslant 0, \\ 0 & \text{otherwise,} \end{cases}$$

where λ is a positive constant.

The random variable Y is the greatest integer less than or equal to X, and $Z = X - Y$.

(i) Show that, for any non-negative integer n,

$$P(Y = n) = (1 - e^{-\lambda})e^{-n\lambda}.$$

(ii) Show that

$$P(Z < z) = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}} \quad \text{for } 0 \leqslant z \leqslant 1.$$

(iii) Evaluate $E(Z)$.

(iv) Obtain an expression for

$$P(Y = n \text{ and } z_1 < Z < z_2),$$

where $0 \leqslant z_1 < z_2 \leqslant 1$ and n is a non-negative integer.

Determine whether Y and Z are independent.

Examiner's report

Comfortably the most popular applied question on the paper attracting slightly more than a third of candidates, it was the second most successful on the whole paper with a mean of 11/20. The quality of attempts for this question was high, with many candidates scoring full or close to full marks. Almost all candidates attempting it dealt with part **(i)** successfully. However, in part **(ii)** candidates often made incorrect conditioning arguments. The most common errors were computing $P(Z < z|Y = n)$ rather than $P(Z < z)$ and confusing $P(Z < z|Y = n)$ with $P(Z < z \text{ and } Y = n)$.

In part **(iii)**, most candidates suitably obtained a probability density function for Z, but there were several computational mistakes in the integration by parts to evaluate the expectation. The independence argument in part **(iv)** was largely well executed, even when candidates had been unsuccessful in answering parts **(ii)** and **(iii)** of the question.

Solution

The relationship $Z = X - Y$ means that Y is the integer part of X, and Z is the fractional part. For example consider 4.34 — here the greatest integer less than or equal to 4.34 is equal to 4, and the fractional part is $4.34 - 4 = 0.34$.

Since X is a continuous random variable, we cannot say that $X = 4.34$ exactly. A better example for this situation is if $4.27 < X < 4.34$ then $Y = 4$ (Y is a discrete random variable as it takes integer values!) and $0.27 < Z < 0.34$.

(i) We are told that Y is the greatest integer less than or equal to X, so if $Y = n$ then this means we have $n < X < n + 1$.

$$\begin{aligned} P(Y = n) &= \int_n^{n+1} \lambda e^{-\lambda x} \, dx \\ &= \left[-e^{-\lambda x}\right]_n^{n+1} \\ &= -e^{-\lambda(n+1)} + e^{-\lambda n} \\ &= e^{-\lambda n}\left(1 - e^{-\lambda}\right) \end{aligned}$$

(ii) Z must take a value in the range $0 \leqslant Z < 1$. If we are to have $Z < z$, then we can have $0 \leqslant X < z$, $1 \leqslant X < 1 + z$ etc. This means we are going to need to consider a sum:

$$\begin{aligned} P(Z < z) &= \sum_{i=0}^{\infty} P(i \leqslant X < i + z) \\ &= \sum_{i=0}^{\infty} \int_i^{i+z} \lambda e^{-\lambda x} \, dx \\ &= \sum_{i=0}^{\infty} \left[-e^{-\lambda x}\right]_i^{i+z} \\ &= \sum_{i=0}^{\infty} \left[e^{-\lambda i}(1 - e^{-\lambda z})\right] \\ &= (1 - e^{-\lambda z}) \sum_{i=0}^{\infty} e^{-\lambda i} \\ &= (1 - e^{-\lambda z}) \left[1 + e^{-\lambda} + e^{-2\lambda} + \cdots\right] \end{aligned}$$

The expression in the square brackets is the sum of an infinite Geometric Progression, and since λ is positive we have $|e^{-\lambda}| < 1$, and so the sum exists. This means we have:

$$P(Z < z) = (1 - e^{-\lambda z}) \times \frac{1}{1 - e^{-\lambda}} = \frac{1 - e^{-\lambda z}}{1 - e^{-\lambda}}$$

as required.

(iii) We have $\mathrm{P}(Z < z) = \dfrac{1-\mathrm{e}^{-\lambda z}}{1-\mathrm{e}^{-\lambda}}$, and so the pdf[4] is given by:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{1-\mathrm{e}^{-\lambda z}}{1-\mathrm{e}^{-\lambda}}\right) = \frac{\lambda\mathrm{e}^{-\lambda z}}{1-\mathrm{e}^{-\lambda}}$$

Therefore the expectation is given by:

$$\begin{aligned}
\mathrm{E}(Z) &= \int_0^1 z \times \frac{\lambda\mathrm{e}^{-\lambda z}}{1-\mathrm{e}^{-\lambda}}\,\mathrm{d}z \\
&= \frac{1}{1-\mathrm{e}^{-\lambda}}\int_0^1 \lambda z\mathrm{e}^{-\lambda z}\,\mathrm{d}z \\
&= \frac{1}{1-\mathrm{e}^{-\lambda}}\left(\left[-z\mathrm{e}^{-\lambda z}\right]_0^1 + \int_0^1 \mathrm{e}^{-\lambda z}\,\mathrm{d}z\right) \\
&= \frac{1}{1-\mathrm{e}^{-\lambda}}\left(-\mathrm{e}^{-\lambda} - \frac{1}{\lambda}\left[\mathrm{e}^{-\lambda z}\right]_0^1\right) \\
&= \frac{1}{1-\mathrm{e}^{-\lambda}}\left(-\mathrm{e}^{-\lambda} - \frac{1}{\lambda}\left[\mathrm{e}^{-\lambda} - 1\right]\right) \\
&= \frac{1}{\lambda} - \frac{\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}
\end{aligned}$$

(iv) If $Y = n$ and $z_1 < Z < z_2$, then this means that we have $n+z_1 < X < n+z_2$. the probability is therefore:

$$\begin{aligned}
\int_{n+z_1}^{n+z_2} \lambda\mathrm{e}^{-\lambda x}\,\mathrm{d}x &= \left[-\mathrm{e}^{-\lambda x}\right]_{n+z_1}^{n+z_2} \\
&= -\mathrm{e}^{-\lambda(n+z_2)} + \mathrm{e}^{-\lambda(n+z_1)} \\
&= \mathrm{e}^{-\lambda n}\left[\mathrm{e}^{-\lambda z_1} - \mathrm{e}^{-\lambda z_2}\right]
\end{aligned}$$

So we have $\mathrm{P}\big(Y = n \text{ and } z_1 < Z < z_2\big) = \mathrm{e}^{-\lambda n}\left[\mathrm{e}^{-\lambda z_1} - \mathrm{e}^{-\lambda z_2}\right]$.

We know that $\mathrm{P}(Y = n) = (1-\mathrm{e}^{-\lambda})\mathrm{e}^{-n\lambda}$, and $\mathrm{P}(z_1 < Z < z_2) = \mathrm{P}(Z < z_2) - \mathrm{P}(Z < z_1)$. Therefore:

$$\begin{aligned}
\mathrm{P}(Y = n) \times \mathrm{p}(z_1 < Z < z_2) &= (1-\mathrm{e}^{-\lambda})\mathrm{e}^{-n\lambda} \times \left[\frac{1-\mathrm{e}^{-\lambda z_2}}{1-\mathrm{e}^{-\lambda}} - \frac{1-\mathrm{e}^{-\lambda z_1}}{1-\mathrm{e}^{-\lambda}}\right] \\
&= \mathrm{e}^{-n\lambda}\left[(1-\mathrm{e}^{-\lambda z_2}) - (1-\mathrm{e}^{-\lambda z_1})\right] \\
&= \mathrm{e}^{-n\lambda}\left[\mathrm{e}^{-\lambda z_1} - \mathrm{e}^{-\lambda z_2}\right]
\end{aligned}$$

which is the same expression as the one for $\mathrm{P}\big(Y = n \text{ and } z_1 < Z < z_2\big)$. Hence Y and Z are independent.

[4]Probability Density Function

Question 12

12 **(i)** In a game, each member of a team of n players rolls a fair six-sided die.

The total score of the team is the number of pairs of players rolling the same number.
For example, if 7 players roll $3, 3, 3, 3, 6, 6, 2$ the total score is 7, as six different pairs of players both score 3 and one pair of players both score 6.

Let X_{ij}, for $1 \leqslant i < j \leqslant n$, be the random variable that takes the value 1 if players i and j roll the same number and the value 0 otherwise.

Show that X_{12} is independent of X_{23}.

Hence find the mean and variance of the team's total score.

(ii) Show that, if Y_i, for $1 \leqslant i \leqslant m$, are random variables with mean zero, then

$$\text{Var}(Y_1 + Y_2 + \ldots + Y_m) = \sum_{i=1}^{m} \text{E}(Y_i^2) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \text{E}(Y_iY_j)\,.$$

(iii) In a different game, each member of a team of n players rolls a fair six-sided die.

The total score of the team is the number of pairs of players rolling the same even number minus the number of pairs of players rolling the same odd number.
For example, if 7 players roll $3, 3, 3, 3, 6, 6, 2$ the total score is -5.

Let Z_{ij}, for $1 \leqslant i < j \leqslant n$, be the random variable that takes the value 1 if players i and j roll the same even number, the value -1 if players i and j roll the same odd number and the value 0 otherwise.

Show that Z_{12} is not independent of Z_{23}.

Find the mean of the team's total score and show that the variance of the team's total score is $\frac{1}{36}n(n^2-1)$.

Examiner's report

A sixth of candidates attempted this, making it the second least popular question, and it was the third least successful with a mean just shy of six and a half. Very few candidates obtained full marks for the very first part of the question, showing that X_{12} and X_{23} are independent; the most common error being to check only that $\text{P}(X_{12} = 1, X_{23} = 1) = \text{P}(X_{12} = 1)\text{P}(X_{23} = 1)$, rather than all four possible cases for the different values of the two random variables. However, in general, candidates engaged well with the combinatorial aspect of this part and provided sound methods for counting pairs of indices in order to obtain the mean and variance, though many did not use the fact that for independent random variables, $\mathrm{Var}\left(\sum_i X_i\right) = \sum_i \mathrm{Var}(X_i)$. However, part **(ii)** was consistently well executed, with most candidates that attempted it being successful. In part **(iii)**, establishing non-independence was well executed, and again, as in part **(i)**, candidates provided sound methods for counting pairs of indices.

Solution

(i) X_{12} and X_{23} can take the values 1 and 0. We have:

$$\mathrm{P}(X_{12} = 1) = \frac{1}{6} \quad \text{and} \quad \mathrm{P}(X_{12} = 0) = \frac{5}{6}$$
$$\mathrm{P}(X_{23} = 1) = \frac{1}{6} \quad \text{and} \quad \mathrm{P}(X_{23} = 0) = \frac{5}{6}$$

The probability that players 1 and 2 roll the same number can either be thought of as there being 6 "successful outcomes" — $(1,1); (2,2); \cdots$ — out of a total of 36 outcomes, or that player 2 as a probability of $\frac{1}{6}$ of getting the same value as player 1. The same argument holds for players 2 and 3.

If X_{12} is independent of X_{23}, then the outcome of X_{23} should not change the probabilities of the outcomes of X_{12}.

Start by considering $\mathrm{P}(X_{12} = 1 | X_{23} = 1)$[5]. In this case we know that players 2 and 3 have rolled the same number. The probability that player 1 also rolls the same number is $\frac{1}{6}$, so we have $\mathrm{P}(X_{12} = 1|X_{23} = 1) = \frac{1}{6} = \mathrm{P}(X_{12} = 1)$. Similarly the probability that player 1 rolls a different number to players 2 and 3 is $\frac{5}{6}$, so we have $\mathrm{P}(X_{12} = 0|X_{23} = 1) = \frac{5}{6} = \mathrm{P}(X_{12} = 0)$.

If $X_{23} = 0$ this means that players 2 and 3 have rolled different numbers. We have $\mathrm{P}(X_{12} = 1|X_{23} = 0) = \frac{1}{6} = \mathrm{P}(X_{12} = 1$ as there is a probability of $\frac{1}{6}$ that player 1 rolls the same number as player 2 (which is different to player 3's number). Similarly $\mathrm{P}(X_{12} = 0|X_{23} = 0) = \frac{5}{6} = \mathrm{P}(X_{12} = 0)$. Therefore the probabilities of the outcomes of X_{12} do not change if you know the outcome of X_{23}, so we know that X_{12} is independent of X_{23}.

Alternatively, you could use the same idea used at the end of question 11.

$$\begin{aligned} \mathrm{P}(X_{12} = 1 \text{ and } X_{23} = 1) &= \mathrm{P}(\text{Players 1, 2, and 3 all roll the same}) \\ &= 1 \times \frac{1}{6} \times \frac{1}{6} \\ &= \mathrm{P}(X_{12} = 1) \times \mathrm{P}(X_{23} = 1) \end{aligned}$$

$$\begin{aligned} \mathrm{P}(X_{12} = 0 \text{ and } X_{23} = 1) &= \mathrm{P}(\text{Player 2 rolls a different number to player 1,} \\ &\qquad \text{and player 3 rolls the same as player 2}) \\ &= 1 \times \frac{5}{6} \times \frac{1}{6} \\ &= \mathrm{P}(X_{12} = 0) \times \mathrm{P}(X_{23} = 1) \end{aligned}$$

and then show the similar results for $\mathrm{P}(X_{12} = 1 \text{ and } X_{23} = 0)$ and $\mathrm{P}(X_{12} = 0 \text{ and } X_{23} = 0)$.

[5]This means the probability that $X_{12} = 1$ given that $X_{23} = 1$, i.e. the probability that players 1 and 2 roll the same number given that you know players 2 and 3 have rolled the same number.

The total score is given by the sum of X_{ij} over all the possible pairs of i, j (remembering that $1 \leqslant i < j \leqslant n$). For example, if there are three players then the total sum is $X_{12} + X_{13} + X_{23}$. The condition $i < j$ ensures that we don't double count any of the pairs.

Since the X_{ij}'s are independent then the the expectation and variance of the total score are equal to the sum of expectation/variance of the individual pairs scores, and each pair will have the same expectation/variance by symmetry.

Consider X_{12}. We have:

$$\begin{aligned}
\mathrm{E}(X_{12}) &= 1 \times \frac{1}{6} + 0 \times \frac{0}{6} = \frac{1}{6} \\
\mathrm{E}(X_{12}^2) &= 1^2 \times \frac{1}{6} + 0^2 \times \frac{0}{6} = \frac{1}{6} \\
\mathrm{Var}(X_{12}) &= \frac{1}{6} - \left(\frac{1}{6}\right)^2 = \frac{5}{36}
\end{aligned}$$

The number of different "Pairs" of players is given by ${}^nC_2$, as we need to find the number of different ways of "choosing" a pair of players from the n players in total.
This is ${}^nC_2 = \dfrac{n(n-1)}{2}$ and so the expectation and variance of the total score are:

$$\begin{aligned}
\mathrm{E}(T) &= \frac{n(n-1)}{2} \times \frac{1}{6} = \frac{n(n-1)}{12} \\
\mathrm{Var}(T) &= \frac{n(n-1)}{2} \times \frac{5}{36} = \frac{5n(n-1)}{72}
\end{aligned}$$

(ii) We don't know that the random variables are independent, but we are told that $\mathrm{E}(Y_1) = \mathrm{E}(Y_2) = \cdots = \mathrm{E}(Y_m)$. Using $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ gives:

$$\begin{aligned}
\mathrm{Var}(Y_1 + Y_2 + \cdots + Y_m) &= \mathrm{E}\big([Y_1 + Y_2 + \cdots + Y_m]^2\big) - \big[\mathrm{E}(Y_1 + Y_2 + \cdots + Y_m)\big]^2 \\
&= \mathrm{E}\big(Y_1^2 + Y_2^2 + \cdots + Y_m^2 + \text{cross-terms}\big) - \big[\mathrm{E}(Y_1) + \mathrm{E}(Y_2) + \cdots + \mathrm{E}(Y_m)\big]^2 \\
&= \mathrm{E}\left(Y_1^2 + Y_2^2 + \cdots + Y_m^2 + \sum_{i \neq j} Y_i Y_j\right) - \big[0 + 0 + \cdots + 0\big]^2 \\
&= \mathrm{E}\left(Y_1^2 + Y_2^2 + \cdots + Y_m^2 + 2\sum_{i<j} Y_i Y_j\right) - 0^2 \\
&= \mathrm{E}(Y_1^2) + \mathrm{E}(Y_2^2) + \cdots + \mathrm{E}(Y_m^2) + 2\sum_{i<j} \mathrm{E}(Y_i Y_j) \\
&= \sum_{i=1}^{m} \mathrm{E}(Y_i^2) + 2\sum_{i=1}^{m-1} \sum_{j=i+1}^{m} \mathrm{E}(Y_i Y_j)
\end{aligned}$$

The result $\sum_{i \neq j} Y_i Y_j = 2\sum_{i<j} Y_i Y_j$ makes use of the fact that the cross-terms come in pairs — $(Y_1 Y_2)$ and $(Y_2 Y_1)$ etc.

(iii) Consider $\mathrm{P}(Z_{12} = 1 \text{ and } Z_{23} = 1)$. This needs players 1, 2 and 3 all to roll the same even number.

$$\mathrm{P}(Z_{12} = 1 \text{ and } Z_{23} = 1) = \frac{1}{2} \times \frac{1}{6} \times \frac{1}{6} = \frac{1}{72}$$

We also have $\mathrm{P}(Z_{12} = 1) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12}$ (as player 1 has to roll an even number, and then player 2 has to roll the same number). In the same way, $\mathrm{P}(Z_{23} = 1) = \frac{1}{12}$. This gives $\mathrm{P}(Z_{12}) \times \mathrm{P}(Z_{23}) = \frac{1}{144}$ and so we have $\mathrm{P}(Z_{12} = 1 \text{ and } Z_{23} = 1) \neq \mathrm{P}(Z_{12}) \times \mathrm{P}(Z_{23})$. Hence Z_{12} and Z_{23} are **not** independent.

Like in part **(i)**, you could use another method. We have $\mathrm{P}(Z_{12}) = \frac{1}{12}$. If we know that $Z_{23} = 1$, then both players 2 and 3 have rolled the same (even) number.
Hence $\mathrm{P}(Z_{12} = 1 | Z_{23} = 1) = \frac{1}{6}$, which is different to $\mathrm{P}(Z_{12} = 1)$, and so the two events are not independent.

The expectation of Z_{12} is $\mathrm{E}(Z_{12}) = 1 \times \frac{1}{12} + (-1) \times \frac{1}{12} + 0 \times \frac{5}{6} = 0$. By symmetry, we have $E(Z_{ij}) = 0$ (and now we begin to suspect that part **(ii)** might be useful here!).

If T is the total score we have:

$$\mathrm{E}(T) = \mathrm{E}\left(\sum_{i<j} Zij\right) = \sum_{i<j} \mathrm{E}(Z_{ij}) = 0$$

Let $Z_{12} = Y_1$, $Z_{13} = Y_2$, $Z_{1n} = Y_{n-1}$, $Z_{23} = Y_n$, ..., $Z_{(n-1)n} = Y_m$, where $m = {}^nC_2 = \dfrac{n(n-1)}{2}$.

Using the result in part **(ii)** (which we can do as we have shown that the expectation/mean of each Z_{ij} is equal to 0) we have:

$$\begin{aligned}\mathrm{Var}(Z_{12} + Z_{13} + \cdots + Z_{(n-1)n}) &= \mathrm{Var}(Y_1 + Y_2 + \cdots + Y_m) \\ &= \sum_{i=1}^{m} \mathrm{E}(Y_i^2) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \mathrm{E}(Y_iY_j)\end{aligned}$$

Consider $\mathrm{E}(Z_{12}^2)$. This is given by:

$$\mathrm{E}(Z_{12}^2) = 1^2 \times \frac{1}{12} + (-1)^2 \times \frac{1}{12} + 0 \times \frac{5}{6} = \frac{1}{6}$$

and the same result holds for all of $\mathrm{E}(Z_{ij}^2)$.

We know that $m = \dfrac{n(n-1)}{2}$, so we have

$$\sum_{i=1}^{m} \mathrm{E}(Y_i^2) = \frac{n(n-1)}{2} \times \frac{1}{6} \qquad (*)$$

Consider $\mathrm{E}(Z_{12}Z_{13})$. If $Z_{12} = 1$ then both player 1 and 2 rolled the same even number, so Z_{13} can be either equal to 1 (player three rolls the same even number as players and 2), or equal to 0. Z_{13} cannot be equal to -1 as that would need player 1 to have rolled an odd number. Similarly if $Z_{12} = -1$ then Z_{13} is either -1 or 0.

The only two non-zero terms of $\mathrm{E}(Z_{12}Z_{13})$ are contributed by $Z_{12} = Z_{13} = 1$ and $Z_{12} = Z_{13} = -1$. Hence we have:

$$\begin{aligned}\mathrm{E}(Z_{12}Z_{13}) &= 1 \times 1 \times \left(\frac{1}{2} \times \frac{1}{6} \times \frac{1}{6}\right) + (-1) \times (-1) \times \left(\frac{1}{2} \times \frac{1}{6} \times \frac{1}{6}\right) \\ &= \frac{1}{72} + \frac{1}{72} = \frac{1}{36}\end{aligned}$$

In a similar way, we have $\mathrm{E}(Z_{12}Z_{23}) = \frac{1}{36}$.

Consider $\mathrm{E}(Z_{12}Z_{34})$, so now we are interested in players 1, 2, 3 and 4. We have:

$$\begin{aligned}\mathrm{P}(Z_{12}Z_{34} = 1) &= \mathrm{P}(Z_{12} = Z_{34} = 1) + \mathrm{P}(Z_{12} = Z_{34} = -1) \\ &= \frac{1}{12} \times \frac{1}{12} + \frac{1}{12} \times \frac{1}{12} = \frac{1}{72} \\ \mathrm{P}(Z_{12}Z_{34} = -1) &= \mathrm{P}(Z_{12} = 1, Z_{34} = -1) + \mathrm{P}(Z_{12} = -1, Z_{34} = 1) \\ &= \frac{1}{12} \times \frac{1}{12} + \frac{1}{12} \times \frac{1}{12} = \frac{1}{72} \\ \mathrm{P}(Z_{12}Z_{34} = 0) &= 1 - \frac{1}{72} - \frac{1}{72} = \frac{35}{36}\end{aligned}$$

Therefore we have:

$$\mathrm{E}(Z_{12}Z_{34}) = 1 \times \frac{1}{72} + (-1) \times \frac{1}{72} + 0 \times \frac{35}{36} = 0$$

More generally, we have $\mathrm{E}(Z_{ij}Z_{jk}) = \mathrm{E}(Z_{ij}Z_{ik}) = \dfrac{1}{36}$ (where $i < j < k$) and $\mathrm{E}(Z_{ij}Z_{kl}) = 0$ (where none of i, j, k, l are the same).

The $2\displaystyle\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \mathrm{E}(Y_iY_j)$ parts of the variance only have contributions when the terms have the form $\mathrm{E}(Z_{ij}Z_{jk})$ or $\mathrm{E}(Z_{ij}Z_{ik})$. The number of ways to choose these pairs can be found by considering the following

$$\begin{aligned}&\text{Ways to pick one repeating index} \times \text{Ways to pick two distinct indices from those left} \\ &= {}^{n}C_1 \times {}^{n-1}C_2 \\ &= n \times \frac{(n-1)(n-2)}{2}\end{aligned}$$

Instead you could have considered the number of ways of picking 2 indices to appear once each, and then picking one from the $n - 2$ left to appear twice - this would give the same answer!

Hence the Variance is given by:

$$\begin{aligned}&\sum_{i=1}^{m} \mathrm{E}(Y_i^2) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \mathrm{E}(Y_iY_j) \\ &= \left(\frac{n(n-1)}{2} \times \frac{1}{6}\right) + \left(\cancel{2} \times \frac{n(n-1)(n-2)}{\cancel{2}} \times \frac{1}{36}\right) \\ &= \frac{n(n-1)}{36}\Big[3 + (n-2)\Big] \\ &= \frac{n(n-1)}{36}(n+1) \\ &= \tfrac{1}{36}n(n^2-1)\end{aligned}$$