STEP Support Programme - Cambridge state school offer-holders day
Workshop 2 Questions

1 **2013 S2 Q11**

Three identical particles lie, not touching one another, in a straight line on a smooth horizontal surface. One particle is projected with speed u directly towards the other two which are at rest. The coefficient of restitution in all collisions is e, where $0 < e < 1$.

(i) Show that, after the second collision, the speeds of the particles are $\frac{1}{2}u(1-e)$, $\frac{1}{4}u(1-e^2)$ and $\frac{1}{4}u(1+e)^2$. Deduce that there will be a third collision whatever the value of e.

(ii) Show that there will be a fourth collision if and only if e is less than a particular value which you should determine.

2 **2013 S2 Q12**

The random variable U has a Poisson distribution with parameter λ. The random variables X and Y are defined as follows.

$$X = \begin{cases} U & \text{if } U \text{ is } 1,\ 3,\ 5,\ 7,\ \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$Y = \begin{cases} U & \text{if } U \text{ is } 2,\ 4,\ 6,\ 8,\ \ldots \\ 0 & \text{otherwise} \end{cases}$$

(i) Find $\mathrm{E}(X)$ and $\mathrm{E}(Y)$ in terms of λ, α and β, where

$$\alpha = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \cdots \quad \text{and} \quad \beta = \frac{\lambda}{1!} + \frac{\lambda^3}{3!} + \frac{\lambda^5}{5!} + \cdots .$$

(ii) Show that

$$\mathrm{Var}\,(X) = \frac{\lambda\alpha + \lambda^2\beta}{\alpha + \beta} - \frac{\lambda^2\alpha^2}{(\alpha+\beta)^2}$$

and obtain the corresponding expression for $\mathrm{Var}\,(Y)$. Are there any non-zero values of λ for which $\mathrm{Var}\,(X) + \mathrm{Var}\,(Y) = \mathrm{Var}\,(X+Y)$?

3 **2013 S3 Q1**

Given that $t = \tan \frac{1}{2}x$, show that $\dfrac{\mathrm{d}t}{\mathrm{d}x} = \frac{1}{2}(1+t^2)$ and $\sin x = \dfrac{2t}{1+t^2}$.

Hence show that

$$\int_0^{\frac{1}{2}\pi} \frac{1}{1+a\sin x}\,\mathrm{d}x = \frac{2}{\sqrt{1-a^2}} \arctan \frac{\sqrt{1-a}}{\sqrt{1+a}} \qquad (0 < a < 1).$$

Let

$$I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x}{2+\sin x}\,\mathrm{d}x \qquad (n \geqslant 0).$$

By considering $I_{n+1} + 2I_n$, or otherwise, evaluate I_3.

4 **2012 S3 Q1**

Given that $z = y^n \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$, show that

$$\frac{\mathrm{d}z}{\mathrm{d}x} = y^{n-1}\frac{\mathrm{d}y}{\mathrm{d}x}\left(n\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}\right).$$

(i) Use the above result to show that the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \sqrt{y} \qquad (y > 0)$$

that satisfies $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$ is $y = \left(\frac{3}{8}x^2 + 1\right)^{\frac{2}{3}}$.

(ii) Find the solution to the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - y\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^2 = 0$$

that satisfies $y = 1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ when $x = 0$.

5 2006 S3 Q4

The function f satisfies the identity

$$\mathrm{f}(x) + \mathrm{f}(y) \equiv \mathrm{f}(x+y) \qquad (*)$$

for all x and y. Show that $2\mathrm{f}(x) \equiv \mathrm{f}(2x)$ and deduce that $\mathrm{f}''(0) = 0$. By considering the Maclaurin series for $\mathrm{f}(x)$, find the most general function that satisfies $(*)$.
[*Do not consider issues of existence or convergence of Maclaurin series in this question.*]

(i) By considering the function G, defined by $\ln\big(\mathrm{g}(x)\big) = \mathrm{G}(x)$, find the most general function that, for all x and y, satisfies the identity

$$\mathrm{g}(x)\mathrm{g}(y) \equiv \mathrm{g}(x+y)\,.$$

(ii) By considering the function H, defined by $\mathrm{h}(\mathrm{e}^u) = \mathrm{H}(u)$, find the most general function that satisfies, for all positive x and y, the identity

$$\mathrm{h}(x) + \mathrm{h}(y) \equiv \mathrm{h}(xy)\,.$$

(iii) Find the most general function t that, for all x and y, satisfies the identity

$$\mathrm{t}(x) + \mathrm{t}(y) \equiv \mathrm{t}(z)\,,$$

where $z = \dfrac{x+y}{1-xy}$.

6 2008 S2 Q8

The points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, relative to the origin O. The points A, B and O are not collinear. The point P lies on AB between A and B such that

$$AP : PB = (1-\lambda) : \lambda\,.$$

Write down the position vector of P in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. Given that OP bisects $\angle AOB$, determine λ in terms of a and b, where $a = |\mathbf{a}|$ and $b = |\mathbf{b}|$.

The point Q also lies on AB between A and B, and is such that $AP = BQ$. Prove that

$$OQ^2 - OP^2 = (b-a)^2\,.$$

7 **2009 S3 Q4**

For any given (suitable) function f, the *Laplace transform* of f is the function F defined by

$$\mathrm{F}(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{f}(t)\,\mathrm{d}t \qquad (s > 0)\,.$$

(i) Show that the Laplace transform of $\mathrm{e}^{-bt}\mathrm{f}(t)$, where $b > 0$, is $\mathrm{F}(s+b)$.

(ii) Show that the Laplace transform of $\mathrm{f}(at)$, where $a > 0$, is $a^{-1}\mathrm{F}(\frac{s}{a})$.

(iii) Show that the Laplace transform of $\mathrm{f}'(t)$ is $s\mathrm{F}(s) - \mathrm{f}(0)$.

(iv) In the case $\mathrm{f}(t) = \sin t$, show that $\mathrm{F}(s) = \dfrac{1}{s^2+1}$.

Using only these four results, find the Laplace transform of $\mathrm{e}^{-pt}\cos qt$, where $p > 0$ and $q > 0$.

8 **2009 S3 Q6**

Show that $\left|\mathrm{e}^{\mathrm{i}\beta} - \mathrm{e}^{\mathrm{i}\alpha}\right| = 2\sin\frac{1}{2}(\beta - \alpha)$ for $0 < \alpha < \beta < 2\pi$. Hence show that

$$\left|\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\beta}\right| \left|\mathrm{e}^{\mathrm{i}\gamma} - \mathrm{e}^{\mathrm{i}\delta}\right| + \left|\mathrm{e}^{\mathrm{i}\beta} - \mathrm{e}^{\mathrm{i}\gamma}\right| \left|\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\delta}\right| = \left|\mathrm{e}^{\mathrm{i}\alpha} - \mathrm{e}^{\mathrm{i}\gamma}\right| \left|\mathrm{e}^{\mathrm{i}\beta} - \mathrm{e}^{\mathrm{i}\delta}\right|,$$

where $0 < \alpha < \beta < \gamma < \delta < 2\pi$.

Interpret this result as a theorem about cyclic quadrilaterals.

9 **2012 S3 Q8**

The sequence F_0, F_1, F_2, $\ldots$ is defined by $F_0 = 0$, $F_1 = 1$ and, for $n \geqslant 0$,

$$F_{n+2} = F_{n+1} + F_n\,.$$

(i) Show that $F_0F_3 - F_1F_2 = F_2F_5 - F_3F_4\,.$

(ii) Find the values of $F_nF_{n+3} - F_{n+1}F_{n+2}$ in the two cases that arise.

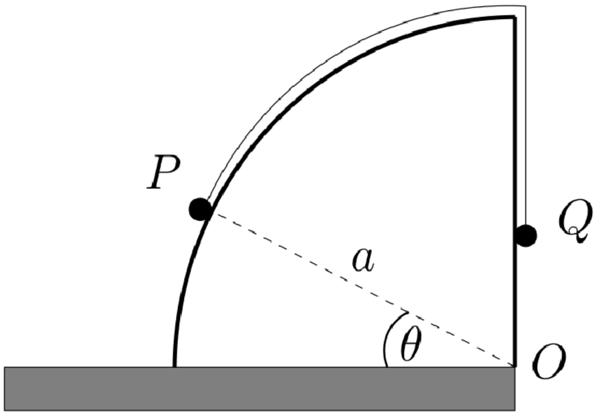
(iii) Prove that, for $r = 1, 2, 3, \ldots,$

$$\arctan\left(\frac{1}{F_{2r}}\right) = \arctan\left(\frac{1}{F_{2r+1}}\right) + \arctan\left(\frac{1}{F_{2r+2}}\right)$$

and hence evaluate the following sum (which you may assume converges):

$$\sum_{r=1}^{\infty} \arctan\left(\frac{1}{F_{2r+1}}\right)\,.$$

10 2010 S3 Q9



The diagram shows two particles, P and Q, connected by a light inextensible string which passes over a smooth block fixed to a horizontal table. The cross-section of the block is a quarter circle with centre O, which is at the edge of the table, and radius a. The angle between OP and the table is θ. The masses of P and Q are m and M, respectively, where $m < M$.

Initially, P is held at rest on the table and in contact with the block, Q is vertically above O, and the string is taut. Then P is released. Given that, in the subsequent motion, P remains in contact with the block as θ increases from 0 to $\frac{1}{2}\pi$, find an expression, in terms of m, M, θ and g, for the normal reaction of the block on P and show that

$$\frac{m}{M} \geqslant \frac{\pi - 1}{3}\,.$$

11 2008 S3 Q13

A box contains n pieces of string, each of which has two ends. I select two string ends at random and tie them together. This creates either a ring (if the two ends are from the same string) or a longer piece of string. I repeat the process of tying together string ends chosen at random until there are none left.

Find the expected number of rings created at the first step and hence obtain an expression for the expected number of rings created by the end of the process. Find also an expression for the variance of the number of rings created.

Given that $\ln 20 \approx 3$ and that $1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \ln n$ for large n, determine approximately the expected number of rings created in the case $n = 40\,000$.