STEP Support Programme - Cambridge state school offer-holders day
Workshop 1 Questions

1 2014 S3 Q6

Starting from the result that

$$\text{h}(t) > 0 \text{ for } 0 < t < x \Longrightarrow \int_0^x \text{h}(t)\,\text{d}t > 0\,,$$

show that, if $\text{f}''(t) > 0$ for $0 < t < x_0$ and $\text{f}(0) = \text{f}'(0) = 0$, then $\text{f}(t) > 0$ for $0 < t < x_0$.

(i) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\cos x \cosh x < 1\,.$$

(ii) Show that, for $0 < x < \frac{1}{2}\pi$,

$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}\,.$$

2 2014 S3 Q7

The four distinct points P_i ($i = 1, 2, 3, 4$) are the vertices, labelled anticlockwise, of a cyclic quadrilateral. The lines P_1P_3 and P_2P_4 intersect at Q.

(i) By considering the triangles P_1QP_4 and P_2QP_3 show that $(P_1Q)(QP_3) = (P_2Q)(QP_4)\,.$

(ii) Let $\mathbf{p}_i$ be the position vector of the point P_i ($i = 1, 2, 3, 4$). Show that there exist numbers a_i, not all zero, such that

$$\sum_{i=1}^{4} a_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{4} a_i\mathbf{p}_i = \mathbf{0}\,. \qquad (*)$$

(iii) Let a_i ($i = 1, 2, 3, 4$) be any numbers, not all zero, that satisfy $(*)$. Show that $a_1 + a_3 \neq 0$ and that the lines P_1P_3 and P_2P_4 intersect at the point with position vector

$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}\,.$$

Deduce that $a_1a_3(P_1P_3)^2 = a_2a_4(P_2P_4)^2\,.$

3 **2014 S3 Q8**

The numbers $\mathrm{f}(r)$ satisfy $\mathrm{f}(r) > \mathrm{f}(r+1)$ for $r = 1, 2, \ldots$. Show that, for any non-negative integer n,

$$k^n(k-1)\,\mathrm{f}(k^{n+1}) \leqslant \sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) \leqslant k^n(k-1)\,\mathrm{f}(k^n)$$

where k is an integer greater than 1.

(i) By taking $\mathrm{f}(r) = 1/r$, show that

$$\frac{N+1}{2} \leqslant \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leqslant N+1\,.$$

Deduce that the sum $\sum\limits_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) By taking $\mathrm{f}(r) = 1/r^3$, show that

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leqslant 1\tfrac{1}{3}\,.$$

(iii) Let $S(n)$ be the set of positive integers less than n which do not have a 2 in their decimal representation and let $\sigma(n)$ be the sum of the reciprocals of the numbers in $S(n)$, so for example $\sigma(5) = 1 + \frac{1}{3} + \frac{1}{4}$. Show that $S(1000)$ contains $9^3 - 1$ distinct numbers.

Show that $\sigma(n) < 80$ for all n.