STEP Support Programme - Cambridge state school offer-holders day
Workshop 1 Solutions

In these solutions I have written more detail than you would expect in a STEP exam question solution. In an exam situation I would use more notation e.g."$\Longrightarrow$", and possibly use arrows etc. to show my reasoning, for example where I have used a result elsewhere in the question. I would also be more inclined to use sketches and diagrams that I am when LaTeX-ing up solutions.

1 2014 S3 Q6

Let $0 < u < x_0$, so we have $\mathrm{f}''(t) > 0$ for $0 < t < u$. Therefore (using the given result) we have:

$$\begin{aligned} &\int_0^u \mathrm{f}''(t)\,\mathrm{d}t > 0 \\ \Longrightarrow\ &\mathrm{f}'(u) - \mathrm{f}'(0) > 0 \\ \Longrightarrow\ &\mathrm{f}'(u) > 0 \qquad \text{since } \mathrm{f}'(0) = 0 \end{aligned}$$

We now have $\mathrm{f}'(u) > 0$ for $0 < u < x_0$. Let $0 < v < x_0$ so we have $\mathrm{f}'(u) > 0$ for $0 < u < v$. Therefore:

$$\begin{aligned} &\int_0^v \mathrm{f}'(u)\,\mathrm{d}u > 0 \\ \Longrightarrow\ &\mathrm{f}(v) - \mathrm{f}(0) > 0 \\ \Longrightarrow\ &\mathrm{f}(v) > 0 \qquad \text{since } \mathrm{f}(0) = 0 \end{aligned}$$

We have now shown that if $\mathrm{f}''(t) > 0$ for $0 < t < x_0$ **and** $\mathrm{f}(0) = \mathrm{f}'(0) = 0$ then $\mathrm{f}(v) > 0$ for $0 < v < x_0$ or equivalently we can say $\mathrm{f}(t) > 0$ for $0 < t < x_0$.

The first time I attempted this question I started by showing that $\mathrm{f}'(x_0) > 0$ but then realised that I really wanted a variable rather than x_0 in the derivative so I had to change my argument. For the very last part we have essentially used a substitution of $t = v$.

(i) Here we are trying to show that something is less than 1, but the stem of the question was about showing that something is greater than 0. It might be a good idea to arrange the given inequality to match what we have in the stem![1]

We want to show that $\cos x \cosh x < 1$ or equivalently that $1 - \cos x \cosh x > 0$. Let $\mathrm{f}(x) = 1 - \cos x \cosh x$.

If we want to use the stem result then we must show that the conditions of the stem are satisfied.

Note that in this case $x_0 = \frac{1}{2}\pi$. We have:

$$\begin{aligned} &\mathrm{f}(x) = 1 - \cos x \cosh x \\ \Longrightarrow\ &\mathrm{f}'(x) = 0 + \sin x \cosh x - \cos x \sinh x \\ \Longrightarrow\ &\mathrm{f}''(x) = \cancel{\cos x \cosh x} + \sin x \sinh x + \sin x \sinh x - \cancel{\cos x \cosh x} \\ &\phantom{\mathrm{f}''(x)} = 2 \sin x \sinh x \end{aligned}$$

[1] As a general rule, when trying to show that an inequality is true it is often easier to rearrange and show that something is positive or negative.

In the range $0 < x < \frac{1}{2}\pi$ we have $\sin x > 0$ and $\sinh x > 0$, so we have $\mathrm{f}''(x) = 2\sin x \sinh x > 0$ for $0 < x < \frac{1}{2}\pi$.

We also have:

$$\begin{aligned} \mathrm{f}(0) &= 1 - \cos 0 \cosh 0 = 1 - 1 = 0 \\ \mathrm{f}'(0) &= \sin 0 \cosh 0 - \cos 0 \sinh 0 = 0 - 0 = 0 \end{aligned}$$

Hence the conditions in the stem apply, and so we have $\mathrm{f}(x) > 0$ and so:

$$1 - \cos x \cosh x > 0 \implies \cos x \cosh x < 1$$

(ii) Here there are two inequalities to consider.

Start by considering $\dfrac{1}{\cosh x} < \dfrac{\sin x}{x}$. In the range $0 < x < \frac{1}{2}\pi$ both $x > 0$ and $\cosh x > 0$ so we can legitimately[2] rearrange the inequality to get $x < \sin x \cosh x$ or (more usefully) $\sin x \cosh x - x > 0$.

Let $\mathrm{f}(x) = \sin x \cosh x - x$.

$$\begin{aligned} \mathrm{f}'(x) &= \sin x \sinh x + \cos x \cosh x - 1 \\ \implies \mathrm{f}''(x) &= \cancel{\sin x \cosh x} + \cos x \sinh x + \cos x \sinh x - \cancel{\sin x \cosh x} \\ &= 2\cos x \sinh x \end{aligned}$$

and as both $\cos x > 0$ and $\sinh x > 0$ for $0 < x < \frac{1}{2}\pi$ we have $\mathrm{f}''(x) > 0$.

We also have:

$$\begin{aligned} \mathrm{f}(0) &= \sin 0 \cosh 0 - 0 = 0 \\ \mathrm{f}'(0) &= \sin 0 \sinh 0 + \cos 0 \cosh 0 - 1 = 0 + 1 - 1 = 0 \end{aligned}$$

Hence we can use the result from the stem and we have

$$\sin x \cosh x - x > 0 \implies \frac{1}{\cosh x} < \frac{\sin x}{x}$$

Now consider $\dfrac{\sin x}{x} < \dfrac{x}{\sinh x}$. Since both $x > 0$ and $\sinh x > 0$ for $0 < x < \frac{1}{2}\pi$ we can rearrange to get $x^2 - \sin x \sinh x > 0$.

Let $\mathrm{f}(x) = x^2 - \sin x \sinh x$.

$$\begin{aligned} \mathrm{f}'(x) &= 2x - \cos x \sinh x - \sin x \cosh x \\ \implies \mathrm{f}''(x) &= 2 + \cancel{\sin x \sinh x} - \cos x \cosh x - \cos x \cosh x - \cancel{\sin x \sinh x} \\ &= 2 - 2\cos x \cosh x \end{aligned}$$

At first it looks like we are going to have to do some work to show that $\mathrm{f}''(x) > 0$ **but** in part **(i)** we showed that $\cos x \cosh x < 1$ which implies that $1 - \cos x \cosh x > 0$ and so we have $\mathrm{f}''(x) = 2(1 - \cos x \cosh x) > 0$.

[2]When rearranging inequalities always consider whether you might be multiplying or dividing by something negative. Sketching a graph can often be a safer way to deal with inequalities.

We also have:

$$\begin{aligned} \mathrm{f}(0) &= 0^2 - \sin 0 \sinh 0 = 0 \\ \mathrm{f}'(0) &= 2 \times 0 - \cos 0 \sinh 0 - \sin 0 \cosh 0 = 0 \end{aligned}$$

Hence we can use the stem result and so we have:

$$x^2 - \sin x \sinh x > 0 \implies \frac{\sin x}{x} < \frac{x}{\sinh x}$$

and therefore we have shown that:

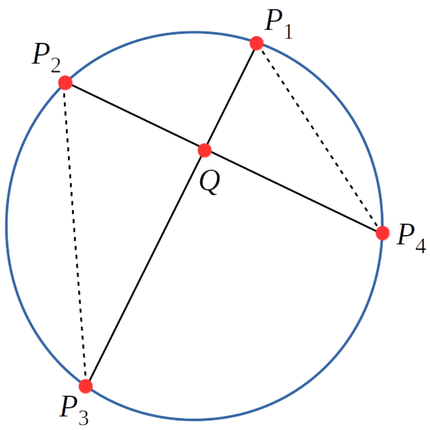
$$\frac{1}{\cosh x} < \frac{\sin x}{x} < \frac{x}{\sinh x}$$

Note that you can use the "stem" result to do the rest of the question even if you could not work out how to show the stem result to be true - and similarly you can use the result in **(i)** to help you to do part **(ii)** even if you couldn't show **(i)** to be true.

Originally this question had another part: "Show that, for $0 < x < \frac{1}{2}\pi$, $\tanh x < \tan x$", but it was decided that this made the question too long and it was cut.

2 **2014 S3 Q7**

Like most vector questions, it is a good idea to start with a diagram, such as below:



(i) Note that the lines P_1P_4 and P_2P_3 are **not** (necessarily) parallel, so we cannot use alternate angles.

We have:

$$\begin{aligned}
\angle P_3P_2Q &= \angle P_4P_1Q \quad \text{angles subtended by the same arc are equal}\\
\angle P_2P_3Q &= \angle P_1P_4Q \quad \text{angles subtended by the same arc are equal}\\
\angle P_3QP_2 &= \angle P_4QP_1 \quad \text{vertically opposite angles are equal}
\end{aligned}$$

Hence the three angles are equal and so the triangles P_1QP_4 and P_2QP3 are similar.

You can use "angles in the same segment are equal" or any other way of stating this theorem!

Since the triangles are similar and P_2Q corresponds to P_1Q etc we have:

$$P_2Q = c \times P_1Q \implies c = \frac{P_2Q}{P_1Q}$$
$$P_3Q = c \times P_4Q \implies c = \frac{P_3Q}{P_4Q}$$

Where c is the scale factor of the enlargement. Equating the expressions for c gives:

$$\frac{P_2Q}{P_1Q} = \frac{P_3Q}{P_4Q} \implies (P_3Q)(P_1Q) = (P_2Q)(P_4Q)$$

as required. My final answer does not look exactly the same as the one in the question, but it is equivalent so this is fine!

(ii) Since Q is on the line P_1P_3 we can write the position vector of Q as:

$$\mathbf{q} = \mathbf{p}_1 + \lambda(\mathbf{p}_3 - \mathbf{p}_1)$$

and since Q also lies on P_2P_4 we have:

$$\mathbf{q} = \mathbf{p}_2 + \mu(\mathbf{p}_4 - \mathbf{p}_2)$$

Equating expressions for $\mathbf{q}$ gives us:

$$\begin{aligned}
\mathbf{p}_1 + \lambda(\mathbf{p}_3 - \mathbf{p}_1) &= \mathbf{p}_2 + \mu(\mathbf{p}_4 - \mathbf{p}_2)\\
(1-\lambda)\mathbf{p}_1 + \lambda\mathbf{p}_3 &= (1-\mu)\mathbf{p}_2 + \mu\mathbf{p}_4\\
(1-\lambda)\mathbf{p}_1 + (\mu-1)\mathbf{p}_2 + \lambda\mathbf{p}_3 - \mu\mathbf{p}_4 = 0 &
\end{aligned}$$

So we have $\displaystyle\sum_{i=1}^{4} a_i\mathbf{p}_i = 0$ where $a_1, a_2, a_3, a_4 = (1-\lambda), (\mu-1), \lambda, -\mu$, and so we also have $a_1 + a_2 + a_3 + a_4 = 0$ (and not all of the a_i can be simultaneously zero).

(iii) If we take $a_1 + a_3 = 0$, then by $(*)$ we also have $a_2 + a_4 = 0$. So we have $a_3 = -a_1$ and $a_4 = -a_2$. Substituting into the second part of $(*)$ gives:

$$\begin{aligned}
& a_1\mathbf{p}_1 + a_2\mathbf{p}_2 - a_1\mathbf{p}_3 - a_2\mathbf{p}_4 = 0\\
\implies & a_1(\mathbf{p}_1 - \mathbf{p}_3) = a_2(\mathbf{p}_4 - \mathbf{p}_2)\\
\implies & P_1P_3 \parallel P_2P_4
\end{aligned}$$

But P_1P_3 cannot be parallel to P_2P_4 as the points are distinct (look at the diagram!), and so we have $a_1 + a_3 \neq 0$ by contradiction (and similarly we know that $a_2 + a_4 \neq 0$).

We know that $a_1 + a_3 = -(a_2 + a_4)$ and that $a_1\mathbf{p}_1 + a_3\mathbf{p}_3 = -(a_2\mathbf{p}_2 + a_4\mathbf{p}_4)$ and so we have:

$$\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3} = \frac{a_2\mathbf{p}_2 + a_4\mathbf{p}_4}{a_2 + a_4}$$

Considering the LHS gives us:

$$\begin{aligned}
\frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3} &= \frac{(a_1 + a_3)\mathbf{p}_1 - a_3\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}\\
&= \mathbf{p}_1 + \frac{a_3}{a_1 + a_3}(\mathbf{p}_3 - \mathbf{p}_1)
\end{aligned}$$

and so this point lies on the line P_1P_3. Using the RHS we have:

$$\begin{aligned}
\frac{a_2\mathbf{p}_2 + a_4\mathbf{p}_4}{a_2 + a_4} &= \frac{(a_2 + a_4)\mathbf{p}_2 - a_4\mathbf{p}_2 + a_4\mathbf{p}_4}{a_2 + a_4}\\
&= \mathbf{p}_2 + \frac{a_4}{a_2 + a_4}(\mathbf{p}_4 - \mathbf{p}_2)
\end{aligned}$$

and so the point also lies on the line P_2P_4, and so it must be where the two lines intersect i.e.:

$$\mathbf{q} = \frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3} = \frac{a_2\mathbf{p}_2 + a_4\mathbf{p}_4}{a_2 + a_4}$$

Substituting this into the result from part **(i)** we have:

$$\begin{aligned}
(P_3Q)(P_1Q) = (P_2Q)(P_4Q) &\implies\\
\left(\mathbf{p}_3 - \frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}\right)\left(\mathbf{p}_1 - \frac{a_1\mathbf{p}_1 + a_3\mathbf{p}_3}{a_1 + a_3}\right) &= \left(\mathbf{p}_2 - \frac{a_2\mathbf{p}_2 + a_4\mathbf{p}_4}{a_2 + a_4}\right)\left(\mathbf{p}_4 - \frac{a_2\mathbf{p}_2 + a_4\mathbf{p}_4}{a_2 + a_4}\right)\\
\cancel{\frac{1}{(a_1 + a_3)^2}}\big(a_1\mathbf{p}_3 - a_1\mathbf{p}_1\big)\big(a_3\mathbf{p}_1 - a_3\mathbf{p}_3\big) &= \cancel{\frac{1}{(a_2 + a_4)^2}}\big(a_4\mathbf{p}_2 - a_4\mathbf{p}_4\big)\big(a_2\mathbf{p}_4 - a_2\mathbf{p}_2\big)\\
-a_1(\mathbf{p}_3 - \mathbf{p}_1) \times a_3(\mathbf{p}_3 - \mathbf{p}_1) &= -a_4(\mathbf{p}_2 - \mathbf{p}_4) \times a_2(\mathbf{p}_2 - \mathbf{p}_4)\\
a_1a_3(P_1P_3)^2 &= a_2a_4(P_2P_4)^2
\end{aligned}$$

3 **2014 S3 Q8**

If we expand the sum we get:

$$\sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) = \mathrm{f}(k^n) + \mathrm{f}(k^n+1) + \mathrm{f}(k^n+2) + \cdots + \mathrm{f}(k^{n+1}-1)$$

There are $(k^{n+1}-1)-(k^n-1) = k^{n+1}-k^n = k^n(k-1)$ terms here and since $\mathrm{f}(r) > \mathrm{f}(r+1)$ they are all less than or equal to $\mathrm{f}(k^n)$. Hence we have $\displaystyle\sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) \leqslant k^n(k-1)\times \mathrm{f}(k^n)$.

Similarly, all of the terms must be greater than $\mathrm{f}(k^{n+1})$ (as the last term is $\mathrm{f}(k^{n+1}-1)$ and so we have $\displaystyle\sum_{r=k^n}^{k^{n+1}-1} \mathrm{f}(r) \geqslant k^n(k-1)\times \mathrm{f}(k^{n+1})$.

The lower limit could actually be given as a strict inequality, but we could have equality for the upper limit in the case when $k=2, n=1$. Worrying about strict or non strict limits is a good habit to get into, but don't spend too much time on this during an exam!

(i) $\mathrm{f}(r) = 1/r$ satisfies the requirements of $\mathrm{f}(r) > \mathrm{f}(r+1)$, so we can use the result shown in the stem.

Comparing the top limits of the sum in this part and the sum in the stem, take $k=2$ (don't worry about the bottom limit for now). This gives us:

$$2^n \times 1 \times \mathrm{f}(2^{n+1}) \leqslant \sum_{r=2^n}^{2^{n+1}-1} \mathrm{f}(r) \leqslant 2^n \times 1 \times \mathrm{f}(2^n)$$

$$\frac{2^n}{2^{n+1}} \leqslant \sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r} \leqslant \frac{2^n}{2^n}$$

$$\frac{1}{2} \leqslant \sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r} \leqslant 1$$

The requested sum runs from $r=1$ to $r=2^{N+1}-1$. We can split this sum up as:

$$\sum_{r=1}^{2^{N+1}-1} = \sum_{r=2^0}^{2^{0+1}-1} + \sum_{r=2^1}^{2^{1+1}-1} + \sum_{r=2^2}^{2^{2+1}-1} + \sum_{r=2^3}^{2^{3+1}-1} + \cdots + \sum_{r=2^N}^{2^{N+1}-1}$$

Each of these sums is bounded below by $\dfrac{1}{2}$ and above by 1, and there are $N+1$ of them, hence:

$$\frac{N+1}{2} \leqslant \sum_{r=1}^{2^{N+1}-1} \frac{1}{r} \leqslant N+1$$

As $N \to \infty$ we have $\frac{N+1}{2} \to \infty$ and hence $\displaystyle\sum_{r=1}^{\infty} \frac{1}{r}$ does not converge.

(ii) Here $\mathrm{f}(r) = \dfrac{1}{r^3}$, and taking $k = 2$ again gives:

$$\sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r^3} \leqslant 2^n \times \frac{1}{(2^n)^3} \implies \sum_{r=2^n}^{2^{n+1}-1} \frac{1}{r^3} \leqslant \frac{1}{2^{2n}}$$

from the stem result.

Splitting the sum in the same way as in part **(i)** gives us:

$$\sum_{r=1}^{2^{N+1}-1} \frac{1}{r^3} \leqslant \frac{1}{2^0} + \frac{1}{2^2} + \frac{1}{2^4} + \cdots + \frac{1}{2^{2N}}$$

This is a geometric series with first term 1 and common ratio $\frac{1}{4}$, so the sum to infinity is $\dfrac{1}{1-\frac{1}{4}} = \frac{4}{3}$. Hence we have:

$$\sum_{r=1}^{\infty} \frac{1}{r^3} \leqslant \frac{4}{3}$$

(iii) $S(1000)$ is the set of positive integers less than 1000 (so at most 999) which do not have a 2 in their decimal representation. So we have 9 options for the first digit (0,1,3,4,5,6,7,8,9), 9 options for the second digit and 9 options for the third digit. However we need to exclude the case of picking "000", so in total there are $9^3 - 1$ distinct numbers in $S(1000)$.

For the last part, start by considering $\sigma(10)$, $\sigma(100)$ etc.

$S(10) = 1, 3, 4, 5, 6, 7, 8, 9$ (i.e. has 8 members) and so
$\sigma(10) = 1 + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{8} + \frac{1}{9} < 8 \times 1 = 8$.

$S(100) = 1, 3, \cdots, 9, 10, 11, 13, \cdots, 19, 30, 31, \ldots$ and so it has $8 + 8 \times 9$ members (it has 8 one-digit members and then 8 sets of two-digit members each with 9 possibilities for the second digit - as 0 is now a possibility!).

We have:

$$\begin{aligned}\sigma(100) &= \left(1 + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{8} + \frac{1}{9}\right) + \left(\frac{1}{10} + \frac{1}{11} + \frac{1}{13} + \cdots + \frac{1}{18} + \frac{1}{19}\right) \\ &\quad + \left(\frac{1}{30} + \frac{1}{31} + \frac{1}{33} + \cdots + \frac{1}{39}\right) + \cdots \\ &\qquad + \left(\frac{1}{90} + \frac{1}{91} + \frac{1}{93} + \cdots + \frac{1}{99}\right) \\ &< 8 \times 1 + 8 \times 9 \times \frac{1}{10} = 8 + 8 \times \frac{9}{10}\end{aligned}$$

Now consider $\sigma(1000)$. This has 8 one-digit members, 8×9 two-digit members and $8 \times 9 \times 9$ three-digit members. In $\sigma(1000)$ the reciprocals of the one-digit members are all less than or equal to 1, the reciprocals of the two-digit members are all less than or equal to $\frac{1}{10}$ and the reciprocals of the three-digit members are all less than or equal to

$\frac{1}{100}$. So we have:

$$\begin{aligned}\sigma(1000) &< 8 \times 1 + 8 \times 9 \times \frac{1}{10} + 8 \times 9 \times 9 \times \frac{1}{100} \\ &= 8 + 8 \times \frac{9}{10} + 8 \times \left(\frac{9}{10}\right)^2\end{aligned}$$

Following the same argument we have:

$$\sigma(10^N) < 8 + 8 \times \frac{9}{10} + 8 \times \left(\frac{9}{10}\right)^2 + \cdots + 8 \times \left(\frac{9}{10}\right)^{N-1}$$

This is another geometric series, and the sum to infinity is given by $\dfrac{8}{1 - \frac{9}{10}} = 80$, hence we have $\sigma(n) < 80$ for all n.

The "point" of this last bit is that in part **(i)** you showed that $\sum \frac{1}{r}$ does not converge (is unbounded above), but if instead you sum all the reciprocals of the integers which do not contain a digit 2 you get a sum which is bounded above by 80. This initially seems rather surprising, but large numbers are very likely to contain a digit of 2 (or any other digit!) so as the sequence goes on more and more numbers are excluded.

You can read more about *Kempner Series* here.