

**STEP Support Programme - Cambridge state school offer-holders day
Workshop 1 Questions**

1 2014 S3 Q6

Starting from the result that

$$h(t) > 0 \text{ for } 0 < t < x \implies \int_0^x h(t) dt > 0,$$

show that, if $f''(t) > 0$ for $0 < t < x_0$ and $f(0) = f'(0) = 0$, then $f(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}.$$

2 2014 S3 Q7

The four distinct points P_i ($i = 1, 2, 3, 4$) are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines P_1P_3 and P_2P_4 intersect at Q .

(i) By considering the triangles P_1QP_4 and P_2QP_3 show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)$.

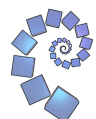
(ii) Let \mathbf{p}_i be the position vector of the point P_i ($i = 1, 2, 3, 4$). Show that there exist numbers a_i , not all zero, such that

$$\sum_{i=1}^4 a_i = 0 \quad \text{and} \quad \sum_{i=1}^4 a_i \mathbf{p}_i = \mathbf{0}. \quad (*)$$

(iii) Let a_i ($i = 1, 2, 3, 4$) be any numbers, not all zero, that satisfy (*). Show that $a_1 + a_3 \neq 0$ and that the lines P_1P_3 and P_2P_4 intersect at the point with position vector

$$\frac{a_1 \mathbf{p}_1 + a_3 \mathbf{p}_3}{a_1 + a_3}.$$

Deduce that $a_1 a_3 (P_1P_3)^2 = a_2 a_4 (P_2P_4)^2$.



3 2014 S3 Q8

The numbers $f(r)$ satisfy $f(r) > f(r+1)$ for $r = 1, 2, \dots$. Show that, for any non-negative integer n ,

$$k^n(k-1)f(k^{n+1}) \leq \sum_{r=k^n}^{k^{n+1}-1} f(r) \leq k^n(k-1)f(k^n)$$

where k is an integer greater than 1.

(i) By taking $f(r) = 1/r$, show that

$$\frac{N+1}{2} \leq \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leq N+1.$$

Deduce that the sum $\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $f(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leq 1\frac{1}{3}.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n .

